<table>
<thead>
<tr>
<th>Title</th>
<th>Dependency of polarity on the drift of Brownian motion of a compact manifold (Regularity and Singularity for Partial Differential Equations with Conservation Laws)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>正宗 哲</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2015), 1962: 45-49</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2015-08</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/224157">http://hdl.handle.net/2433/224157</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Dependency of polarity on the drift of Brownian motion of a compact manifold

Jun Masamune

Division of Mathematics Research Center for Pure and Applied Mathematics
Graduate School of Information Sciences, Tohoku University

CONTENTS
1. Introduction 1
2. Closed forms 1
3. Capacity associated to $\mathcal{E}_\alpha$ 4
References 5

1. INTRODUCTION

It is well-known that the Brownian motion on a Riemannian manifold $M$ will not hit a subset $\Sigma$ of $M$ if and only if the capacity related to the Brownian motion of $\Sigma$ is zero [2]. However, the situation is not clear for a Brownian motion with a drift; in particular, it would be interesting to know if the capacity of $\Sigma$ associated to the Brownian motion with a drift being zero is independent of the drift. In this note, we will study this problem. A lower bounded non-symmetric semi Dirichlet form generates a non-symmetric Markov process [3, 5], and this relationship will be the foundation for our study. The main aim of this note is to answer the following two questions:

- Does the operator $\Delta + \langle F, \nabla \cdot \rangle + V$, where $\Delta$ is the sub-Laplacian, $F$ is a one-form, and $V$ is a non-negative continuous function, generate a lower bounded semi Dirichlet form?
- Find a characterisation of the capacity for a lower bounded semi Dirichlet form in terms of that for the Dirichlet integral.

The structure of the note is the following. Section 2 will be devoted for the first question and the second question will be studied in Section 3.

2. CLOSED FORMS

Let $(M, g)$ be a compact smooth Riemannian manifold without boundary. Let $\sigma > 0$ be a positive continuous function on $M$. We consider the weighted space, $L^2 = L^2(M, dm)$, where $dm = \sigma dv_g$ and $v_g$ is the Riemannian volume associated with the metric $g$. Let $F \in \Gamma(TM^*)$ be a smooth 1-form and $V \in C(M)$, the space of continuous functions on $M$, with $V \geq 0$. Suppose that $TM$ admits a system of Hörmander vector fields $\{X_i\}$ and the $X_x \subset T_x M$ is the subspace spanned by $\{X_i\}$ at point $x \in M$. Let $\pi$ be the orthogonal projection $T_x M \rightarrow X_x$. The sub-gradient $\nabla$ is then defined pointwise as $\nabla u = \pi \circ \text{grad}(u)$, where grad is the gradient operator associated to $g$. The energy form $\mathcal{E}$ is

$$\mathcal{E}(u) = \int_M (g(\nabla u, \nabla v) + \langle F, \nabla u \rangle v + V uv) \, dm, \quad u, v \in C^\infty(M)$$

where $\langle \cdot, \cdot \rangle$ is the pairing between cotangent and tangent vector spaces. We will denote $\mathcal{E}(u) = \mathcal{E}(u, u)$ and $\mathcal{E}_\alpha(u, v) = \mathcal{E}(u, v) + \alpha(u, v)$ for some $\alpha > 0$, where $\langle u, v \rangle = \int_M uv \, dm$ and $\|u\| = (u, u)^{1/2}$, for short. The symbol $| \cdot |$ stands for the pointwise norm. The weighted divergence, which is the negative of the formal joint of $\nabla$, will be denoted by div. We will employ $W^{1,2}$ =
DEPENDENCY OF POLARITY ON THE DRIFT OF BROWNIAN MOTION OF A COMPACT MANIFOLD

\{u \in L^2 \mid \nabla u \in L^2(TM, dm)\}. Let us recall some basic definitions regarding with semi-Dirichlet forms stated in the current setting.

