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Multicriteria Multipliers of Banach-valued Functions on Locally Compact Abelian Group

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Abstract

Let $G$ be a locally compact Abelian (LCA) group, $A$ a commutative Banach algebra, "$X"$ and "$Y"$ denote the Banach spaces of $A$-module. $L^1(G,A)$ stands for the space of all $A$-valued commutative Banach algebra with convolution product. $L^p(G,X)$, $1 \leq p \leq \infty$, for each $p$, is a Banach space. In this note, we study the multipliers of $L^1(G,A)$ and the representation of the homomorphism $L^1(G,A)$ module multipliers of $L^1(G,A)$ to $L^p(G,Y)$ which can be identified by $L^1(G,A) \otimes L^q(G,Y^*)^*$ under reasonable conditions, where $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. The multipliers of $L^1(G,A)$ to $C_0(G,X)$ is also subscribed.

Key words and phrases: locally compact Abelian (LCA) group, separable Banach space, Radon Nikodym property, multipliers, invariant operator, projective tensor product space.

1 Introduction and preliminaries

Let $G$ be a locally compact Abelian (LCA) group with Haar measure $dt$ and dual group $\hat{G}$. Let $A$ be a commutative Banach algebra with a bounded approximate identity. A continuous linear map $T \in \mathcal{L}(A) \cong \mathcal{L}(A,A)$ is called a multiplier of $A$ if

$$T(a \cdot b) = a \cdot Tb = (Ta) \cdot b \text{ for all } a,b \in A.$$ 

Denote by $\mathfrak{M}(A)$ the space of all multipliers for $A$. Clearly, $\mathfrak{M}(A)$ is a Banach subalgebra of $\mathcal{L}(A)$. In particular, if $A = L^1(G)$, a

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commutative group algebra under convolution product, then the multiplier algebra $\mathfrak{M}(L^1(G))$ has the following equivalent statements (i)~(iv). (See Larsen [7], cf also Lai, Lee and Liu [1]):

**Theorem 1.** Let $T \in \mathcal{L}(L^1(G))$. Then the following statements are equivalent.

(i) $T$ commutes with convolution (call $T$ a multiplier)

$$T(f \ast g) = Tf \ast g = f \ast T(g), \text{ for all } f, g \in L^1(G)$$

(ii) $T$ commutes with translation operator $\tau_a$ ($a \in G$). (call $T$ an invariant operator)

$$T\tau_a = \tau_a T, \tau_a f(t) = f(t-a), \text{ for all } a \in G,$$

(iii) $\exists ! \mu \in M_b(G)$, space of all bounded regular Borel measures such that,

$$Tf = \mu \ast f, \text{ for all } f \in L^1(G).$$

(iv) there exists a bounded function $\phi$ on $\hat{G}$ such that

$$\hat{Tf} = \phi \hat{f} \text{ or } \phi = \hat{\mu} \in \overline{M_b(G)} \subset C^b(\hat{G}).$$

It is remarkable that

(a) the Fourier transforms $\hat{L^1(\hat{G})} = A(\hat{G}) \subsetneq C_0(\hat{G})$ is dense of 1st category in $C_0(\hat{G})$, the continuous function on $\hat{G}$, vanishing at infinite.

(b) Similarly, it is known that the Fourier – Stieltjes transforms:

$$\hat{\mu} \in M_b(G) \subsetneq C^b(\hat{G}),$$

the space of all bounded continuous functions on $\hat{G}$.

By Theorem 1, we see that the definition of multipliers is in various types. Actually the concept of multiplier comes from *Fourier Series* of a function $f$ by using a bounded sequence $\phi(n)$ multiply the *Fourier coefficient* $c_n$ of $f$, it still approve as a *Fourier coefficient* of another function of $g$. This ideal leads to study for multipliers in harmonic analysis on *locally compact Abelian group* $G$.

In this Note, we would like to extend the multipliers of $L^1(G)$ to the multipliers of $L^1(G,A)$ as well as multipliers of $L^1(G,X)$ to $L^1(G,Y)$ under module homomorphism of *Banach vector – valued functions* defined on *LCA group* $G$, and compare
the Banach algebras $L^1(G,A)$ and $L^1(G)$, do have the same properties as in the Theorem 1? Actually, the invariant operator $T$ in $\mathcal{L}(L^1(G,A))$ can not be a multiplier of $L^1(G,A)$ provided $\dim A > 1$. (See Tewari, Dutta and Vaidya [9]). That is, in Theorem 1, (ii) $\Rightarrow$ (i) is false, the other implications are true.

