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Generalized Beckner’s inequalities and its applications to new geometric properties

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1 Introduction

This note is a survey on [7, 8]. For a Banach space $X$, let

$$\delta_X(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : x, y \in S_X, \|x-y\| = \varepsilon \right\}$$

for each $\varepsilon \in (0, 2]$, and let

$$\rho_X(\tau) = \sup \left\{ \frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 : x, y \in S_X \right\}$$

for each $\tau \geq 0$. These constants are, respectively, the moduli of convexity and smoothness of $X$. Let $1 < p \leq 2 \leq q < \infty$. Then a Banach space $X$ is said to be

(i) uniformly convex if $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$,

(ii) $q$-uniformly convex if there exists $C > 0$ such that $\delta_X(\varepsilon) \geq C\varepsilon^q$ for each $\varepsilon \in (0, 2]$,

(iii) uniformly smooth if $\lim_{\tau \to 0^+} \rho_X(\tau)/\tau = 0$, and

(iv) $p$-uniformly smooth if there exists $K > 0$ such that $\rho_X(\tau) \leq K\tau^p$ for all $\tau \geq 0$.

Obviously the implications (ii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (iii) hold. These properties are called geometric properties of Banach spaces as well as strict convexity and uniform non-squareness, and play important roles in the study of Banach space geometry. For basic facts of $p$-uniform smoothness and $q$-uniform convexity, the readers are referred to [1, 9].

A norm $\| \cdot \|$ on $\mathbb{R}^2$ is said to be absolute if $\| (x, y) \| = \| (x, y) \|$ for all $(x, y) \in \mathbb{R}^2$, normalized if $\| (1, 0) \| = \| (0, 1) \| = 1$, and symmetric if $\| (x, y) \| = \| (y, x) \|$. The set of all absolute normalized norms on $\mathbb{R}^2$ is denoted by $AN_2$. Bonsall and Duncan [3] showed the following characterization of absolute normalized norms on $\mathbb{R}^2$. Namely, the set $AN_2$ of all absolute normalized norms on $\mathbb{R}^2$ is in a one-to-one correspondence with the set $\Psi_2$ of all convex functions $\psi$ on $[0, 1]$ satisfying $\max \{ 1 - t, t \} \leq \psi(t) \leq 1$ for each $t \in [0, 1]$ (cf.
The correspondence is given by the equation $\psi(t) = \|(1 - t, t)\|$ for each $t \in [0, 1]$. Remark that the norm $\| \cdot \|_{\psi}$ associated with the function $\psi \in \Psi_{2}$ is given by

$$
\| (x, y) \|_{\psi} =\begin{cases}
(|x| + |y|)\psi \left( \frac{|y|}{|x| + |y|} \right) & \text{if } (x, y) \neq (0, 0), \\
0 & \text{if } (x, y) = (0, 0).
\end{cases}
$$

We also remark that the norm $\| \cdot \| \in AN_{2}$ is symmetric if and only if $\psi(1 - t) = \psi(t)$ for each $t \in [0, 1]$. For example, the function $\psi_{p}$ corresponding to $\| \cdot \|_{p}$ is given by

$$
\psi_{p}(t) =\begin{cases}
((1 - t)^{p} + t^{p})^{1/p} & \text{if } 1 \leq p < \infty, \\
\max\{1 - t, t\} & \text{if } p = \infty,
\end{cases}
$$

and satisfies $\psi_{p}(1 - t) = \psi_{p}(t)$ for each $t \in [0, 1]$. Let $\Psi_{2}^{S} = \{ \psi \in \Psi_{2} : \psi(1 - t) = \psi(t) \text{ for each } t \in [0, 1] \}$. The aim of this note is to present generalized Beckner inequalities, and to introduce new geometric properties of Banach spaces that generalize $p$-uniform smoothness and $q$-uniform convexity using absolute normalized norms.