**Definition 1** (closed forms). A quadratic form \( Q \) defined on a dense subspace \( D(Q) \subset L^2 \) will be called closed on \( L^2 \) provided the following three conditions:

\( (E.1) \) \( Q \) is lower bounded: There exists \( \alpha_0 \geq 0 \) such that

\[ Q_{\alpha_0}(u) \geq 0, \quad \forall u \in D(Q). \]

\( (E.2) \) Sector condition: There exists \( K \geq 1 \) such that

\[ |Q(u, v)|^2 \leq KQ_{\alpha_0}(u)\mathcal{E}_{\alpha_0}(v), \quad \forall u, v \in D(Q). \]

\( (E.3) \) \( D(Q) \) is a Hilbert space with respect to the inner product

\[ Q_{\alpha}^{(s)}(u, v) = \frac{1}{2}(Q_{\alpha_0}(u, v) + Q_{\alpha_0}(v, u)), \quad \forall \alpha \geq \alpha_0. \]

**Theorem 1.** The form \((\mathcal{E}, W^{1,2})\) is a closed form.

**Proof.** The proof follows from Propositions 1 and 2.

**Proposition 1.** The energy \((\mathcal{E}_\alpha, C^\infty(M))\) is closable in \( L^2 \) whenever

\[ \alpha > \sup \left( \frac{1}{2}(\text{div} F) - V \right). \]

**Proof.** We must show:

\[ \lim_{m,n \to \infty} \mathcal{E}_\alpha(u_n - u_m) = 0, \quad \lim_{n \to \infty} \|u_n\| = 0 \implies \lim_{n \to \infty} \mathcal{E}_\alpha(u_n) = 0. \]

Let use denote the sub-Dirichlet integral by \( \mathcal{D}(u) = \|\nabla u\|^2 \). By Green's formula,

\[ \mathcal{E}_\alpha(u) = \mathcal{D}(u) + \int_M \frac{1}{2}(F, \nabla(u^2)) + (\alpha + V)u^2 \, dm = \mathcal{D}(u) + \int_M \left( -\frac{1}{2}(\text{div} F) + \alpha + V \right) u^2 \, dm. \]

Letting \( \alpha \) so that \( 0 < \lambda_1 = \inf(-\frac{1}{2}(\text{div} F) + \alpha + V) \), we get

\[ \lambda_1 \mathcal{D}_1(u) \leq \mathcal{E}_\alpha(u) \leq \lambda_2 \mathcal{D}_1(u), \]

where \( \lambda_2 = \sup(-\frac{1}{2}(\text{div} F) + \alpha + V) \). The assertion will follow from the fact that \( \mathcal{D} \) is closable, which is well known and proved for the sake of completeness: As \((\nabla u_n)\) is a Cauchy sequence in \( L^2(TM, dm) \), we denote its limit by \( X \). For any smooth vector field \( Y \),

\[ \int_M g(X, Y) \, dm = \lim_{n \to \infty} \int_M g(\nabla u_n, Y) \, dm = -\lim_{n \to \infty} \int_M u_n \text{div} Y \, dm = 0. \]

**Proposition 2.** The energy \((\mathcal{E}_\alpha, C^\infty(M))\) satisfies the sector condition, that is, there exists a constant \( K \geq 1 \) such that

\[ |\mathcal{E}(u, v)|^2 \leq K\mathcal{E}_\alpha(u)\mathcal{E}_\alpha(v), \quad \forall u, v \in C^\infty(M). \]
Proof. Let $u, v \in C^\infty(M)$. Denoting $C = \sup(|F| + |V|)$ and $C' = 2(1 + 2C^2)$,

$$|\mathcal{E}(u, v)|^2 = \left| \int_M (g(\nabla u, \nabla v) + (\langle F, \nabla u \rangle + Vu)v)dm \right|^2$$

$$\leq \left| \int_M (|\nabla u||\nabla v| + |F||\nabla u| + |Vu|)|v|dm \right|^2$$

$$\leq \left| \int_M (|\nabla u||\nabla v| + C(|\nabla u| + |u|)|v|)dm \right|^2$$

$$\leq 2 \left( \left( \int_M |\nabla u||\nabla v|dm \right)^2 + \left( \int_M (|\nabla u| + |u|)|v|dm \right)^2 \right)$$

$$\leq C' \left( \left( \int_M |\nabla u||\nabla v|dm \right)^2 + \left( \int_M |\nabla u||v|dm \right)^2 + \left( \int_M |u||v|dm \right)^2 \right).$$

By the Cauchy-Schwarz inequality,

$$|\mathcal{E}(u, v)|^2 \leq C' (\|\nabla u\|^2 \|\nabla v\|^2 + (\|\nabla u\|^2 + \|u\|^2) \|v\|^2)$$

