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Kyoto University
On Kato's inequality for the relativistic Schrödinger operators with magnetic fields*

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This lecture deals with whether Kato’s inequality holds for the magnetic relativistic Schrödinger operator $H_A$ with vector potential $A(x)$ and mass $m \geq 0$ associated with the classical relativistic Hamiltonian symbol $\sqrt{\xi - A(x)^2 + m^2}$ such as

$$\text{Re}[(\text{sgn } u)H_A u] \geq \sqrt{-\Delta + m^2} |u|, \quad (1)$$

in the distribution sense, for $u$ is in $L^2(\mathbb{R}^d)$ with $H_A u$ in $L^1_{loc}(\mathbb{R}^d)$.

In the literature there are three magnetic relativistic Schrödinger operators associated with the classical symbol (1) (e.g. [112], [113]). The first two $H_A^{(1)}$ and $H_A^{(3)}$ are to be defined as pseudo-differential operators: for $f \in C_0^\infty(\mathbb{R}^d)$,

$$(H_A^{(1)} f)(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{i(x-y) \cdot \xi} \sqrt{\left(\xi - A(\frac{x+y}{2})^2 + m^2\right)} f(y) dy d\xi, \quad (2)$$

$$(H_A^{(2)} f)(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{i(x-y) \cdot \xi} \sqrt{\left(\xi - \int_0^1 A((1-\theta)x+\theta y) d\theta\right)^2 + m^2} f(y) dy d\xi. \quad (3)$$

The third $H_A^{(3)}$ is defined as the square root of the nonnegative selfadjoint (nonrelativistic Schrödinger) operator $(-i\nabla - A(x))^2 + m^2$ in $L^2(\mathbb{R}^d)$:

$$H_A^{(3)} := \sqrt{(-i\nabla - A(x))^2 + m^2}. \quad (4)$$

$H_A^{(1)}$ is the so-called Weyl pseudo-differential operator ([ITa 86], [I89]). $H_A^{(2)}$ is a modification of $H_A^{(1)}$ given in [IfMP 07], and $H_A^{(3)}$ used in [LSei 10] to discuss relativistic stability of matter.

All these three operators are nonlocal operators, and, under suitable condition on $A(x)$, become selfadjoint. For $A = 0$ we put $H_0 = \sqrt{-\Delta + m^2}$, where $-\Delta$ is the minus-signed Laplacian in $\mathbb{R}^d$. $H_A^{(2)}$ and $H_A^{(3)}$ are gauge-covariant, but not $H_A^{(1)}$.

Inequality (1) for $H_A^{(1)}$ has been shown in [I89], [ITs 76], and similarly will be for $H_A^{(2)}$.

For $H_A^{(3)}$, we assume that $d \geq 2$, as in case $d = 1$ any magnetic vector potential can be removed by a gauge transfromation. We want to show

**Theorem 1** (Kato’s inequality). Let $m \geq 0$ and assume that $A \in [L^2_{loc}(\mathbb{R}^d)]^d$. Then if $u$ is in $L^2(\mathbb{R}^d)$ with $H_A^{(3)} u$ in $L^1_{loc}(\mathbb{R}^d)$, then the distributional inequality holds:

$$\text{Re}[(\text{sgn } u)H_A^{(3)} u] \geq \sqrt{-\Delta + m^2} |u|, \quad (5)$$

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or
\[ \text{Re}((\text{sgn } u) H_A^{(3)} u) \geq [\sqrt{-\Delta + m^2} - m] |u|. \]  
(6)

Here \((\text{sgn } u)(x) := \frac{u(x)}{|u(x)|}\), if \(u(x) \neq 0\); \(= 0\), if \(u(x) = 0\).

From Theorem 1 follows the following corollary.

**Corollary** (Diamagnetic inequality) (cf. [FLSei08], [HILo12, 13]) Let \(m \geq 0\) and assume that \(A \in [L^2_{\text{loc}}(\mathbb{R}^d)]^d\). Then \(f, g \in L^2(\mathbb{R}^d)\)
\[ |(f, e^{-i[H_A^{(3)}-m]}g)| \leq (|f|, e^{-i[H_0-m]}|g|). \]  
(7)

Once Theorem 1 is established, we can apply it to show the following theorem on essential selfadjointness of the relativistic Schrödinger operator with both vector and scalar potentials \(A(x)\) and \(V(x)\):
\[ H := H_A^{(3)} + V. \]  
(8)

**Theorem 2.** Let \(m \geq 0\) and assume that \(A \in [L^2_{\text{loc}}(\mathbb{R}^d)]^d\). If \(V(x)\) is in \(L^2_{\text{loc}}(\mathbb{R}^d)\) with \(V(x) \geq 0\) a.e., then \(H = H_A^{(3)} + V\) is essentially selfadjoint on \(C_0^\infty(\mathbb{R}^d)\) and its unique selfadjoint extension is bounded below by \(m\).

The characteristic feature is that, unlike \(H_A^{(1)}\) and \(H_A^{(2)}\), \(H_A^{(3)}\) is, since being defined as an operator square root (4), neither an integral operator nor a pseudo-differential operator associated with a certain tractable symbol. \(H_A^{(3)}\) is, under the condition of the theorem, essentially selfadjoint on \(C_0^\infty(\mathbb{R}^d)\) so that \(H_A^{(3)}\) has domain
\[ D[H_A^{(3)}] = \{u \in L^2(\mathbb{R}^d); (i\nabla + A(x))u \in L^2(\mathbb{R}^d)\}, \]
which contains \(C_0^\infty(\mathbb{R}^d)\) as an operator core. Although we can know the domain of \(H_A^{(3)}\) is determined, the point which becomes crucial is in how to derive regularity of the weak solution \(u \in L^2(\mathbb{R}^d)\) of equation
\[ H_A^{(3)} u \equiv \sqrt{(-i\nabla - A(x))^2 + m^2} u = f, \quad \text{for given } f \in L^1_{\text{loc}}(\mathbb{R}^d). \]

We shall show inequality (5)/(6), modifying the method used in the case ([I89], [ITs92]) for the Weyl pseudo-differential operator \(H_A^{(1)}\), basically along the idea of Kato’s original proof for the magnetic nonrelativistic Schrödinger operator \(\frac{1}{2}(-i\nabla - A(x))^2\) in [K72]. However, the present case seems to be not so simple as to need much further modification within “operator theory plus alpha”.

**References**


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