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The Arcsine law and an asymptotic behavior of orthogonal polynomials

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Abstract

In the present papar we generalize "quantum-classical correspondence" for harmonic oscillators to the context of interacting Fock spaces. Under a simple condition for Jacobi sequences, it is shown that the Arcsine law is the unique probability distribution corresponding to the "Classical limits (large quantum number limits)". As a corollary, we obtain that the squared n-th orthogonal polynomials for a probability distribution corresponding to such kinds of interacting Fock spaces, multiplied by the probability distribution and normalized, weakly converge to the Arcsine law as n tends to infinity.

1 Introduction

The distribution μ_{As} defined as

$$\mu_{As}(dx)=\frac{1}{\pi}\frac{dx}{\sqrt{2-x^{2}}}\quad(-\sqrt{2}<x<\sqrt{2}).$$

is called the Arcsine law, which plays lots of crucial roles both in pure and applied probability theory. The n-th moment $M_n := \int_{\mathbb{R}} x^n \mu_{As}(dx)$ is given

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by

$$M_{2m+1} = 0, \quad M_{2m} = \frac{1}{2^m}\binom{2m}{m}.$$

The moment problem for the Arcsine law is determinate, that is, the moment sequence $\{M_n\}$ characterizes μ_{As}. In [8] we have proved that the Arcsine law appears as the "Classical limit distribution" of quantum harmonic oscillator, in the framework of algebraic probability thoery (also known as "noncommutative probability theory" or "quantum probability theory").

The purpose of this paper is to extend this "quantum-classical correspondence" in general interacting Fock spaces [1]. It implies asymptotic behavior of orthogonal polynomials for certain kind of symmetric probability measures.

2 Basic notions

2.1 Algebraic Probability Space

Let $\mathcal{A}$ be a $*$-algebra. We call a linear map $\varphi : \mathcal{A} \to \mathbb{C}$ a state on $\mathcal{A}$ if it satisfies

$$\varphi(1) = 1, \quad \varphi(a^*a) \geq 0.$$

A pair $(\mathcal{A}, \varphi)$ of a $*$-algebra and a state on it is called an algebraic probability space. An element of $\mathcal{A}$ is called an algebraic random variable. Here we adopt a notation for a state $\varphi : \mathcal{A} \to \mathbb{C}$, an element $X \in \mathcal{A}$ and a probability distribution μ on $\mathbb{R}$.

Notation 2.1. We use the notation $X \sim_\varphi \mu$ when $\varphi(X^m) = \int_{\mathbb{R}} x^m \mu(dx)$ for all $m \in \mathbb{N}$.

Remark 2.2. Existence of μ for X which satisfies $X \sim_\varphi \mu$ always holds.

2.2 Interacting Fock space

Definition 2.3 (Jacobi sequence). A sequence $\{\omega_n\}$ is called a Jacobi sequence if it satisfies one of the conditions below:

- (finite type) There exist a number m such that $\omega_n > 0$ for $n < m$ and $\omega_n = 0$ for $n \geq m$;
- (infinite type) $\omega_n > 0$ for all n.

Definition 2.4 (Interacting Fock space). Let $\{\omega_n\}$ be a Jacobi sequence and $\{\alpha_n\}$ be a real sequence. An interacting Fock space (IFS) $\Gamma_{\{\omega_n\},\{\alpha_n\}}$ is a quadruple $(\Gamma(\mathbb{C}), a, a^*, a^o)$ where $\Gamma(\mathbb{C})$ is a Hilbert space $\Gamma(\mathbb{C}) := \oplus_{n=0}^{\infty}\mathbb{C}\Phi_n$ with inner product given by $\langle\Phi_n, \Phi_m\rangle = \delta_{n,m}$, and a, a^* are operators defined as follows:

$$a\Phi_0 = 0, \quad a\Phi_n = \sqrt{\omega_n}\Phi_{n-1}(n \geq 1),$$
$$a^*\Phi_n = \sqrt{\omega_{n+1}}\Phi_{n+1},$$
$$a^o\Phi_n = \alpha_{n+1}\Phi_n.$$

Let $\mathcal{A}$ be the $*$-algebra generated by $\{a, a^*, a^o = (a^o)^*\}$, and φ_n be the state defined as $\varphi_n(\cdot) := \langle\Phi_n, (\cdot)\Phi_n\rangle$. Then $(\mathcal{A}, \varphi_n)$ is an algebraic probability space.

