数理解析研究所講究録
第 1961 巻 2015 年 42-55

Dynamics for non-symmetric Hamiltonians, and Gupta-Bleuler formalism for Dirac-Maxwell operator

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Abstract

The Gupta-Bleuler formalism for the Dirac-Maxwell model in the Lorenz gauge is investigated. A full description in detail will appear in [7].

1 Introduction

To quantize canonically gauge theories in a Lorentz covariant gauge, we have to adopt an indefinite metric space as the state space in order to realize the canonical commutation relation. Then it is important to identify a positive definite subspace as a physical state space, and show that an ordinary quantum system is defined in the physical state space. There are several well-known procedures which provides a covariant quantization; the Gupta-Bleuler formalism [5, 9] is the most basic one. The purpose of the present article is to apply the Gupta-Bleuler formalism to the Dirac-Maxwell model in the Feynman (Lorenz) gauge.

Dirac-Maxwell model describes a quantum system of Dirac particles under an external potential V interacting with a quantum gauge field. By using this model and the informal perturbation theory, we can derive some quantitative predictions such as the Klein-Nishina formula [11], and thus the Dirac-Maxwell model is realistic and worth investigating, even though it may suffer from the negative energy problem. The first mathematically rigorous study of this model was given by Arai in Ref. [1], and there are several preceding studies so far (see, e.g., [2], [3], [4], [12], and [13]).

In the Gupta-Bleuler formalism, to impose the *Gupta subsidiary condition*

$$[\partial_\mu A^\mu]^+(t,\boldsymbol{x})\Psi=0, \tag{1.1}$$

on the state vectors is one of the most important steps, where A^μ's denote gauge fields, $[\partial_\mu A^\mu]^+$ denotes the positive frequency part of the free field $\partial_\mu A^\mu$. However, when we perform this procedure rigorously, several problems arise. The first problem is the existence of the time-evolution of the gauge field $t\mapsto A^\mu(t,\boldsymbol{x})$. Since the present state space is infinite dimensional indefinite metric space, it is far from trivial whether there is a solution of quantum Heisenberg equations of motion. The second one is the identification of the positive frequency part of the operator satisfying Klein-Gordon equation in an indefinite metric space. In the present paper, we solve the first problem by the general construction method of time evolution operator generated by a non-self-adjoint operator given in [6]. We emphasize that there are some models for which the time-evolution of the gauge fields can be solved explicitly (e.g., [10, 14]), but for the Dirac-Maxwell model, we can *not* explicitly solve it. As to the second problem, we give a definition of the positive frequency part of a free field safisfying Klein-Gordon equation in an abstract setup. Our definition is different from that given in Ref [10], but results in the same consequence when applied to the concrete models.

2 Abstract results

2.1 Construction of dynamics for non-symmetric Hamiltonians

We begin by summarizing the results obtained in Ref. [6].

Let $\mathcal{H}$ be a Complex Hilbert space and $\langle\cdot,\cdot\rangle$ its inner product, and $\|\cdot\|$ its norm. The inner product is linear in the second variable. For a linear operator T in $\mathcal{H}$, we denote its domain (resp. range) by $D(T)$ (resp. $R(T)$). We also denote the adjoint of T by T^* and the closure by $\overline{T}$ if these exist. For a self-adjoint operator T, $E_T(\cdot)$ denotes the spectral measure of T.

Let H_0 be a self-adjoint operator on $\mathcal{H}$. Suppose that there is a non-negative self-adjoint operator A which is strongly commuting with H_0. We use the notations

$$V_L := E_A([0,L]), \quad L \geq 0, \tag{2.1}$$

$$D := \bigcup_{L\geq 0} V_L, \tag{2.2}$$

$$D' = D \cap D(H_0). \tag{2.3}$$

Definition 2.1. *We say that a linear operator B is in $\mathcal{C}_0$-class if B satisfies*

(i) B is densely defined and closed.

(ii) B and B^* are $A^{1/2}$- bounded.

(iii) There is a constant $b > 0$ such that $\xi \in V_L$ implies $B\xi$ and $B^*\xi$ belong to V_{L+b}.

The set of all $\mathcal{C}_0$-class operators is also denoted by the same symbol $\mathcal{C}_0$. We consider an operator

$$H = H_0 + H_1 \tag{2.4}$$

with $H_1 \in \mathcal{C}_0$.

Proposition 2.1. *For each $t, t' \in \mathbb{R}$, $\xi \in D$, the series:*

$$U(t,t')\xi := \xi + (-i)\int_{t'}^{t} d\tau_1\, H_1(\tau_1)\xi + (-i)^2\int_{t'}^{t} d\tau_1 \int_{t'}^{\tau_1} d\tau_2\, H_1(\tau_1)H_1(\tau_2)\xi + \cdots \tag{2.5}$$

converges absolutely, where each of integrals are strong integrals, and $H_1(\tau) := e^{i\tau H_0} H_1 e^{-i\tau H_0}$ $(\tau \in \mathbb{R})$.

