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Multicriteria Multipliers of Banach-valued Functions on Locally Compact Abelian Group*

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Abstract

Let $G$ be a locally compact Abelian (LCA) group, $A$ a commutative Banach algebra, “$X$” and “$Y$” denote the Banach spaces of $A$-module. $L^1(G,A)$ stands for the space of all $A$-valued commutative Banach algebra with convolution product. $L^p(G,X)$, $1 \leq p \leq \infty$, for each $p$, is a Banach space. In this note, we study the multipliers of $L^1(G,A)$ and the representation of the homomorphism $L^1(G,A)$ module multipliers of $L^1(G,A)$ to $L^p(G,Y)$ which can be identified by $L^1(G,A) \otimes L^q(G,Y^*)^*$ under reasonable conditions, where $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. The multipliers of $L^1(G,A)$ to $C_0(G,X)$ is also subscribed.

Key words and phrases: locally compact Abelian (LCA) group, separable Banach space, Radon Nikodym property, multipliers, invariant operator, projective tensor product space.

1 Introduction and preliminaries

Let $G$ be a locally compact Abelian (= LCA) group with Haar measure $dt$ and dual group $\hat{G}$. Let $A$ be a commutative Banach algebra with a bounded approximate identity. A continuous linear map $T \in \mathfrak{L}(A) \cong \mathfrak{L}(A,A)$ is called a multiplier of $A$ if

$$T(a \cdot b) = a \cdot Tb = (Ta) \cdot b \text{ for all } a, b \in A.$$  

Denote by $\mathfrak{M}(A)$ the space of all multipliers for $A$.

Clearly, $\mathfrak{M}(A)$ is a Banach subalgebra of $\mathfrak{L}(A)$. In particular, if $A = L^1(G)$, a

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commutative group algebra under convolution product, then the multiplier algebra \( \mathfrak{M}(L^1(G)) \) has the following equivalent statements (i)\(~\)~(iv). (See Larsen [7], cf also Lai, Lee and Liu [1]):

**Theorem 1.** Let \( T \in \mathcal{L}(L^1(G)) \). Then the following statements are equivalent.

(i) \( T \) commutes with convolution (call \( T \) a multiplier)

\[ T(f * g) = Tf * g = f * T(g), \text{ for all } f, g \in L^1(G) \]

(ii) \( T \) commutes with translation operator \( \tau_a \) (\( a \in G \)). (call \( T \) an invariant operator)

\[ T \tau_a = \tau_a T, \tau_a f(t) = f(t - a), \text{ for all } a \in G, \]

(iii) \( \exists ! \) a \( \mu \in M_b(G) \), space of all bounded regular Borel measures such that,

\[ Tf = \mu * f, \text{ for all } f \in L^1(G). \]

(iv) there exists a bounded function \( \phi \) on \( \hat{G} \) such that

\[ \hat{Tf} = \phi \hat{f} \text{ or } \phi = \hat{\mu} \in \overline{M_b(G)} \subsetneq C^b(\hat{G}). \]

It is remarkable that

(a) the Fourier transforms \( \hat{L^1(G)} = A(\hat{G}) \subsetneq C_0(\hat{G}) \) is dense of 1st category in \( C_0(\hat{G}) \), the continuous function on \( \hat{G} \), vanishing at infinite.

(b) Similarly, it is known that the Fourier – Stieltjes transforms:

\( \hat{\mu} \in \overline{M_b(G)} \subsetneq C^b(\hat{G}) \), the space of all bounded continuous functions on \( \hat{G} \).

By Theorem 1, we see that the definition of multipliers is in various types. Actually the concept of multiplier comes from Fourier Series of a function \( f \) by using a bounded sequence \( \phi(n) \) multiply the Fourier coefficient \( c_n \) of \( f \), it still approve as a Fourier coefficient of another function of \( g \). This ideal leads to study for multipliers in harmonic analysis on locally compact Abelian group \( G \).

In this Note, we would like to extend the multipliers of \( L^1(G) \) to the multipliers of \( L^1(G,A) \) as well as multipliers of \( L^1(G,X) \) to \( L^1(G,Y) \) under module homomorphism of Banach vetor – valued functions defined on LCA group \( G \), and compare
2 Multipliers of Banach algebra.

Let $A$ be a commutative Banach algebra, we say that a Banach space $X$ is $A$-module if

$$AX \subset X, \text{ and } \| a \cdot x \|_X \leq \| a \|_A \| x \|_X \text{ for each } a \in A, x \in X.$$ 

and $X$ is said to be an essential $A$-module if

$$AX = X, \text{ and } \| ax \|_X \leq \| a \|_A \| x \|_X \text{, for each } a \in A, x \in X.$$ 

For convenience, we give following Theorem to check that an $A$-module Banach space to be essential.