2.3 Interacting Fock Spaces and orthogonal polynomials for probability measures

Theorems for interacting Fock spaces often have interesting interpretation in terms of orthogonal polynomials. To see this we review the relation between interacting Fock spaces, probability measures and orthogonal polynomials. Let μ be a probability measure on $\mathbb{R}$ having finite moments. (For the rest of the present paper, we always assume that all the moments are finite.) Then the space of polynomial functions is contained in the Hilbert space $L^2(\mathbb{R}, \mu)$. A Gram-Schmidt procedure provides orthogonal polynomials which only depend on the moment sequence.

Let $\{p_n(x)\}_{n=0,1,\cdots}$ be the monic orthogonal polynomials of μ such that the degree of p_n equals to n. Then there exist sequences $\{\alpha_n\}_{n=0,1,\cdots}$ and Jacobi sequence $\{\omega_n\}_{n=1,2,\cdots}$ such that

$$xp_n(x) = p_{n+1}(x) + \alpha_{n+1}p_n(x) + \omega_n p_{n-1}(x) \quad (p_{-1}(x) \equiv 0).$$

$\alpha_n \equiv 0$ if μ is symmetric, i.e., $\mu(-dx) = \mu(dx)$. We call $\{\{\omega_n\}, \{\alpha_n\}\}$ Jacobi sequences corresponding to μ.

It is known that there exist an isometry $U : \Gamma_{\{\omega_n\}} \to L^2(\mathbb{R}, \mu)$ which sends $X := a + a^* + a^o$ to the multiplication operator, and through which we obtain

$$X := a + a^* + a^o \sim_{\varphi_N} |P_N(x)|^2\mu(dx)$$

where P_n denotes the normalized orthogonal polynomial of degree n [1, 6].

In other words, through U, we can "decompose" a (measure-theoretic/classical) random variable into the sum of non-commutative algebraic random variables. (This crucial idea in algebraic probability theory is called "quantum decomposition" [5, 6].)

3 Quantum-Classical Correspondence for interacting Fock spaces

3.1 Quantum-Classical Correspondence for Harmonic Oscillator

The interacting Fock space corresponding to $\omega_n = n, \alpha_n \equiv 0$ is called "Quantum Harmonic Osillator(QHO)". For quantum harmonic oscillator, it is well known that

$$X := a + a^* + a^o \equiv a + a^*$$

represents the "position" and that

$$X \sim_{\varphi_0} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx.$$

That is, in $n = 0$ case, the distribution of position is Gaussian.

On the other hand, the asymptotic behavior of the distributions of position as n tends to infinity is nontrivial. In other words, what is the "Classical limit" of quantum harmonic oscillator? This question, which is related to fundamental problems in Quantum theory and asymptotic analysis [3], was analyzed in [8] from the viewpoint of noncommutative algebraic probability with quite a simple combinatorial argument. The answer for this question is that the "Classical Limit" for quantum harmonic oscillator is nothing but the Arcsine law.

Theorem 3.1 ([8]). *Let $\Gamma_{\{\omega_n=n\},\{\alpha_n\equiv 0\}} := (\Gamma(\mathbb{C}), a, a^*, a^o \equiv 0)$ be the Quantum harmonic oscillator, $X := a + a^*$ and μ_N be a probability distribution on $\mathbb{R}$ such that*

$$\frac{X}{\sqrt{2N-1}} \sim_{\varphi_N} \mu_N.$$

Then μ_N weakly converges to μ_{As}.