Proof. See [6, Theorem 2.1]. □

Let

$$W(t) := e^{-itH_0}\overline{U(t,0)}, \quad t \in \mathbb{R}. \tag{2.6}$$

Proposition 2.2. *For each* $\xi \in D'$, *the vector valued function* $t \mapsto \xi(t) := W(t)\xi$ *is strongly differentiable in* $t \in \mathbb{R}$, *and*

$$\frac{d}{dt}\xi(t) = -iH\xi(t) = -iW(t)H\xi, \tag{2.7}$$

Proof. See [6, Theorem 2.5]. □

Proposition 2.3 (weak Heisenberg equation)**.** *Let* $B \in \mathcal{C}_0$. *Then the operator-valued function* $B(t)$ *defined as*

$$D(B(t)) := D, \quad B(t)\xi := W(-t)BW(t)\xi, \quad \xi \in D, \quad t \in \mathbb{R}, \tag{2.8}$$

is a solution of weak Heisenberg equation:

$$\frac{d}{dt}\langle \eta, B(t)\xi \rangle = \langle (iH)^*\eta, B(t)\xi \rangle - \langle B(t)^*\eta, iH\xi \rangle, \quad \xi, \eta \in D'. \tag{2.9}$$

Proof. See [6, Theorem 2.7]. □

Proposition 2.4. *Let* $H_1 \in \mathcal{C}_0$ *and symmetric. Then, for the symmetric operator* H, *exactly one of the following (a) and (b) holds:*

(a) H *has no self-adjoint extension.*

(b) H *is essentially self-adjoint.*

Proof. See [8, Theorem 2.1]. □

2.2 N-th derivatives and Taylor expansion

To identify the physical state space in later sections, we need to extend the results obtained in Ref. [6]. The proofs of the following theorems will appear in [7].

Definition 2.2. *We say an operator* B *is in* $\mathcal{C}_1$*-class if it satisfies*

(i) B *is in* $\mathcal{C}_0$*-class.*

(ii) There is an operator $C \in \mathcal{C}_0$ *such that*

$$\langle (iH)^*\xi, B\eta \rangle - \langle B^*\xi, iH\eta \rangle = \langle \xi, C\eta \rangle \tag{2.10}$$

for all $\xi, \eta \in D'$.

We denote

$$\mathrm{ad}(B) := \overline{C \restriction D(A^{1/2})}. \tag{2.11}$$

Theorem 2.1. *For $B \in \mathcal{C}_1$ and $\xi \in D'$ the mapping $t \mapsto B(t) = W(-t)BW(t)\xi \in \mathcal{H}$ is strongly continuously differentiable in $t \in \mathbb{R}$ and satisfies the Heisenberg equation of motion*

$$\frac{d}{dt}B(t)\xi = W(-t)\mathrm{ad}(B)W(t)\xi. \tag{2.12}$$

Definition 2.3. *We define $\mathcal{C}_n$-class and $\mathrm{ad}^n(B)$ for $n = 0, 1, \ldots$ inductively. That is, we say that an operator B is in $\mathcal{C}_n$-class if B is in $\mathcal{C}_{n-1}$-class and $\mathrm{ad}(B)$ is in $\mathcal{C}_{n-1}$-class. For $B \in \mathcal{C}_n$, we write*

$$\mathrm{ad}^n(B) := \mathrm{ad}(\mathrm{ad}^{n-1}(B)), \quad n = 1, 2, \ldots. \tag{2.13}$$

We define $\mathrm{ad}^0(B) := B$. An operator B is said to be in $\mathcal{C}_\infty$-class if B is in $\mathcal{C}_n$ for all $n \in \mathbb{N}$.

Definition 2.4. *We say that an operator B is in class $\mathcal{C}_\omega$ if*

(i) $B \in \mathcal{C}_\infty$,

(ii) The operator norm $a_n := \left\|\mathrm{ad}^n(B)(A+1)^{1/2}\right\|$ satisfies

$$\lim_{n\to\infty} \frac{t^n a_n}{n!} = 0, \quad t > 0. \tag{2.14}$$

(iii) There exists some constant $b > 0$ such that for all $n \geq 0$, $\xi \in V_L$ implies that $\mathrm{ad}^n(B)\xi$ belongs to $\mathcal{V}_{L+b}$.

Theorem 2.2. *Let B is in $\mathcal{C}_n$-class. Then, for all $\xi \in D'$, $B(t)\xi$ is n-times strongly continuously differentiable in $t \in \mathbb{R}$ and*

$$\frac{d^k}{dt^k}B(t)\xi = W(-t)\mathrm{ad}^k(B)W(t)\xi, \quad k = 0, 1, 2, \ldots, n. \tag{2.15}$$

From Theorem 2.2, we immediately have

Theorem 2.3. *Let $B \in \mathcal{C}_n$ and $\xi \in D'$. Then, there is a $\theta \in (0,1)$ such that*

$$B(t)\xi = \sum_{k=0}^{n-1} \frac{t^k}{k!}\mathrm{ad}^k(B)\xi + \frac{t^n}{n!}W(-\theta t)\mathrm{ad}^n(B)W(\theta t)\xi. \tag{2.16}$$

Theorem 2.4. *Suppose that $B \in \mathcal{C}_\omega$. Then, for each $\xi \in D'$, $B(t)\xi$ has the norm-converging power series expansion formula*

$$B(t)\xi = \sum_{n=0}^{\infty} \frac{t^n}{n!}\mathrm{ad}^n(B)\xi, \quad t \in \mathbb{R}. \tag{2.17}$$

2.3 Klein-Gordon equation

To find the positive frequency part of the solution of Klein-Gordon equation, we investigate a general theory on an abstract Klein-Gordon equation which is suitable to the present context.