**Theorem 2.** Let $A$ be a commutative Banach algebra with uniform bounded approximate identity. Then any $A$-module Banach space is essential.

For example, the group algebra $L^1(G)$ has bounded approximate identity: $\{e_\alpha\}$, where $e_\alpha = e_\alpha = \frac{\chi_{V_\alpha}}{|V_\alpha|}$, where $\{V_\alpha\}$ is defined by an open neighborhood system of the identity $\theta \in G$ with ordered by $\alpha < \beta$ if $V_\beta \subset V_\alpha$, then $\| e_\alpha \|_1 = \int_G \frac{\chi_{V_\alpha}}{|V_\alpha|} dt = 1$. Thus by Theorem 2, directly we get easily that

$$L^1(G) * L^p(G) = L^p(G), \text{ if } 1 < p < \infty$$

if $p = \infty$, we choose $C_0(G)$, the space of continuous functions vanishing at infinite on $G$, we also have

$$L^1(G) * C_0(G) = C_0(G)$$

**Remark 1** It is remarkable that not every Banach algebra has a bounded approximate identity. For example, the space

$$A^p(G) = \{ f \in L^1(G) | \hat{f} \in L^p(\hat{G}), 1 < p \leq \infty \} \subset L^1(G)$$
for each \( p, \, 1 \leq p < \infty \). But there is an approximate identity \( \{ e_{\alpha} \} \) in \( L^{1}(G) \) with Fourier transform \( \hat{e}_{\alpha} \) having compact support in \( \hat{G} \) for each \( \alpha \), then \( \hat{e}_{\alpha} \in L^{p}(\hat{G}) \) shows that \( \{ e_{\alpha} \} \) is also an approximate identity of \( A^{p}(G) \), but this system \( \{ e_{\alpha} \} \) of approximate identity is not uniform bounded in \( A^{p}(G) \). (cf. Lai [2, p254])

3 Multipliers of Banach Module Homomorphism

Let \( A \) be a commutative Banach algebra and \( X, Y \) \( A \)-module Banach spaces. A bounded linear operator \( T \in \mathcal{L}(X, Y) \) satisfying

\[
T(ax) = a(Tx) \quad \text{for all} \quad a \in A, \, x \in X,
\]

is called a multiplier of \( X \) to \( Y \) under \( A \)-module. The space of such multipliers is \( A \)-module homomorphisms from \( X \) to \( Y \) and is denoted by

\[
\mathfrak{M}_{A}(X, Y) = \text{Hom}_{A}(X, Y) = \{ T \in \mathcal{L}(X, Y) \mid T(ax) = a(Tx), \, a \in A, \, x \in X \}.
\]

It is a closed subalgebra of \( \mathcal{L}(X, Y) \), the space of all bounded linear mappings of \( X \) into \( Y \). In particular, if \( A = X = Y = \mathcal{L}^{1}(G) \), then the multiplier space \( \mathfrak{M}(\mathcal{L}^{1}(G)) \) coincides with the expression of isometrically isomorphic relations “\( \cong \)” as follows.

\[
\mathfrak{M}(\mathcal{L}^{1}(G)) = \text{Hom}_{\mathcal{L}^{1}(G)}(\mathcal{L}^{1}(G), \mathcal{L}^{1}(G)) \cong (\mathcal{L}^{1}(G), \mathcal{L}^{1}(G)) \cong M_{b}(G).
\]

where \( (E(G), F(G)) \) stands for the space of all invariant operators commute with translation operator \( \tau_{a} \) on the function spaces of \( E(G) \) to \( F(G) \).

In general, the multiplier space \( \text{Hom}_{A}(X, Y^{*}) \) was characterized by Rieffel [8] as the following dual space of the module tensor product \( X \otimes_{A} Y \):

\[
\text{Hom}_{A}(X, Y^{*}) \cong (X \otimes_{A} Y)^{*},
\]

where \( \otimes_{A} \) denotes the \( A \)-module tensor product defined by \( X \otimes_{A} Y = X \otimes_{\gamma} Y / K \). \( K \) is the closed linear subspace of the complete projective tensor product space \( X \otimes_{\gamma} Y \) generating by elements: \( ax \otimes y - x \otimes ay \), for \( a \in A, \, x \in X, \, y \in Y \)

Here \( \otimes_{\gamma} \) is the completion of the algebra tensor \( X \otimes Y \) under the largest reasonable cross norm \( \gamma \), and

\[
X \otimes Y = \{ u = \sum_{i} x_{i} \otimes y_{i} \mid \sum_{i} \| x_{i} \|_{X} \| y_{i} \|_{Y} < \infty \}\]
with norm $\gamma(u) \equiv \|u\| = \inf_{u} \sum_{i} \|x_{i} \otimes y_{i}\| = \inf_{u} \sum_{i} \|x_{i}\|_{x}\|y_{i}\|_{y}$, \(\inf\) means that the infimum is taken by all representations of \(u = \sum_{i} x_{i} \otimes y_{i}\) in \(X \otimes Y\).