2 Multipliers of Banach algebra.

Let $A$ be a commutative Banach algebra, we say that a Banach space $X$ is an $A$-module if

$$AX \subset X, \text{ and } \|a \cdot x\|_X \leq \|a\|_A \|x\|_X \text{ for each } a \in A, x \in X.$$ 

and $X$ is said to be an essential $A$-module if

$$AX = X, \text{ and } \|ax\|_X \leq \|a\|_A \|x\|_X \text{, for each } a \in A, x \in X.$$ 

For convenience, we give following Theorem to check that an $A$-module Banach space to be essential.

**Theorem 2.** Let $A$ be a commutative Banach algebra with uniform bounded approximate identity. Then any $A$-module Banach space is essential.

For example, the group algebra $L^1(G)$ has bounded approximate identity: $\{e_\alpha\}$, where $e_\alpha = e_\alpha = \frac{\chi_{V_\alpha}}{|V_\alpha|}$, where $\{V_\alpha\}$ is defined by an open neighborhood system of the identity $\theta \in G$ with ordered by $\alpha < \beta$ if $V_\beta \subset V_\alpha$, then $\|e_\alpha\|_1 = \int_G \frac{\chi_{V_\alpha}}{|V_\alpha|} dt = 1$. Thus by Theorem 2, directly we get easily that

$$L^1(G) * L^p(G) = L^p(G), \text{ if } 1 < p < \infty$$ 

if $p = \infty$, we choose $C_0(G)$, the space of continuous functions vanishing at infinite on $G$, we also have

$$L^1(G) * C_0(G) = C_0(G)$$

**Remark 1** It is remarkable that not every Banach algebra has a bounded approximate identity. For example, the space

$$A^p(G) = \{f \in L^1(G) \mid \hat{f} \in L^p(\hat{G}), 1 < p \leq \infty\} \subset L^1(G)$$
with norm defined by \( \| f \|_{A^p} = \| f \|_1 + \| \hat{f} \|_p \) is a \textit{commutative Banach algebra} for each \( p, 1 \leq p < \infty \). But there is an approximate identity \( \{ e_\alpha \} \) in \( L^1(G) \) with Fourier transform \( \hat{e_\alpha} \) having compact support in \( \hat{G} \) for each \( \alpha \), then \( \hat{e_\alpha} \in L^p(\hat{G}) \) shows that \( \{ e_\alpha \} \) is also an approximate identity of \( A^p(G) \), but this system \( \{ e_\alpha \} \) of approximate identity is not uniform bounded in \( A^p(G) \). (cf. Lai [2, p254])

\section{Multipliers of Banach Module Homomorphism}

Let \( A \) be a \textit{commutative Banach algebra} and \( X, Y \) \( A \)-\textit{module Banach spaces}. A bounded linear operator \( T \in \mathcal{L}(X,Y) \) satisfying

\begin{equation}
T(ax) = a(Tx) \quad \text{for all } a \in A, x \in X,
\end{equation}

is called a \textbf{multiplier} of \( X \) to \( Y \) under \( A \)-module. The space of such multipliers is \( A \)-\textit{module homomorphisms} from \( X \) to \( Y \) and is denoted by

\begin{equation}
\mathfrak{M}_A(X,Y) = \text{Hom}_A(X,Y) = \{ T \in \mathcal{L}(X,Y) \mid T(ax) = a(Tx), a \in A, x \in X \}.
\end{equation}

It is a closed subalgebra of \( \mathcal{L}(X,Y) \), the space of all bounded linear mappings of \( X \) into \( Y \). In particular, if \( A = X = Y = L^1(G) \), then the multiplier space \( \mathfrak{M}(L^1(G)) \) coincides with the expression of isometrically isomorphic relations \( \cong \) as follows.

\begin{equation}
\mathfrak{M}(L^1(G)) = \text{Hom}_{L^1(G)}(L^1(G), L^1(G)) \cong (L^1(G), L^1(G)) \cong M_b(G).
\end{equation}

where \( (E(G), F(G)) \) stands for the space of all invariant operators commute with translation operator \( \tau_a \) on the function spaces of \( E(G) \) to \( F(G) \).