Let $\mathfrak{h}$ be a complex Hilbert space and T be a nonnegative self-adjoint operator on $\mathfrak{h}$.

Definition 2.5 (generalized Klein-Gordon equation). *A mapping $\mathfrak{h} \to \mathcal{C}_0$ is said to be T-free field if and only if for $f \in D(T^2)$, $\phi(f)$ belongs to $\mathcal{C}_2$-class, and $\phi(t,f) := W(-t)\phi(f)W(t)$ satisfies the differential equation:*

$$\frac{d^2}{dt^2}\phi(t,f)\xi - \phi(t,-T^2 f)\xi = 0, \quad \xi \in D', \tag{2.18}$$

where the differentiation is the strong one.

We denote

$$C^{\infty}(T) := \bigcap_{n=1}^{\infty} D(T^n). \tag{2.19}$$

Definition 2.6. *A T-free field $\phi(\cdot)$ is said to be analytic if*

(i) For all f which belongs to the subspace

$$\bigcup_{N\in\mathbb{N}} E_T\left(\left[\frac{1}{N}, N\right]\right), \tag{2.20}$$

$\phi(f)$ is in $\mathcal{C}_\omega$ class.

(ii) For $f \in D(T)$, $\phi(f) \in \mathcal{C}_1$.

(iii) $f_n \to f$ implies

$$\phi(f_n)\xi \to \phi(f)\xi, \quad \xi \in D', \tag{2.21}$$

and $Tf_n \to Tf$ implies

$$\mathrm{ad}[\phi(f_n)]\xi \to \mathrm{ad}[\phi(f)]\xi, \quad \xi \in D'. \tag{2.22}$$

Theorem 2.5. *Let ϕ be an analytic T-free field. Then, for all $f \in D(T)$, we find*

$$\phi(t,f) = \phi((\cos tT)f) + \mathrm{ad}\left[\phi\left(\left(\frac{\sin tT}{T}\right)f\right)\right] \tag{2.23}$$

on D'.

Theorem 2.5 enables us to define positive and negative frequency parts of ϕ:

Definition 2.7. *Let ϕ be an analytic T-free field. We define for $f \in D(T^{-1})$, on D'*

$$\phi^+(t,f) := \phi\left(\frac{e^{-itT}}{2}f\right) - \mathrm{ad}\left[\phi\left(\frac{e^{-itT}}{2iT}f\right)\right], \tag{2.24}$$

$$\phi^-(t,f) := \phi\left(\frac{e^{itT}}{2}f\right) + \mathrm{ad}\left[\phi\left(\frac{e^{itT}}{2iT}f\right)\right] \tag{2.25}$$

and call ϕ^+ (resp. ϕ^-) positive (resp. negative) frequency part of ϕ.

3 Dirac-Maxwell Hamiltonian in the Lorenz gauge

3.1 Definitions

We use the unit system in which the speed of light and $\hbar$, the Planck constant devided by 2π, are set to be unity. We denote the mass and the charge of the Dirac particle by $M > 0$ and $q \in \mathbb{R}$, respectively. The Hilbert space of state vectors for the Dirac particle is taken to be

$$\mathcal{H}_{\mathrm{D}} := L^2(\mathbb{R}^3; \mathbb{C}^4). \tag{3.1}$$

The target space $\mathbb{C}^4$ realizes a representation of the four dimensional Clifford algebra accompanied by the four dimensional Minkowski vector space. The Minkowski metric tensor $\eta = (\eta_{\mu\nu})$ is given by $\eta = \mathrm{diam}(-1, 1, 1, 1)$. We set $\eta^{-1} = (\eta^{\mu\nu})$, the inverse matrix of η, Then we have $\eta^{\mu\nu} = \eta_{\mu\nu}$, $\mu, \nu = 0, 1, 2, 3$.

The Hamiltonian of one Dirac particle under the influence of an external potential V is given by the Dirac operator

$$H_{\mathrm{D}}(V) := \boldsymbol{\alpha} \cdot \boldsymbol{p} + M\beta + V \tag{3.2}$$

acting in $\mathcal{H}_{\mathrm{D}}$, with the domain $D(H_{\mathrm{D}}(V)) := H^1(\mathbb{R}^3; \mathbb{C}^4) \cap D(V)$, where $H^1(\mathbb{R}^3; \mathbb{C}^4)$ denotes the $\mathbb{C}^4$-valued Sobolev space of order one, V denotes the multiplication operator defined by a 4×4 Hermitian matrix-valued function on $\mathbb{R}^3$ with each matrix components being Borel measurable.

Let C be the conjugation operator in $\mathcal{H}_{\mathrm{D}}$ defined by

$$(Cf)(\boldsymbol{x}) = f(\boldsymbol{x})^*, \quad f \in \mathcal{H}_{\mathrm{D}}, \quad x \in \mathbb{R}^3,$$

where $*$ means the usual complex conjugation. By Pauli's lemma [15], there is a 4×4 unitary matrix U satisfying

$$U^2 = 1, \quad UC = CU, \tag{3.3}$$

$$U^{-1}\alpha^j U = \overline{\alpha^j}, \quad j = 1, 2, 3, \quad U^{-1}\beta U = -\beta, \tag{3.4}$$

where for a matrix A, $\overline{A}$ denotes its complex-conjugated matrix and 1 the identity matrix. We assume that the potential V satisfies the following conditions :

Assumption 3.1. *(I) Each matrix component of V belongs to*

$$L^2_{\mathrm{loc}}(\mathbb{R}^3) := \left\{ f : \mathbb{R}^3 \to \mathbb{C} \,\middle|\, \text{Borel measurable and } \int_{|\boldsymbol{x}| \leq R} |f(\boldsymbol{x})|^2 < \infty \text{ for all } R > 0. \right\}.$$

(II) V is Charge-Parity (CP) invariant in the following sense:

$$U^{-1}V(\boldsymbol{x})U = V(-\boldsymbol{x})^*, \quad \text{a.e.}\, x \in \mathbb{R}^3. \tag{3.5}$$

(III) $H_{\mathrm{D}}(V)$ is essentially self-adjoint.