The reasonable crossnorm means that
\[ u \in X \otimes Y, \ u = x \otimes y \text{ implies } \|u\| = \|x \otimes y\| = \|x\|_{X} \|y\|_{Y}; \]
and \(u = \sum_{i} x_{i} \otimes y_{i}, \ \|u\| = \inf \sum_{i} \|x_{i}\|_{X} \|y_{i}\|_{Y}\).

Note that a bounded linear operator \(T \in \text{Hom}_{A}(X,Y^{*})\) in (3.4) corresponding a continuous linear functional \(\psi\) on \(X \otimes_{A} Y\) is given by
\[(Tx)(y) = \psi(x \otimes y) \text{ for all } x \in X, y \in Y.\]

Here \(\text{Hom}_{A}(X,Y^{*}) = \mathcal{M}_{A}(X,Y^{*})\) is the space of all \(A\)–module homomorphisms from \(X\) to \(Y^{*}\), the topological dual of \(Y\), that is, each \(T \in \text{Hom}_{A}(X,Y^{*})\) satisfies
\[T(ax) = a(Tx) \text{ for all } a \in A, x \in X, Tx \in Y^{*}.\]

where \(T\) is a bounded linear operator from \(X\) to \(Y^{*}\); \(X \otimes_{A} Y\) denotes the \(A\)–module tensor product space of \(X\) and \(Y\).

There are some known results in scalar-valued function space of \(L^{1}(G)\)–module by convolution. We state three typical \(L^{1}(G)\)–module multiplier problems as follows.

**Theorem 3.** (i) \(\text{Hom}_{G}(L^{1}(G),L^{1}(G)) \cong M_{b}(G)\), (by Theorem 1, (iii) \(\iff\) (i))

where \(\text{Hom}_{G} = \text{Hom}_{L^{1}(G)}\), and \(M_{b}(G)\) is the space of all bounded regular Borel measures on \(G\).

(ii) \(\text{Hom}_{G}(L^{1}(G),L^{p}(G)) \cong (L^{1}(G) \otimes_{G} L^{q}(G))^{*} = (L^{q}(G))^{*} = L^{p}(G)\),
for \(1 < p < \infty, \ \frac{1}{p} + \frac{1}{q} = 1\) where \(\otimes_{G} = \otimes_{L^{1}(G)}\).

(iii) \(\text{Hom}_{G}(L^{p}(G),L^{p}(G)) \cong (L^{p}(G) \otimes_{G} L^{q}(G))^{*} \cong S_{p}(G)^{*},\)
where \(S_{p}(G)\) is a Banach algebra generated by
\[\{u = \sum_{i} f_{i} g_{i} : f_{i} \in L^{p}(G), g_{i} \in L^{q}(G), \sum_{i} \|f_{i}\|_{p} \|g_{i}\|_{q} < \infty\}\]
under pointwise product and the norm is defined by (cf. Larsen [7])
\[\|u\| = \inf \sum_{i} \|f_{i}\|_{p} \|g_{i}\|_{q}; \ u = \sum_{i} f_{i} \cdot g_{i} \in S_{p}(G)\}.\]
4 Multipliers of Banach-valued Functions on $G$.

Let $A$ be a commutative semi-simple Banach algebra with bounded approximate identity. Assume $X$ is on $A$-module Banach space. It is not hard to prove that $L^1(G,A) = L^1(G)\otimes_A A$. Since both $L^1(G)$ and $A$ have bounded approximate identity, thus $L^1(G,A)$ is a commutative Banach algebra with bounded approximate identity.

By Theorem 2
\[
L^1(G,A) \ast L^p(G,X) = L^p(G,X), \quad 1 < p < \infty
\]

Denote by
\[
L^1(G,A) = \{ f : G \rightarrow A \mid f \text{ is measurable and is Bochner integrable on } G \}
\]

Then $L^1(G,A)$ is a commutative Banach algebra, under convolution. Actually
\[
|f \ast g(t)|_A \leq \int_G |f(s-t)|_A |g(s)|_A \, ds = \|g\|_1 \int_G |f(s-t)|_A \, ds = \|g\|_1 \|f\|_1,
\]

\[
\|f \ast g\|_1 = \int_G |f \ast g(t)|_A \, dt \leq \|g\|_1 \int_G |f(s-t)|_A \, dt \leq \|g\|_1 \|f\|_1.
\]