Here $\sqrt{2N-1}$ is the normalization factor to make the variance of μ_N to be equal to 1. Since it is easy to see that th Arcsine law gives "time-averaged behavior" of classical harmonic oscillator, the result can be viewed as "Quantum-Classical Correspondence" for harmonic oscillators!

3.2 The notion of Classical limit distribution of IFS

As the case for QHO, we define the notion of classical limit distribution for IFS. It is a distribution to which the distribution for X under φ_N, after normalization, converges in moment.

Definition 3.2 (Classical Limit distribution). Let $\Gamma_{\{\omega_n\},\{\alpha_n\}} := (\Gamma(\mathbb{C}), a, a^*, a^o)$ be an interacting Fock space, $X := a + a^* + a^o$ and μ_N be a probability distribution on $\mathbb{R}$ such that

$$\frac{X - \alpha_{N+1}}{\sqrt{\omega_N + \omega_{N-1}}} \sim_{\varphi_N} \mu_N.$$

A probability distribution μ on $\mathbb{R}$ is called a classical limit distribution if μ_N converge μ in moment.

By normalization, μ_N has mean 0 and variance 1.

Remark 3.3. The uniqueness of classical limit distribution depends on the moment problem. Note that convergence in moment implies weak convergence in case the limit distribution is the solution of a determinate moment problem [2, 4].

3.3 The Arcsine law as classical limit distribution

The Arcsine law is the classical limit distribution for certain kinds of IFSs.

Theorem 3.4. *Let $\Gamma_{\{\omega_n\},\{\alpha_n\}} := (\Gamma(\mathbb{C}), a, a^*, a^o)$ be an interacting Fock space satisfying*

$$\lim_{n\to\infty} \frac{\omega_{n+1}}{\omega_n} = 1, \quad \lim_{n\to\infty} \frac{\alpha_n - \alpha_{n+1}}{\sqrt{\omega_n + \omega_{n+1}}} = 0.$$

Then the classical limit distribution is the Arcsine law μ_{As}.

Proof. We will show that $\varphi_N((\frac{X-\alpha_{N+1}}{\sqrt{\omega_N+\omega_{N-1}}})^n) = \varphi_N((\frac{a+a^*+a^o-\alpha_{N+1}}{\sqrt{\omega_N+\omega_{N-1}}})^n)$ converge to m-th moment of the Arcsine law. It is easy to show that

$$\frac{\omega_{N+k}}{\omega_N} \to 1 \quad (N \to \infty),$$

and the effect of $\frac{a^o-\alpha_{N+1}}{\sqrt{\omega_N+\omega_{N-1}}}$ tends to 0. Hence it suffices to calculate $\varphi_N((\frac{a+a^*}{\sqrt{\omega_N+\omega_{N-1}}})^n)$.

First, it is clear that

$$\varphi_N((\frac{a + a^*}{\sqrt{\omega_N + \omega_{N-1}}})^{2m+1}) = \langle \Phi_N, (\frac{a + a^*}{\sqrt{\omega_N + \omega_{N-1}}})^{2m+1} \Phi_N \rangle = 0$$

since $\langle \Phi_N, \Phi_M \rangle = 0$ when $N \neq M$. To consider the moments of even degrees, we introduce the following notations:

- $\Lambda^{2m} := \{\text{maps from } \{1, 2, ..., 2m\} \text{ to } \{1, *\}\}$,
- $\Lambda^{2m}_m := \{\lambda \in \Lambda^{2m}; |\lambda^{-1}(1)| = |\lambda^{-1}(*)| = m\}$.

Note that the cardinality $|\Lambda_m^{2m}|$ equals to $\binom{2m}{m}$ because the choice of λ is equivalent to the choice of m elements which consist the subset $\lambda^{-1}(1)$ from $2m$ elements in $\{1, 2, ..., 2m\}$.