The Hilbert space for N Dirac particles is given by

$$\wedge^N \mathcal{H}_{\rm D} := \bigotimes_{\rm as}^{N} L^2(\mathbb{R}^3;\mathbb{C}^4) = L^2_{\rm as}((\mathbb{R}^3 \times \{1,2,3,4\})^N), \tag{3.6}$$

where $\otimes_{\rm as}^N$ denotes the N-fold anti-symmetric tensor product. The a-th component in

$$\boldsymbol{X} = (\boldsymbol{x}^1, l^1; \ldots; \boldsymbol{x}^N, l^N) \in (\mathbb{R}^3 \times \{1,2,3,4\})^N$$

represents the position and the spinor of the a-th Dirac particle. For notational simplicity, we denote the position-spinor space of one electron by $\mathcal{X} = \mathbb{R}^3 \times \{1,2,3,4\}$ in what follows. We regard $\mathcal{X}$ as a topological space with the product topology of the ordinary one on $\mathbb{R}^3$ and the discrete one on $\{1,2,3,4\}$. The N particle Hamiltonian is then given by

$$H_D(V,N) = \sum_{a=1}^{N} \big(1 \otimes \cdots \otimes \overset{a\text{-th}}{H_D(V)} \otimes \cdots \otimes 1\big) \tag{3.7}$$

$$\equiv \sum_{a=1}^{N} \big(\alpha^a \cdot \boldsymbol{p}^a + \beta^a M + V^a\big). \tag{3.8}$$

Next, we introduce the free gauge field Hamiltonian in the Lorenz gauge. We adopt as the one-photon Hilbert space

$$\mathcal{H}_{\rm ph} := L^2(\mathbb{R}^3;\mathbb{C}^4). \tag{3.9}$$

The Hilbert space for the quantized gauge field in the Lorenz gauge is given by

$$\mathcal{F}_{\rm ph} := \bigoplus_{n=0}^{\infty} \bigotimes_{\rm s}^{n} \mathcal{H}_{\rm ph} = \left\{ \Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \,\middle|\, \Psi^{(n)} \in \bigotimes_{\rm s}^{n} \mathcal{H}_{\rm ph},\ \|\Psi\|^2 := \sum_{n=0}^{\infty} \|\Psi^{(n)}\|^2 < \infty \right\}, \tag{3.10}$$

the Boson Fock space over $\mathcal{H}_{\rm ph}$, where $\otimes_{\rm s}^n$ denotes the n-fold symmetric tensor product with the convention $\otimes_{\rm s}^0 \mathcal{H}_{\rm ph} := \mathbb{C}$. Let $\omega(\boldsymbol{k}) := |\boldsymbol{k}|$, $\boldsymbol{k} \in \mathbb{R}^3$, the energy of a photon with momentum $\boldsymbol{k} \in \mathbb{R}^3$. The free Hamiltonian of the quantum gauge field is given by its second quantization

$$H_{\rm ph} := d\Gamma_{\rm b}(\omega) := \bigoplus_{n=0}^{\infty} \overline{\left(\sum_{j=1}^{n} 1 \otimes \cdots \otimes 1 \otimes \overset{j\text{-th}}{\omega} \otimes 1 \otimes \cdots \otimes 1\right) \restriction \hat{\otimes}^{n} D(\omega)}. \tag{3.11}$$

The operator $H_{\rm ph}$ is self-adjoint.

We introduce the indefinite metric which is needed for realizing the canonical commutation relation. We denote the second quantization of η by the same symbol:

$$\eta := \Gamma_{\rm b}(\eta) := \bigoplus_{n=0}^{\infty} \big(\eta \otimes \cdots \otimes \eta\big).$$

Then η is unitary and satisfies $\eta^* = \eta$, $\eta^2 = I$. The physical inner product can be written as

$$\langle \Psi | \Phi \rangle = \langle \Psi, \eta \Phi \rangle.$$

For a densely defined linear operator T, we denote

$$T^\dagger = \eta T^* \eta. \tag{3.12}$$

Definition 3.1. *(i) A densely defined linear operator T is self-adjoint with respect to $\langle\cdot|\cdot\rangle$ or η-self-adjoint if $T^\dagger = T$.*

(ii) A densely defined linear operator T is essentially self-adjoint with respect to $\langle\cdot|\cdot\rangle$ or essentially η-self-adjoint if $\overline{T}$ is η-self-adjoint.

Note that the free Hamiltonian H_{ph} is self-adjoint and η-self-adjoint.