Denote by
\[
L^p_X = \{ f : G \rightarrow X \mid f \text{ is measurable and } |f(\cdot)|_X \in L^p(G) \}, 1 \leq p < \infty,
\]

\[
\|f\|_p = \left( \int_G |f(t)|_X^p \, dt \right)^{\frac{1}{p}}, \text{ for } f \in L^p_X, 1 \leq p < \infty \quad (2.1)
\]

and for $p = \infty$, \[ \|f\|_{\infty} = \text{esssup}_{t \in G} |f(t)|_X \] for $f \in L^\infty_X \quad (2.2)

Show that $L^p_X, 1 \leq p \leq \infty$ are Banach spaces with the norm $\|f\|_p, 1 \leq p < \infty$, as (2.1) and if $p = \infty$, the norm is taken $\|\cdot\|_{\infty}$ as (2.2). If $X = \mathbb{C}$, the complex numbers, then

\[
L^p_X = L^p = L^p(G), 1 \leq p \leq \infty.
\]

If $X$ and $Y$ are $A$-module Banach space, the multiplier space of $X$ to $Y$ is given by
\[
\text{Hom}_A(X,Y) = \{ T \in \mathcal{L}(X,Y) \mid T(ax) = aT(x), a \in A, x \in X \}.
\]
Recall [8], Rieffel characterized the homomorphism module multiplier is represented by the dual space of module tensor product as the following form:

$$Hom_{A}(X,Y^{*}) \cong (X \otimes A Y)^{*} \text{ or } Hom_{A}(X,Y) \cong (X \otimes A Y^{*})^{*}. \text{ (if } Y \text{ is reflexive)}$$

where $\otimes_{A}$ is namely module tensor product of $X$ into $Y^{*}$ or of $X$ into $Y$ and $Z^{*}$ denotes the dual space of the Banach space $Z$. The space $\otimes_{A}$ is the complete projective tensor product $X \otimes_{\gamma} Y^{*}$ quotients by $K$, that is, $X \otimes_{A} Y = X \otimes_{\gamma} Y^{*}/K$.

Here $K$ is the closed linear subspace of the projective tensor product space $X \otimes_{\gamma} Y$ generated by the elements $ax \otimes y - x \otimes ay$; for $a \in A, x \in X, y \in Y$ and $X \otimes_{\gamma} Y$ is the completion of the algebra tensor $x \otimes y$ under the $\gamma-$norm, and

$$X \otimes Y = \{u = \sum_{i} x_i \otimes y_i \mid x_i \in X, y_i \in Y, \sum_{i} \|x_i\| \|y_i\| < \infty\}$$

$$\gamma(u) = \inf_{u} \left\{ \sum_{i} \|x_i\| \|y_i\| \mid u = \sum_{i} x_i \otimes y_i \in Y \right\}$$

$$= \|\|u\|| = \inf_{u} \left\{ \sum_{i} \|x_i \otimes y_i, x_i \in X, y_i \| \right\} = \inf_{u} \sum_{i} \|x_i\| \|y_i\|$$

where $\inf_{u}$ means that the infimum is taken by all representations of $u = \sum_{i} x_i \otimes y_i$ in $X \otimes Y$, and the tensor norm. We state the following Theorem for the characterization of the invariant operators. For detail, we consult Lai [3,4] and [6] cf. also Lai [5].

**Theorem 4.** Let $X$ and $Y$ be Banach spaces. Then the following two statements are equivalent.

(a) $T \in (L^{1}(G,Y),L^{1}(G,X))$ is an invariant operator.

(b) There exists a unique continuous linear map $L \in \mathcal{L}(Y,M_{b}(G,X))$ such that $T(f \otimes y) = f * L_{y}$ for all $f \in L^{1}(G), y \in Y$.

Moreover, $(L^{1}(G,Y),L^{1}(G,X)) \cong \mathcal{L}(Y,M_{b}(G,X))$.

**Theorem 5.** Let $A$ be a commutative semi-simple Banach algebra (not necessarily with identity) and $X$ a Banach $A$-module. Then

(5.1) $Hom_{L^{1}(G,A)}(L^{1}(G,A),L^{1}(G,X)) \cong Hom_{A}(A,M_{b}(G,X))$. 
In Lai [6], he showed that an invariant operator is also a multiplier if and only if the $A$ in $L^1(G,A)$ must be scalar space $\mathbb{C}$.

**Theorem 6.** Let $A$ be a commutative Banach algebra with identity of norm 1. $X$ be a unit linked, order-free, Banach-module and $A$ a faithful representation on $X$, then each invariant operator $T : L^1(G,A) \rightarrow F(G,X)$ is a multiplier if and only if $A \cong \mathbb{C}$. Here $F(G,X) = L^p(G,X)$ for each $p$, $1 \leq p \leq \infty$, or $F(G,X) = C_0(G,X)$.

**References**