It is clear that for any $\lambda \notin \Lambda_m^{2m}$

$$\langle \Phi_N, a^{\lambda_1} a^{\lambda_2} \cdots a^{\lambda_{2m}} \Phi_N \rangle = 0$$

since $\langle \Phi_N, \Phi_M \rangle = 0$ when $N \neq M$. On the other hand, for any $\lambda \in \Lambda_m^{2m}$

$$\frac{1}{\omega_N^m} \langle \Phi_N, a^{\lambda_1} a^{\lambda_2} \cdots a^{\lambda_{2m}} \Phi_N \rangle \to 1 \quad (N \to \infty)$$

holds since $\langle \Phi_N, a^{\lambda_1} a^{\lambda_2} \cdots a^{\lambda_{2m}} \Phi_N \rangle$ becomes the product of $2m$ terms having the form $\sqrt{\omega_{N+k}}$ (k is an integer and $-m + 1 \leq k \leq m$) and

$$\frac{\omega_{N+k}}{\omega_N} \to 1 \quad (N \to \infty)$$

as we have mentioned. Hence,

$$\varphi_N\left(\left(\frac{a + a^*}{\sqrt{\omega_N + \omega_{N-1}}}\right)^{2m}\right) = \langle \Phi_N, \left(\frac{a + a^*}{\sqrt{(1 + \frac{\omega_{N-1}}{\omega_N})\omega_N}}\right)^{2m} \Phi_N \rangle$$

$$= \frac{1}{(1+\frac{\omega_{N-1}}{\omega_N})^m} \sum_{\lambda \in \Lambda^{2m}} \frac{1}{\omega_N^m} \langle \Phi_N, a^{\lambda_1} a^{\lambda_2} \cdots a^{\lambda_{2m}} \Phi_N \rangle$$

$$= \frac{1}{(1+\frac{\omega_{N-1}}{\omega_N})^m} \sum_{\lambda \in \Lambda_m^{2m}} \frac{1}{\omega_N^m} \langle \Phi_N, a^{\lambda_1} a^{\lambda_2} \cdots a^{\lambda_{2m}} \Phi_N \rangle$$

$$\to \frac{1}{2^m} |\Lambda_m^{2m}| = \frac{1}{2^m} \binom{2m}{m} \quad (N \to \infty).$$

□

Remark 3.5. It is quite interesting to compare Kerov's theorem on his "Arcsine Law"which is different from our μ_{As} but closely related to it [7].

Remark 3.6. Since the Arcsine law is the solution of a determinate moment problem, moment convergence implies weak convergence.

The theorem means that μ_{As} is turned out to be the Classical limit distribution of many kinds of IFS. For example, IFSs corresponding to uniform distribution, exponential distribution or "q-Gaussians"($-1 < q \leq 1$, $\omega_n = [n]_q := 1 + q + q^2 + \cdots + q^{n-1}$, $\alpha_n \equiv 0$. $q = 1$ is Gaussian and $q = 0$ is Wigner Semicircle Law) satisfy the condition above.

3.4 An asymptotic behavior of orthogonal polynomials

The theorem above implies description of an asymptotic behavior of orthogonal polinomials:

Corollary 3.7. *Let μ be a prabablity measure such that the corresponting Jacobi sequences $\{\omega_n\}, \{\alpha_n\}$ satisfies the conditions above,*

Then the measure μ_n defined as $\mu_n(dx) := |P_n(\sqrt{\omega_N + \omega_{N-1}}x)|^2 \mu(\sqrt{\omega_N + \omega_{N-1}}dx)$ weakly converge to μ_{As}.

Many kinds of orthogonal polynomials such as Jacobi polynomials(e.g. Legendre polynomials), Laguerre polynomials or q-Hermite polynomials($-1 < q \leq 1$) satisfy the condition above.

4 Open questions

Here we have two fundamental questions:

Q1 How can we characterize the family of classical limit distributions?

Q2 When the classical limit distributions are solutions of determinate moment problem?

These questions will be discussed in [9].

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