The creation operator $c^\dagger(F)$ with $F \in \mathcal{H}_{\text{ph}}$ is defined to be a densely defined closed linear operator on $\mathcal{F}_{\text{ph}}$ given by

$$(c^\dagger(F)\Psi)^{(0)} = 0, \quad (c^\dagger(F)\Psi)^{(n)} = \sqrt{n} S_n(F \otimes \Psi^{(n-1)}), \quad n \geq 1, \quad \Psi \in D(c^\dagger(F)), \tag{3.13}$$

where S_n denotes the symmetrization operator on $\otimes^n \mathcal{H}_{\text{ph}}$, i.e. $S_n(\otimes^n \mathcal{H}_{\text{ph}}) = \otimes_{\text{s}}^n \mathcal{H}_{\text{ph}}$. We note that $c^\dagger(F)$ linear in F. The annihilation operator $c(\overline{F})$ ($\overline{F} = \eta F$, $F \in \mathcal{H}_{\text{ph}}$) is then given by

$$c(\overline{F}) := (c^\dagger(F))^\dagger. \tag{3.14}$$

For $f \in L^2(\mathbb{R})$, we introduce the components of $c^\dagger(\cdot)$ with *lower* indices by

$$c_0^\dagger(f) := c^\dagger(f,0,0,0), \quad c_1^\dagger(f) := c^\dagger(0,f,0,0), \tag{3.15}$$

$$c_2^\dagger(f) := c^\dagger(0,0,f,0), \quad c_3^\dagger(f) := c^\dagger(0,0,0,f), \tag{3.16}$$

and the components of $c(\cdot)$ with *upper* indices by

$$c^0(f) := c(f,0,0,0), \quad c^1(f) := c(0,f,0,0), \tag{3.17}$$

$$c^2(f) := c(0,0,f,0), \quad c^3(f) := c(0,0,0,f). \tag{3.18}$$

Let $\{e_\lambda\}_{\lambda=0,1,2,3}$ be the photon polarization vectors, that is, each $e_\lambda(\cdot) = (e^\mu{}_\lambda(\cdot))_{\mu=0}^3$ is $\mathbb{R}^4$-valued measurable function defined on $\mathbb{R}^4$ satisfying

$$e_\lambda(\boldsymbol{k}) \cdot e_\sigma(\boldsymbol{k}) = \eta_{\lambda\sigma}, \quad e_\lambda(\boldsymbol{k}) \cdot k = 0, \quad \text{a.e. } \boldsymbol{k}, \quad \lambda = 1,2. \tag{3.19}$$

For each $f \in L^2(\mathbb{R}^3)$ and $\mu = 0,1,2,3$, we define

$$a^\mu(f) := c^\nu(e^\mu{}_\nu f), \quad a_\mu^\dagger(f) := (a_\mu(f))^\dagger. \tag{3.20}$$

Then we have the Lorentz covariant canonical commutation relations:

$$[a_\mu(f), a_\nu^\dagger(g)] = \eta_{\mu\nu} \langle f, g \rangle,$$

$$[a_\mu(f), a_\nu(g)] = [a_\mu^\dagger(f), a_\nu^\dagger(g)] = 0.$$

For all $f \in L^2(\mathbb{R}^3)$ satisfying $\widehat{f}/\sqrt{\omega} \in L^2(\mathbb{R}_{\boldsymbol{k}}^3)$, we set

$$A_\mu(0,f) := \frac{1}{\sqrt{2}}\Big(a_\mu\Big(\frac{\widehat{f^*}}{\sqrt{\omega}}\Big) + a_\mu^\dagger\Big(\frac{\widehat{f}}{\sqrt{\omega}}\Big)\Big), \tag{3.21}$$

where $\widehat{f}$ denotes the Fourier transform of f, and f^* denotes the complex conjugate of f. Now fix $\chi_{\text{ph}} \in L^2(\mathbb{R}^3_{\boldsymbol{x}})$ which is real and satisfies $\chi_{\text{ph}}(\boldsymbol{x}) = \chi_{\text{ph}}(-\boldsymbol{x})$ and $\widehat{\chi_{\text{ph}}}/\sqrt{\omega} \in L^2(\mathbb{R}^3_{\boldsymbol{k}})$. We set

$$A_\mu(\boldsymbol{x}) := A_\mu(0, \chi^{\boldsymbol{x}}_{\text{ph}}), \tag{3.22}$$

$$\chi^{\boldsymbol{x}}_{\text{ph}}(\boldsymbol{y}) := \chi_{\text{ph}}(\boldsymbol{y} - \boldsymbol{x}), \quad \boldsymbol{y} \in \mathbb{R}^3. \tag{3.23}$$

Next, we introduce the total Hamiltonian in the Hilbert space of state vectors for the coupled system, which is taken to be

$$\mathcal{F}_{\text{DM}}(N) := \wedge^N \mathcal{H}_{\text{D}} \otimes \mathcal{F}_{\text{ph}} = L^2_{\text{as}}(\mathcal{X}^N; \mathcal{F}_{\text{ph}}). \tag{3.24}$$

We remark that this Hilbert space can be naturally identified as

$$\mathcal{F}_{\text{DM}}(N) == A_N \int^{\oplus}_{\mathcal{X}^N} d\boldsymbol{X}\, \mathcal{F}_{\text{ph}}, \tag{3.25}$$

the Hilbert space of $\mathcal{F}_{\text{ph}}$-valued functions on $\mathcal{X}^N = \mathcal{X} \times \cdots \times \mathcal{X}$ which are square integrable with respect to the Borel measure (the product measure of Lebesgue measure on $\mathbb{R}^3$ and counting measure on $\{1, 2, 3, 4\}$) and which are anti-symmetric in the arguments.