## 2 Generalized Beckner inequalities

We first consider generalized Beckner inequalities. The original Becker inequality is the following: Let $1 < p \leq q < \infty$, and let $\gamma_{p,q} = \sqrt{(p - 1)/(q - 1)}$. Then the inequality

$$
\left( \frac{|u + \gamma_{p,q}v|^{q} + |u - \gamma_{p,q}v|^{q}}{2} \right)^{1/q} \leq \left( \frac{|u + v|^{p} + |u - v|^{p}}{2} \right)^{1/p}
$$

holds for each $u, v \in \mathbb{R}$. This was shown in 1975 by Beckner [2]. It is also known that $\gamma_{p,q}$ in the above inequality is the best constant, that is, if $\gamma \in [0, 1]$ and the inequality

$$
\left( \frac{|u + \gamma v|^{q} + |u - \gamma v|^{q}}{2} \right)^{1/q} \leq \left( \frac{|u + v|^{p} + |u - v|^{p}}{2} \right)^{1/p}
$$

holds for each $u, v \in \mathbb{R}$, then we have $\gamma \leq \gamma_{p,q}$. In [10], we constructed an elementary proof of these facts. Beckner's inequality is easily extended to Banach spaces; see [4, Corollary 1.e.15] for the proof.

**Theorem 2.1.** Let $1 < p \leq q < \infty$, and let $\gamma_{p,q} = \sqrt{(p - 1)/(q - 1)}$. Then the inequality

$$
\left( \frac{\|x + \gamma_{p,q}y\|^{q} + \|x - \gamma_{p,q}y\|^{q}}{2} \right)^{1/q} \leq \left( \frac{\|x + y\|^{p} + \|x - y\|^{p}}{2} \right)^{1/p}
$$

holds for each $x, y \in X$.

Using the functions $\psi_{p}$ and $\psi_{q}$, Beckner's inequality can be viewed as follows: Let $1 < p \leq q < \infty$, and let $\gamma_{p,q} = \sqrt{(p - 1)/(q - 1)}$. Then the inequality

$$
\left( \frac{\|(u + \gamma_{p,q}v, u - \gamma_{p,q}v)\|_{q}}{2\psi_{q}(1/2)} \right)^{1/q} \leq \left( \frac{\|(u + v, u - v)\|_{p}}{2\psi_{p}(1/2)} \right)^{1/p}
$$
holds for each \( u, v \in \mathbb{R} \). From this observation, we considered in [7] generalized Beckner's inequality. Namely, for each \( \varphi, \psi \in \Psi_2 \), let

\[
\Gamma(\varphi, \psi) = \left\{ \gamma \in [0, 1] : \frac{\varphi\left(\frac{1-\gamma u}{2}\right)}{\psi\left(\frac{1-u}{2}\right)} \leq \frac{\varphi\left(\frac{1}{2}\right)}{\psi\left(\frac{1}{2}\right)} \text{ for all } u \in [0, 1] \right\},
\]

and let \( \gamma_{\varphi, \psi} = \max \Gamma(\varphi, \psi) \). Then we have the following result. Suppose that \( X \) is a Banach space. Then for each \( \varphi \) the \( \psi \)-direct sum of \( X \), denoted by \( X \oplus_{\psi} X \), is the space \( X \times X \) equipped with the norm \( \| (x, y) \|_\psi = \| (\| x \|, \| y \|) \|_\psi \).

**Theorem 2.2** (Generalized Beckner's inequality [7]). Let \( X \) be a Banach space. Suppose that \( \varphi, \psi \in \Psi_2^S \), and that \( \gamma \in \Gamma(\varphi, \psi) \). Then the inequality

\[
\frac{\| (x + \gamma y, x - \gamma y) \|_\varphi}{2\varphi\left(\frac{1}{2}\right)} \leq \frac{\| (x + y, x - y) \|_\psi}{2\psi\left(\frac{1}{2}\right)}
\]

holds for each \( x, y \in X \).

We present some conditions that \( \gamma_{\varphi, \psi} > 0 \); see [7] for details. For each \( \psi \in \Psi_2^S \), let \( \psi'_L \) denote the left derivative of \( \psi \).

**Theorem 2.3.** Let \( \varphi, \psi \in \Psi_2^S \). Then the following hold:

(i) If \( \varphi'_L(1/2) = 0 \) and \( \psi'_L(1/2) < 0 \), then \( \gamma_{\varphi, \psi} > 0 \).

(ii) If \( \varphi'_L(1/2) < 0 \) and \( \psi'_L(1/2) = 0 \), then \( \gamma_{\varphi, \psi} = 0 \).