In general, the multiplier space \( \text{Hom}_A(X,Y^*) \) was characterized by Rieffel [8] as the following dual space of the module tensor product \( X \otimes_A Y \):

\begin{equation}
\text{Hom}_A(X,Y^*) \cong (X \otimes_A Y)^*,
\end{equation}

where \( \otimes_A \) denotes the \( A \)-\textit{module tensor product} defined by \( X \otimes_A Y = X \otimes_Y Y/K \). \( K \) is the closed linear subspace of the complete projective tensor product space \( X \hat{\otimes} Y \) generating by elements: \( ax \otimes y - x \otimes ay \), for \( a \in A, x \in X, y \in Y \). Here \( \hat{\otimes} \) is the completion of the algebra tensor \( X \otimes Y \) under the largest reasonable cross norm \( \gamma \), and

\[ X \otimes Y = \{ u = \sum_i x_i \otimes y_i \mid \sum_i \| x_i \|_X \| y_i \|_Y < \infty \} \]
with norm \( \gamma(u) = \inf \sum_i \| x_i \otimes y_i \| = \inf \sum_i \| x_i \|_x \| y_i \|_y \), inf means that the infimum is taken by all representations of \( u = \sum_i x_i \otimes y_i \) in \( X \otimes Y \).

The *reasonable crossnorm* means that
\[
\inf_{u} \sum_{i} \| x_{i} \otimes y_{i} \| = \| u \| = \inf_{u} \sum_{i} \| x_{i} \otimes y_{i} \|.
\]

Note that a bounded linear operator \( T \in Hom_A(X,Y^*) \) in (3.4) corresponding a continuous linear functional \( \psi \) on \( X \otimes_A Y \) is given by
\[
(Tx)(y) = \psi(x \otimes y) \quad \text{for all } x \in X, y \in Y.
\]

Here \( Hom_A(X,Y^*) = M_A(X,Y^*) \) is the space of all \( A \)-module homomorphisms from \( X \) to \( Y^* \), the topological dual of \( Y \), that is, each \( T \in Hom_A(X,Y^*) \) satisfies
\[
T(ax) = a(Tx) \quad \text{for all } a \in A, x \in X, Tx \in Y^*.
\]

where \( T \) is a bounded linear operator from \( X \) to \( Y^* \); \( X \otimes_A Y \) denotes the \( A \)-module tensor product space of \( X \) and \( Y \).

There are some known results in scalar-valued function space of \( L^1(G) \)-module by convolution. We state three typical \( L^1(G) \)-module multiplier problems as follows.

**Theorem 3.** (i) \( Hom_G(L^1(G),L^1(G)) \cong M_b(G) \), (by Theorem 1, (iii) \( \iff (i) \))

where \( Hom_G = Hom_{L^1(G)} \), and \( M_b(G) \) is the space of all bounded regular Borel measures on \( G \).

(ii) \( Hom_G(L^1(G),L^p(G)) \cong (L^1(G) \otimes_G L^q(G))^* \cong (L^q(G))^* = L^p(G) \),

for \( 1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1 \) where \( \otimes_G = \otimes_{L^1(G)} \).

(iii) \( Hom_G(L^p(G),L^p(G)) \cong (L^p(G) \otimes_G L^q(G))^* \cong S_p(G)^* \),

where \( S_p(G) \) is a Banach algebra generated by
\[
\{ u = \sum_{i}^\infty f_i g_i : f_i \in L^p(G), g_i \in L^q(G), \sum_{i} \| f_i \|_p \| g_i \|_q < \infty \}
\]

under pointwise product and the norm is defined by (cf. Larsen [7])
\[
\| u \| = \inf \sum_{i} \| f_i \|_p \| g_i \|_q ; u = \sum_{i} f_i \cdot g_i \in S_p(G) \}.
\]
4 Multipliers of Banach-valued Functions on $G$.

Let $A$ be a commutative semi-simple Banach algebra with bounded approximate identity. Assume $X$ is on $A$-module Banach space. It is not hard to prove that $L^{1}(G,A) = L^{1}(G) \otimes_{\gamma} A$. Since both $L^{1}(G)$ and $A$ have bounded approximate identity, thus $L^{1}(G,A)$ is a commutative Banach algebra with bounded approximate identity.

By Theorem 2

$$L^{1}(G,A) \ast L^{p}(G,X) = L^{p}(G,X), \quad 1 < p < \infty$$

Denote by

$$L^{1}(G,A) = \{ f : G \rightarrow A \mid f \text{ is measurable and is Bochner integrable on } G \}$$

Then $L^{1}(G,A)$ is a commutative Banach algebra, under convolution. Actually

$$|f \ast g(t)|_{A} \leq \int_{G} |f(s-t)|_{A} |g(s)|_{A} \, ds = \| g \|_{1} \int_{G} |f(s-t)|_{A} \, ds = \| g \|_{1} \| f \|_{1},$$

$$\| f \ast g \|_{1} = \int_{G} |f \ast g(t)|_{A} \, dt \leq \| g \|_{1} \int_{G} |f(s-t)|_{A} \, dt \leq \| g \|_{1} \| f \|_{1}.$$