The mapping $\boldsymbol{X} \mapsto \chi^{\boldsymbol{x}^a}_{\text{ph}}$ (a=1,2,...,N) from $\mathcal{X}^N$ to $\mathcal{H}_{\text{ph}}$ is strongly continuous, and thus we can define a decomposable operator A_μ by

$$A^a_\mu := \int^{\oplus}_{\mathcal{X}^N} d\boldsymbol{X}\, A_\mu(\boldsymbol{x}^a), \quad \mu = 0, 1, 2, 3, \quad a = 1, 2, \ldots, N, \tag{3.26}$$

acting in $\int^{\oplus}_{\mathcal{X}^N} d\boldsymbol{X}\, \mathcal{F}_{\text{ph}}$.

The total Hamiltonian of the coupled system is then given by

$$H_{\text{DM}}(V, N) := H_0 + H_1, \tag{3.27}$$

$$H_0 := \overline{H_{\text{D}}(V, N) + H_{\text{ph}}}, \tag{3.28}$$

$$H_1 := q \sum_{a=1}^{N} \alpha^{a\mu} A^a_\mu \tag{3.29}$$

This is the N-particle *Dirac-Maxwell Hamiltonian* in the Feynman (Lorenz) gauge.

3.2 η-self-adjointness, existence of time-evolution

Proposition 3.1. *H_1 is in $\mathcal{C}_0$-class with $A = N_{\text{b}} := 1 \otimes d\Gamma_{\text{b}}(1)$.*

Proof. It is straightforward to see that H_1 satisfies the conditions of $\mathcal{C}_0$-class in Definition 2.1, so we omit the proof. □

Proposition 2.2 and 3.1 ensure that the operator-valued function $W(t) = e^{-itH_0}U(t, 0)$ provide a solution of Schrödinger equation.

Theorem 3.1. *Under Assumption 3.1, $H_{\text{DM}}(V, N)$ is essentially η-self-adjoint.*

Proof. By Assumption 3.1, we can show that $\eta H_{\text{DM}}(V, N)$ has a self-adjoint extension in the same manner as in the proof of [1, Theorem 1.2]. Then, applying Theorem 2.4 with $A = N_{\text{b}}$, it follows that $\eta H_{\text{DM}}(V, N)$ is essentially self-adjoint, that is, $H_{\text{DM}}(V, N)$ is essentially η-self-adjoint. □

3.3 Current conservation

Definition 3.2. *Electro-magnetic current density operator is a operator-valued tempered distribution defined by*

$$j^{\mu}(f) := q \sum_{a=1}^{N} \alpha^{a\mu} \int_{\mathcal{X}^N}^{\oplus} f(\boldsymbol{x}^a)\, d\boldsymbol{X}, \quad f \in \mathscr{S}(\mathbb{R}^3). \tag{3.30}$$

We remark that $\alpha^{a\mu}$ is the μ-component of the four-component-velocity of the a-th electron.

Theorem 3.2. *The current density $j^{\mu}(f)$ is in C_0-class for all $f \in D(\sqrt{-\Delta})$ and thus the time-dependent current density $j^{\mu}(t,f) := W(-t)j^{\mu}(f)W(t)$ exists. The zero-th component $j^0(f)$ is in C_1-class and thus satisfies the strong Heisenberg equation of motion. Furthermore, the current density satisfies the conservation equation*

$$\partial_{\mu} j^{\mu}(t,f) := \frac{\partial j^0(t,f)}{\partial t} + \sum_{k=1,2,3} \frac{\partial j^k(t,f)}{\partial x^k} = 0, \tag{3.31}$$

on D'.

Proof. It is not difficult to check that $j^0(f)$ is in C_1-class. Applying Theorem 2.1, the desired result follows. □

3.4 Time evolution of the gauge field

Theorem 3.3. *The gauge field $A_{\mu}(f)$ is in C_2 class for all $f \in D(-\Delta)$ and the time-dependent field $A_{\mu}(t,f) := W(-t)A_{\mu}(f)W(t)$ satisfies the equation of motion*

$$\Box A_{\mu}(t,f) := \partial_{\nu}\partial^{\nu} A_{\mu}(t,f) = j_{\mu}(t, \chi_{\mathrm{ph}} * f), \tag{3.32}$$

where $\chi_{\mathrm{ph}} * f$ is the ordinary convolution of χ_{ph} and f.

Proof. It is straightforward to see $A_{\mu}(f) \in C_2$. By Theorem 2.2, $A_{\mu}(t,f) = W(-t)A_{\mu}(f)W(t)$ is twice differentiable in t on D' and

$$\frac{d^2}{dt^2} A_{\mu}(t,f) = A_{\mu}(t, \Delta f) - j_{\mu}(t, \chi_{\mathrm{ph}} * f), \tag{3.33}$$

which is equivalent to (3.32). □

It is clear by Theorem 3.2 and (3.32) that $A_0(t,f)$ $(f \in D((-\Delta)^{5/4}))$ is three times differentiable in t and