(iii) If \( \varphi'_L(1/2) < 0 \) and \( \psi'_L(1/2) < 0 \), then \( \gamma_{\varphi, \psi} > 0 \).

In particular, if \( \varphi'_L(1/2) < 0 \) then

\[
\gamma_{\varphi, \psi} \leq \frac{\varphi\left(\frac{1}{2}\right)\psi'_L\left(\frac{1}{2}\right)}{\psi\left(\frac{1}{2}\right)\varphi'_L\left(\frac{1}{2}\right)}.
\]

**Theorem 2.4.** Let \( \varphi, \psi \in \Psi_2^S \). Suppose that the second derivatives \( \varphi'' \) and \( \psi'' \) are continuous on \( (\delta, 1-\delta) \) for some \( 0 \leq \delta < 1/2 \). Then the following hold:

(i) If \( \varphi''(1/2) = 0 \) and \( \psi''(1/2) > 0 \), then \( \gamma_{\varphi, \psi} > 0 \).

(ii) If \( \varphi''(1/2) > 0 \) and \( \psi''(1/2) = 0 \), then \( \gamma_{\varphi, \psi} = 0 \).

(iii) If \( \varphi''(1/2) > 0 \) and \( \psi''(1/2) > 0 \), then \( \gamma_{\varphi, \psi} > 0 \).

In particular, if \( \varphi''(1/2) > 0 \) then

\[
\gamma_{\varphi, \psi} \leq \sqrt{\frac{\varphi\left(\frac{1}{2}\right)\psi''\left(\frac{1}{2}\right)}{\psi\left(\frac{1}{2}\right)\varphi''\left(\frac{1}{2}\right)}}.
\]

**Remark 2.5.** We remark that

\[
\sqrt{\frac{\psi_q\left(\frac{1}{2}\right)\psi''_p\left(\frac{1}{2}\right)}{\psi_p\left(\frac{1}{2}\right)\psi''_q\left(\frac{1}{2}\right)}} = \sqrt{\frac{p-1}{q-1}} = \gamma_{p,q},
\]

where \( \gamma_{p,q} \) is the best constant for Beckner's inequality.
For each \( \psi \in \Psi_2 \), define the function \( \psi^* \) by
\[
\psi^*(t) = \max_{0 \leq s \leq 1} \frac{(1-s)(1-t)+st}{\psi(s)}
\]
for each \( t \in [0,1] \). Then \( \psi^* \in \Psi_2 \) and \((\mathbb{R}^2, \|\cdot\|_\psi)^* = (\mathbb{R}, \|\cdot\|_{\psi^*})\), and so the function \( \psi^* \) is called the dual function of \( \psi \); see [5]. Clearly, \( \psi \in \Psi_2^S \) if and only if \( \psi^* \in \Psi_2^S \).

Generalized Beckner inequalities have the following duality property.

**Theorem 2.6.** Let \( \varphi, \psi \in \Psi_2^S \). Then \( \gamma_{\varphi, \psi} = \gamma_{\psi^*, \varphi^*} \).

### 3 New geometric properties

We now consider new geometric properties of Banach spaces. First, we present the following characterizations of \( p \)-uniform smoothness and \( q \)-uniform convexity.

**Proposition 3.1.** Let \( X \) be a Banach space, and let \( 1 < p \leq 2 \). Then \( X \) is \( p \)-uniformly smooth if and only if there exists \( M > 0 \) such that \( \rho_X(\tau) \leq \| (1, M\tau) \|_p - 1 \) for each \( \tau \in [0,1] \).

**Proof.** Suppose that \( X \) is \( p \)-uniformly smooth. Then there exists a \( K > 0 \) satisfying \( \rho_X(\tau) \leq K\tau^p \) for each \( \tau > 0 \). Since the function \( f \) on \([0,1]\) given by
\[
f(\tau) = 1 + pK(1 + K)^{p-1}\tau^p - (1 + K\tau^p)^p
\]
is nondecreasing, it follows that \( f \geq 0 \). Putting \( M = p^{1/p}K^{1/p}(1 + K)^{1-1/p} \) we have
\[
\rho_X(\tau) \leq 1 + K\tau^p - 1 \\
\leq (1 + pK(1 + K)^{p-1}\tau^p)^{1/p} - 1 \\
= \| (1, M\tau) \|_p - 1
\]
for each \( \tau \in [0,1] \).