Denote by

$$L^{p}_{X} = \{ f : G \rightarrow X \mid f \text{ is measurable and } |f(\cdot)|_{X} \in L^{p}(G) \}, 1 \leq p < \infty,$$

$$\| f \|_{p} = \left( \int_{G} |f(t)|_{X}^{p} \, dt \right)^{\frac{1}{p}}, \text{ for } f \in L^{p}_{X}, 1 \leq p < \infty \quad (2.1)$$

and for $p = \infty$, $\| f \|_{\infty} = \esssup_{t \in G} |f(t)|_{X}$ for $f \in L^{\infty}_{X}$ \quad (2.2)

Show that $L^{p}_{X}, 1 \leq p \leq \infty$ are Banach spaces with the norm $\| f \|_{p}, 1 \leq p < \infty$, as (2.1) and if $p = \infty$, the norm is taken $\| \cdot \|_{\infty}$ as (2.2). If $X = \mathbb{C}$, the complex numbers, then

$$L^{p}_{X} = L^{p} = L^{p}(G), 1 \leq p \leq \infty.$$
Recall [8], Rieffel characterized the homomorphism module multiplier is represented by the dual space of module tensor product as the following form:

\[ \text{Hom}_A(X,Y^*) \cong (X \otimes_A Y)^* \text{ or } \text{Hom}_A(X,Y) \cong (X \otimes_A Y^*)^*. \] (if Y is reflexive)

where \( \otimes_A \) is namely module tensor product of X into Y* or of X into Y and \( Z^* \) denotes the dual space of the Banach space \( Z \). The space \( \otimes_A \) is the complete projective tensor product \( X \hat{\otimes}_A Y^* \) quotients by \( K \), that is, \( X \otimes_A Y = X \hat{\otimes}_A Y / K \).

Here \( K \) is the closed linear subspace of the projective tensor product space \( X \hat{\otimes}_A Y \) generated by the elements \( ax \otimes y - x \otimes ay \); for \( a \in A, x \in X, y \in Y \) and \( X \hat{\otimes}_A Y \) is the completion of the algebra tensor \( x \otimes y \) under the \( \gamma \)-norm, and

\[ X \otimes Y = \{ u = \sum_i x_i \otimes y_i \mid x_i \in X, y_i \in Y, \sum_i \| x_i \| \| y_i \| < \infty \} \]

\[ \gamma(u) = \inf \{ \sum_i \| x_i \| \| y_i \| \mid u = \sum_i x_i \otimes y_i \in Y \} = \| u \| = \inf \sum_i \| x_i \otimes y_i \| = \inf \sum_i \| x_i \| \| y_i \| \]

where \( \inf_u \) means that the infimum is taken by all representations of \( u = \sum_i x_i \otimes y_i \) in \( X \otimes Y \), and the tensor norm. We state the following Theorem for the characterization of the invariant operators. For detail, we consult Lai [3,4] and [6] cf. also Lai [5].

**Theorem 4.** Let \( X \) and \( Y \) be Banach spaces. Then the following two statements are equivalent.

(a) \( T \in (L^1(G,Y),L^1(G,X)) \) is an invariant operator.

(b) There exists a unique continuous linear map \( L \in \mathcal{L}(Y,M_b(G,X)) \) such that

\[ T(f \otimes y) = f \ast L_y \] for all \( f \in L^1(G), y \in Y. \)

Moreover, \( (L^1(G,Y),L^1(G,X)) \cong \mathcal{L}(Y,M_b(G,X)). \)

**Theorem 5.** Let \( A \) be a commutative semi-simple Banach algebra (not necessarily with identity) and \( X \) a Banach \( A \)-module. Then

(5.1) \( \text{Hom}_{L^1(G,A)}(L^1(G,A),L^1(G,X)) \cong \text{Hom}_A(A,M_b(G,X)). \)
In Lai [6], he showed that an invariant operator is also a multiplier if and only if the $A$ in $L^1(G,A)$ must be scalar space $\mathbb{C}$.

**Theorem 6.** Let $A$ be a commutative Banach algebra with identity of norm 1. $X$ be a unit linked, order-free, Banach-module and $A$ a faithful representation on $X$, then each invariant operator $T : L^1(G,A) \to F(G,X)$ is a multiplier if and only if $A \cong \mathbb{C}$. Here $F(G,X) = L^p(G,X)$ for each $p$, $1 \leq p \leq \infty$, or $F(G,X) = C_0(G,X)$.

**References**