$$\Box \partial^{\mu} A_{\mu}(t,f)\Psi = 0, \quad \Psi \in D'. \tag{3.34}$$

It is straightforward to obtain

$$\partial^{\mu} A_{\mu}(t,f)\Big|_{t=0} \Psi = -\frac{1}{\sqrt{2}} \left[a_{\mu}\left(ik^{\mu} \frac{\widehat{f^*}}{\sqrt{\omega}} \right) + a_{\mu}^{\dagger}\left(ik^{\mu} \frac{\widehat{f}}{\sqrt{\omega}} \right) \right] \Psi, \quad \Psi \in D', \tag{3.35}$$

where $k^0 := \omega(\boldsymbol{k})$. Thus, it is natural to define

$$\Omega(f) := -\frac{1}{\sqrt{2}}\left[a_\mu\left(ik^\mu \frac{\widehat{f^*}}{\sqrt{\omega}}\right) + a_\mu^\dagger\left(ik^\mu \frac{\widehat{f}}{\sqrt{\omega}}\right)\right] \tag{3.36}$$

for each $f \in D((-\Delta)^{1/4})$. Then it is obvious that $\Omega(f) \in \mathcal{C}_0$. Let $\mathfrak{h}$ be $D((-\Delta)^{1/4})$ with the inner product

$$\langle f, g\rangle_{\mathfrak{h}} := \left\langle (-\Delta)^{1/4} f, (-\Delta)^{1/4} g\right\rangle_{L^2(\mathbb{R}^3)}. \tag{3.37}$$

Then $\mathfrak{h}$ becomes a Hilbert space with this inner product. We regard Ω as a mapping from $\mathfrak{h}$ into $\mathcal{C}_0$.

Theorem 3.4. *The mapping $f \mapsto \Omega(f)$ defines an analytic $\sqrt{-\Delta}$-free field with*

$$\begin{aligned}\mathrm{ad}[\Omega(f)] &= -\frac{1}{\sqrt{2}}\left[a_\mu\left(ik^\mu(i\omega)\frac{\widehat{f^*}}{\sqrt{\omega}}\right) + a_\mu^\dagger\left(ik^\mu(i\omega)\frac{\widehat{f}}{\sqrt{\omega}}\right)\right] \\ &\quad - \frac{iq}{2}\sum_a \alpha^{a\mu}\int_{\mathcal{X}^N}^{\oplus} d\boldsymbol{X}\left[\left\langle \frac{\widehat{\chi_{\mathrm{ph}}^{\boldsymbol{x}^a}}}{\sqrt{\omega}}, \frac{ik_\mu \widehat{f}}{\sqrt{\omega}}\right\rangle - \left\langle \frac{ik_\mu\widehat{f^*}}{\sqrt{\omega}}, \frac{\widehat{\chi_{\mathrm{ph}}^{\boldsymbol{x}^a}}}{\sqrt{\omega}}\right\rangle\right]\end{aligned} \tag{3.38}$$

for $f \in D((-\Delta)^{3/4})$. In particular,

$$\Omega(t, f) = \Omega((\cos t\sqrt{-\Delta})f) + \mathrm{ad}\left[\Omega\left(\frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}}f\right)\right] \tag{3.39}$$

on D'.

Proof. (3.38) follows from a direct calculation. (3.34) implies that $\Omega(\cdot)$ is a $\sqrt{-\Delta}$-free field. By a suitable estimation, we see that $\Omega(\cdot)$ is analytic in the sense of Definition 2.6. Therefore the assertion follows from Theorem 2.5. □

3.5 The Gupta subsidiary condition and physical subspace

From Theorem 3.4 and (2.24), we can define the positive frequency part of $\Omega(t, f)$ by

$$\begin{aligned}\Omega^+(t, f) &:= \Omega\left(\frac{e^{-i\sqrt{-\Delta}t}}{2}f\right) - \mathrm{ad}\left[\Omega\left(\frac{e^{-i\sqrt{-\Delta}t}}{2i\sqrt{-\Delta}}f\right)\right] \\ &= -\frac{1}{\sqrt{2}}a_\mu\left(\frac{ik^\mu e^{i\omega t}\widehat{f^*}}{\sqrt{\omega}}\right) + \frac{iq}{2}\sum_a\int_{\mathcal{X}^N}^{\oplus} d\boldsymbol{X}\left\langle\frac{\widehat{\chi_{\mathrm{ph}}^{\boldsymbol{x}^a}}}{\sqrt{\omega}}, \frac{e^{-i\omega t}\widehat{f}}{\sqrt{\omega}}\right\rangle,\end{aligned} \tag{3.40}$$

for $f \in \mathfrak{h} \cap D((-\Delta)^{-1/4})$ on D'. Then we define the physical subspace as follows:

$$V_{\mathrm{phys}} = \overline{\{\Psi \in D' \,|\, \Omega^+(t, f)\Psi = 0, \text{ for all } t \in \mathbb{R},\, f \in \mathfrak{h} \cap D((-\Delta)^{-1/4})\}}. \tag{3.41}$$

To identify the physical subspace V_{phys}, we define two unitary operators following Refs. [10, 14]. The first one is:

$$W := \bigoplus_{n=0}^{\infty} \bigotimes^{n} \overline{w}, \tag{3.42}$$

with

$$\overline{w} = (w_\mu{}^\nu) = \begin{pmatrix} 1/\sqrt{2} & 0 & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{pmatrix} : \mathcal{H}_{\mathrm{ph}} \to \mathcal{H}_{\mathrm{ph}}. \tag{3.43}$$

To define the other unitary operator we have to assume that $\widehat{\chi_{\mathrm{ph}}} \in D(\omega^{-3/2})$. Set

$$G := -q\sum_a \int_{\mathcal{X}^N}^{\oplus} d\boldsymbol{X} \frac{1}{\sqrt{2}} \overline{\left[c^3 \left(\frac{i\widehat{\chi_{\mathrm{ph}}^{\boldsymbol{x}^a}}}{\omega^{3/2}} \right) + c_3^\dagger \left(\frac{i\widehat{\chi_{\mathrm{ph}}^{\boldsymbol{x}^a}}}{\omega^{3/2}} \right) \right]}. \tag{3.44}$$

Then G is self-adjoint.