Conversely, let \( M \) be a positive real number such that
\[
\rho_X(\tau) \leq \| (1, M\tau) \|_p - 1
\]
for each \( \tau \in [0,1] \). Then for each \( \tau \in [0,1] \) one has
\[
\rho_X(\tau) \leq \| (1, M\tau) \|_p - 1 = (1 + M^{p-1}\tau^p)^{1/p} - 1 \leq 1 + \frac{1}{p}M^p\tau^p - 1 = \frac{1}{p}M^p\tau^p.
\]
On the other hand, if \( \tau \geq 1 \) then \( \rho_X(\tau) \leq \tau \leq \tau^p \). Hence we obtain
\[
\rho_X(\tau) \leq \max\{M^p/p, 1\}\tau^p
\]
for each \( \tau \geq 0 \), that is, the space \( X \) is \( p \)-uniformly smooth. \( \square \)

**Proposition 3.2.** Let \( 2 \leq q < \infty \). Then a Banach space \( X \) is \( q \)-uniformly convex if and only if it is \( K > 0 \) such that \( \| (1 - \delta_X(\varepsilon), K\varepsilon) \|_q \leq 1 \) for each \( \varepsilon \in [0,2] \).
Proof. Suppose that $X$ is $q$-uniformly convex. Then there exists $C > 0$ such that $\delta_X(\epsilon) \geq C\epsilon^q$ for each $\epsilon \in [0, 2]$. One can easily check that

$$(1 - x)^q \leq 1 - \frac{x}{2}$$

for each $x \in [0, 1]$. Hence, by $0 \leq C\epsilon^q \leq \delta_X(\epsilon) \leq 1$, we have

$$(1 - \delta_X(\epsilon))^q \leq (1 - C\epsilon^q)^q \leq 1 - \frac{C\epsilon^q}{2}.$$ 

Putting $K = (C/2)^{1/q}$, we obtain

$$\| (1 - \delta_X(\epsilon), K\epsilon) \|_{q} = \left( (1 - \delta_X(\epsilon))^q + K^q\epsilon^q \right)^{1/q} \leq 1$$

for each $\epsilon \in [0, 2]$. Conversely, assume that there exists $K > 0$ such that $\| (1 - \delta_X(\epsilon), K\epsilon) \|_{q} \leq 1$ for each $\epsilon \in [0, 2]$. Then $(1 - \delta_X(\epsilon))^q \leq 1 - K^q\epsilon^q$, and so

$$1 - \delta_X(\epsilon) \leq (1 - K^q\epsilon^q)^{1/q} \leq 1 - \frac{1}{q}K^q\epsilon^q.$$ 

Thus, for $C = K^q/q$, we have $\delta_X(\epsilon) \geq C\epsilon^q$ for each $\epsilon \in [0, 2]$. This shows $X$ is $q$-uniformly convex.

These propositions allows us to consider new geometric properties using absolute normalized norms. We now introduce $\psi$-uniform smoothness and $\psi^*$-uniform convexity as follows: Let $\psi \in \Phi_2$. Then a Banach space $X$ is said to be

(i) $\psi$-uniformly smooth if there exists $M > 0$ such that $\rho_X(\tau) \leq \|(1, M\tau)\|_{\psi} - 1$ for each $\tau \in [0, 1]$.

(ii) $\psi^*$-uniformly convex if there exists $K > 0$ such that $\|(1 - \delta_X(\epsilon), K\epsilon)\|_{\psi^*} \leq 1$ for each $\epsilon \in [0, 2]$.

Then Propositions 3.1 and 3.2 guarantee that a Banach space $X$ is

(a) $p$-uniformly smooth if and only if it is $\psi_p$-uniformly smooth, and

(b) $q$-uniformly convex if and only if it is $\psi_q$-uniformly smooth.

Naturally, one has $\psi_q = (\psi_p)^*$ provided that $1/p + 1/q = 1$. Hence the above new geometric properties are natural generalizations of that of $p$-uniform smoothness and $q$-uniform convexity.

For further results in this direction, the readers are referred to [8].

References