On the other hand, for any $a > 0$,

$$\mathcal{E}_\alpha(u) = \|\nabla u\|^2 + \int_M \frac{1}{2} \langle F, \nabla(u^2) \rangle + (\alpha + V)u^2dm$$

$$\geq \|\nabla u\|^2 - \int_M |F||u||\nabla u| + (\alpha + V)u^2dm$$

$$\geq \|\nabla u\|^2 - 2 \left( \frac{1}{a} \int_M |F|^2|u|^2 dm + a \int_M |\nabla u|^2 dm \right) + \int_M (\alpha + V)u^2dm$$

$$= (1 - 2a)\|\nabla u\|^2 + \int_M \left( -\frac{2}{a}|F|^2 + \alpha + V \right)u^2dm$$

$$= \frac{1}{2}\|\nabla u\|^2 + \int_M (-8|F|^2 + \alpha + V)u^2dm$$

by letting $a = 1/4$. Setting $\beta \leq \sup(8|F|^2 + \alpha + V)$, we have

$$\mathcal{E}_\alpha(u)\mathcal{E}_\alpha(v) \geq \left( \frac{1}{2}\|\nabla u\|^2 + \beta \int_M u^2dm \right) \left( \frac{1}{2}\|\nabla v\|^2 + \beta \int_M v^2dm \right)$$

$$\geq \frac{1}{4}\|\nabla u\|^2\|\nabla v\|^2 + \beta \left( \frac{1}{2}\|\nabla u\|^2 + \|u\|^2 \right) \|v\|^2.$$

This together with (5), and by the fact that we may take $\beta$ arbitrary large, we get the desired conclusion. \qed

By a standard semigroup theory, Theorem 1 yields

Corollary 1. There exists a strongly semigroup $\{T_t\}_{t \geq 0}$ on $L^2$ such that $\|T_t\| \leq e^{\alpha_0}$ whose resolvent $G_{\alpha}u = \int_0^\infty e^{-\alpha t}T_tudu$ with $\alpha > \alpha_0$ satisfying

$$\mathcal{E}_\alpha(G_{\alpha}u, v) = (u, v), \ \forall u \in L^2, \ v \in \mathcal{F}.$$

Definition 2 (Dirichlet forms). A closed form $(Q, D(Q))$ is called a lower-bounded semi-Dirichlet form if it satisfies

$$u \in D(Q), \ a \geq 0 \implies v = u \land a \in D(Q), \ Q(v, u - v) \geq 0.$$

Theorem 2. The form $(\mathcal{E}, \mathcal{F})$ is a lower-bounded semi-Dirichlet form.
Proof. We need to prove (6). The fact that $u \wedge a \in W^{1,2}$ whenever $u \in W^{1,2}$ and $a \in \mathbb{R}$ can be proved as in the Euclidean case (see, e.g., [2]). It suffices to prove the second statement only for $u \in C^\infty(M)$ by the density argument. Setting $D_+ = \{u > a\}$ and $D_- = \{u < a\}$, we note: $u - u \wedge a = 0$ on $D_-$ and $u \wedge a = a$ on $D_+$. Taking into account that the measures of the boundaries of these sets are 0,

$$
\mathcal{E}(u \wedge a, u - u \wedge a)
= \int_M g(\nabla(u \wedge a), \nabla(u - u \wedge a)) \, dm
+ \int_M \langle F, \nabla(u \wedge a) \rangle (u - u \wedge a) \, dm + \int_M V(u \wedge a)(u - u \wedge a) \, dm = \int_{D_+} V(a - u) \, dm \geq 0.
$$

\square

An important consequence of Theorem 2 is

**Corollary 2** (see, e.g., Theorem 3.3.4 [5]). There exists a Hunt process whose resolvent is a q.e. modification of $G_\alpha$ in $L^\infty$

**Remark 1.** I. Shigekawa [6] obtained a condition for $F$ so that the operator $\Delta + \langle F, \nabla \cdot \rangle$ without the sector condition generates a Markovian semigroup on a complete Riemannian manifold. We will need the sector condition for the existence of equilibrium potential in the next section.

3. Capacity associated to $\mathcal{E}_\alpha$

Hereafter, $\alpha_0 > 0$ is the constant which was specified in the previous section and $\alpha > \alpha_0$. For an open set $A \subset M$, set a subset $\mathcal{L}_A \subset F$ by

$$
\mathcal{L}_A = \{u \in F | u|_A \geq 1 \text{ m-a.e.}\}.
$$

Clearly, $\mathcal{L}_A$ is a non-empty closed convex set. For arbitrary fixed $u \in F$, set:

$$
J(w) = \mathcal{E}_\alpha(u, w), \quad w \in F.
$$

Since $J$ is a continuous linear functional on $F$, we may apply Stampaccia's theorem and find a unique $v \in F$ such that

$$
\mathcal{E}_\alpha(v, w - v) \geq J(w - v), \quad \forall w \in F.
$$

This determines a projection $\pi : F \to \mathcal{L}_A$ by $\pi(u) = v$. The element $\pi(0)$ is called the equilibrium potential of $A$ denoted by $e_A$. It follows that

$$
(7) \quad \mathcal{E}_\alpha(e_A) \leq \mathcal{E}_\alpha(e_A, w) \leq K\mathcal{E}_\alpha(e_A)^{1/2}\mathcal{E}_\alpha(w)^{1/2}, \quad \forall w \in F.
$$