Let $\mathcal{F}_{\mathrm{TL}}$ be the closed subspace

$$\mathcal{F}_{\mathrm{TL}} := \left(\wedge^N \mathcal{H}_{\mathrm{D}}\right) \otimes \left(\mathbb{C}\Omega \otimes \mathcal{F}_{\mathrm{b}}(L^2(\mathbb{R}^3)) \otimes \mathcal{F}_{\mathrm{b}}(L^2(\mathbb{R}^3)) \otimes \mathcal{F}_{\mathrm{b}}(L^2(\mathbb{R}^3))\right). \tag{3.45}$$

Theorem 3.5. *(i)*

$$V_{\mathrm{phys}} = e^{-iG} W \mathcal{F}_{\mathrm{TL}}. \tag{3.46}$$

(ii) The physical subspace V_{phys} is non-negative. That is, for all $\Psi \in V_{\mathrm{phys}}$, $\langle \Psi | \Psi \rangle \geq 0$.

We denote the zero-norm subspace in V_{phys} by $\mathcal{N}$:

$$\mathcal{N} := \{\Psi \in V_{\mathrm{phys}} \mid \langle \Psi | \Psi \rangle = 0\}. \tag{3.47}$$

The quotient vector space $V_{\mathrm{phys}}/\mathcal{N}$ becomes a pre-Hilbert space with respect to the naturally induced metric from η, and its completion

$$\mathcal{H}_{\mathrm{phys}} := \overline{V_{\mathrm{phys}}/\mathcal{N}} \tag{3.48}$$

is called physical Hilbert space. Then we can see that the time-evolution $\{W(t)\}_{t\in\mathbb{R}}$ naturally defines a strongly continuous one-parameter unitary group $\{U(t)\}_{t\in\mathbb{R}}$ on $\mathcal{H}_{\mathrm{phys}}$. Hence the physical Hamiltonian H_{phys} should be defined as:

Definition 3.3. *The physical Hamiltonian H_{phys} is the generator of $\{U(t)\}_t$.*

Set

$$\mathcal{D}_1 := C_0^\infty(\mathbb{R}^3; \mathbb{C}^4) \hat{\otimes} \mathcal{F}_{\mathrm{b,fin}}(D_1). \tag{3.49}$$

Theorem 3.6. *The operator $\tilde{H}$ which defined by*

$$D(\tilde{H}) := [e^{-iG}\mathcal{D}_1], \tag{3.50}$$

$$\tilde{H}[\Psi] := [H_{\mathrm{DM}}(V,N)\Psi], \tag{3.51}$$

is essentially self-adjoint and the unique self-adjoint extension is equal to H_{phys}. Here $[\Psi]$ denotes the equivalent class in $\mathcal{H}_{\mathrm{phys}} = \overline{V_{\mathrm{phys}}/\mathcal{N}}$ to which Ψ belongs.

3.6 Triviality of the physical subspace

If $\widehat{\chi_{\rm ph}}$ does not satisfy the IR regularity condition $\widehat{\chi_{\rm ph}} \in D(\omega^{3/2})$, the definition of G in (3.44) makes no sense. In this case, the physical subspace is trivial:

Theorem 3.7. *Suppose that $\widehat{\chi_{\rm ph}}$ does not belong to $D(\omega^{-3/2})$. Then,*

$$V_{\rm phys} = \{0\}. \tag{3.52}$$

4 Conclusion

Using the method of constructing the time-evolution operator via the time-ordered exponential $U(t, t')$ given in [6] and some extended results (Theorems 2.1-2.4), we can perform the procedures of the Gupta-Bleuler formalism for the Dirac-Maxwell model for which the time-evolution of the gauge fields can not be solved explicitly. However, this scheme does not work for the models of which the Hamiltonian is not divided into the free and the interaction part such as the non-relativistic quantum electrodynamics (in the Lorenz gauge):

$$H_{\rm NRQED} := \frac{1}{2M}\sum_{a=1}^{N}(\boldsymbol{p}_a - q\mathbf{A}_a)^2 + H_{\rm ph} + q\sum_{a=1}^{N} A_a^0, \tag{4.1}$$

and the semi-relativistic quantum electrodynamics:

$$H_{\rm SRQED} := \sum_{a=1}^{N}\sqrt{\left(\boldsymbol{\sigma}\cdot(\boldsymbol{p}_a - q\mathbf{A}_a)\right)^2 + M^2} + H_{\rm ph} + q\sum_{a=1}^{N} A_a^0. \tag{4.2}$$

For the ordinary quantum electrodynamics in the Lorenz gauge, we can construct the time-evolution *if we introduce the spatial cutoff and momentum cutoff* [6]. However, these cutoffs break the current conservation, and $\partial_\mu A^\mu$ cannot be free field.

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