Changing $J$ to $\hat{J}$, where $\hat{J}(w) = \mathcal{E}_\alpha(w, u)$, we find the co-equilibrium potential of $A$ in $\mathcal{L}_A$ denoted by $\hat{e}_A$ and satisfying

$$
\mathcal{E}_\alpha(\hat{e}_A) \leq K^2\mathcal{E}_\alpha(w), \quad \forall w \in F.
$$

Moreover, (see, e.g., Lemma 2.1.1 in [5]),

$$
e_A|_A = 1, \text{ m-a.e.}
$$

and for $u \in F$ such that $u|_A = 1$ m-a.e.,

$$
\mathcal{E}_\alpha(e_A, u) = \mathcal{E}_\alpha(e_A), \quad \mathcal{E}_\alpha(u, \hat{e}_A) = \mathcal{E}_\alpha(e_A, \hat{e}_A)
$$

The $(\alpha)$-capacity of $A$ is defined as

$$
\text{Cap}(A) = \mathcal{E}_\alpha(e_A, \hat{e}_A).
$$

By (3) and (7),

$$
(8) \quad \lambda_1\mathcal{D}(e_A) \leq \mathcal{E}_\alpha(e_A) \leq \text{Cap}(A) \leq K^2\mathcal{E}_\alpha(e_A) \leq K^2\lambda_2\mathcal{D}(e_A).
$$

The capacity of an arbitrary set $B \subset M$ is defined as

$$
\text{Cap}(B) = \inf_{BCA} \{\text{Cap}(A) | A \text{ is open an set in } M\}.
$$

Now we answer the second question in
Theorem 3. For any set $B \subset M$,

$$\text{Cap}(B) = 0 \iff \text{Cap}_D(B) = 0,$$

where $\text{Cap}_D(B)$ is the capacity of $B$ associated to $D$.

Proof. First, let us suppose that $\text{Cap}(B) = 0$. Then (8) implies that

$$0 \leq \text{Cap}_D(B) \leq \liminf_{n \to \infty} D(e_{A_n}) \leq \lambda_1^{-1} \liminf_{n \to \infty} \mathcal{E}_\alpha(e_{A_n}) \leq \lambda_1^{-1} \lim_{n \to \infty} \text{Cap}(A_n) = 0,$$

where $(A_n)$ is a sequence of open sets in $M$ approximating $\text{Cap}(B)$.

Next, let us suppose that $\text{Cap}_D(B) = 0$ and let $(A_n)$ be its approximation sequence. Denoting by $\eta_n \in \mathcal{L}_{A_n}$ the equilibrium potential of $A_n$ associated with $\mathcal{D}$,

$$0 \leq \text{Cap}(B) \leq \liminf_{n \to \infty} \text{Cap}(A_n)$$

where $\mathcal{E}_\alpha(e_{A_n})$ is the energy of $e_{A_n}$ associated with $\mathcal{D}$.

Therefore, we have the assertion. \qed

Remark 2. In closing this note, let us mention two related questions to our study.

- As we have studied in this note, it turned out that the capacity of a closed set of a compact manifold being 0 is independent of drifts. Clearly, the situation will be different for a non-compact Riemannian manifold. In particular, it would be interesting to extend the theory of Cauchy boundary and polarity of a singular set of a singular manifold (see, e.g., [4]) to non-symmetric case.

- It is known that a capacity of a symmetric Dirichlet form is related to a quantum mechanical tunnelling phenomena [1]. Can one formulate a non-symmetric quantum mechanical tunnelling, and if yes, how is it related with the capacity of a non-symmetric Dirichlet form?

Acknowledgements. I wish to show my gratitude toward Professor Kawashita for having invited the author to his stimulating workshop at Research Institute for Mathematical Sciences, Kyoto.

References


E-mail address: masamune@ms.tohoku.ac.jp