On SCT automorphism groups
of divisible designs

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In this talk we consider automorphism groups SCTs of divisible designs acting regularly on the set of point classes and determine the relations among SCTs, RDSs and $\lambda$-planar functions.

§1 Divisible Designs and class regularity

A divisible design $(m, u, k, \lambda)$-DD is an incidence structure $(\mathbb{P}, \mathbb{B})$, where

(i) $\mathbb{P}$ is a set of $mu$ points partitioned into $m$ classes $\mathcal{C}$ (called point classes), each of size $u$,

(ii) $\mathbb{B}$ is a collection of $k$-subsets of $\mathbb{P}$ (called blocks),

(iii) Any two distinct points in the same point class are incident with no blocks and any two points in distinct point classes are incident with exactly $\lambda$ blocks.

We can show the following : $|\mathbb{P}| = mu, \quad |\mathbb{B}| = u^2m(m-1)\lambda/k(k-1)$

An $(m, u, k, \lambda)$-DD with $k = m$ is called a transversal design and denoted by $TD_\lambda(k, u)$. A $TD_\lambda(k, u)$ is called a symmetric transversal design and denoted by $STD_\lambda(k, u)$ with $k = u\lambda$ if its dual is also a $TD_\lambda(k, u)$. We note that an $(m, 1, k, \lambda)$-DD is just a $2-(m, k, \lambda)$ design.

Partial difference matrices

Definition. (Jungnickel [2]) Let $U$ be a group of order $u$. An $m \times t$ matrix $D = [d_{ij}]$ with entries from $U \cup \{0\}$ is called an $(m, u, k, \lambda)$-partial difference matrix (PDM) over $U$ if the following conditions are satisfied:

(i) Each column of $D$ has exactly $k$ nonzero entries.

(ii) $\sum_{1 \leq j \leq t} d_{ij}d_{\ell j}^{-1} = \lambda U, \forall i \neq \ell$, where $0^{-1} = 0$, $0 \cdot g = g \cdot 0 = 0 \forall g \in U$ and $t = |\mathbb{B}|/|G| = m(m-1)u\lambda/k(k-1))$. 
An \((m, u, k, \lambda)\)-PDM with \(m = k\) over a group \(U\) of order \(u\) is called a \((u, k, \lambda)\)-difference matrix (DM). Moreover, a \((u, u, \lambda)\)-DM, denoted by \(GH(u, \lambda)\), is called a generalized Hadamard matrix.

**Example.** Set \(U = \langle a \rangle \simeq \mathbb{Z}_3\).

\[
\begin{array}{cccc}
0 & 1 & 1 & 1 \\
1 & a & 0 & a^2 \\
a & 1 & 0 & a^2 \\
1 & 0 & a^2 & 1 \\
\end{array}
\]

\((5, 3, 4, 1)\)-PDM

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & a & a & a^2 \\
a & a^2 & a & 1 \\
1 & 1 & a^2 & a \\
\end{array}
\]

\((3, 3, 2)\)-DM

\(
GH(3, 1)
\)

**Class regularity**

Following results are known.

**Result.** (D. Jungnickel [3]) The existence of an \((m, u, k, \lambda)\)-DD admitting a class regular automorphism group \(U\)

\[\iff\] The existence of a \((m, u, k, \lambda)\)-partial difference matrix over \(U\)

**Result.** (D.A. Drake [2]) Assume \(U\) is a group of even order \(u\) and \(2 \nmid \lambda\). If a Sylow 2-subgroup of \(U\) is cyclic then there exists no \((u, k, \lambda)\)-DM over \(U\) for \(k \geq 3\).

We now consider the regular action of a subgroup \(G\) of \(\text{Aut}(\mathcal{D})\) on the set of point classes \(\mathcal{C} = \{C_i | i \in I_m\}\), where \(I_m = \{1, 2, \cdots, m\}\).

§ 2 SCT groups and SCT matrices

Let \((\mathbb{P}, \mathbb{B})\) be a \((m, u, k, \lambda)\)-DD and \(G \leq \text{Aut}(\mathbb{P}, \mathbb{B})\). We say \(G\) is an SCT\((m, u, k, \lambda)\) group if \(G\) is semiregular on \(\mathbb{P} \cup \mathbb{B}\) and regular on the set of point classes \(\mathcal{C} = \{C_1, \cdots, C_m\}\). (Note that \(|G| = m\).)

Assume that \(G\) is an SCT\((m, u, k, \lambda)\) group of a \((m, u, k, \lambda)\)-DD \(\mathcal{D}(= (\mathbb{P}, \mathbb{B}))\).

Choose a point class \(C = \{p_1, \cdots, p_u\} \in \mathcal{C}\). Then \(\mathbb{P} = \bigcup_{i \in I_u} p_i^G\) and \(\mathbb{B} = \bigcup_{j \in I_s} B_j^G\), where \(s = |\mathbb{B}|/|G|\).

A \(u \times s\) matrix \(M_D = [D_{ij}] (D_{ij} \subset G)\) over \(G\) is defined by

\[
D_{ij} = \{g \in G | p_i^g \in B_j\} \quad (i \in I_u, \ j \in I_s)
\]

**Theorem 1.** The following holds.

\[
\sum_{j \in I_s} D_{ij} D_{ij}^{(-1)} = \begin{cases} 
\rho + \lambda(G-1) & \text{if } i = \ell, \\
\lambda(G-1) & \text{otherwise}, 
\end{cases}
\]

where \(\rho = (m-1)u\lambda/(k-1)\).

\[
\sum_{i \in I_u} |D_{ij}| = k \quad \forall j \in I_s
\]

**Definition.** Let \(G\) be a group of order \(m\). Let \(u, s \in \mathbb{N}\). For subsets \(D_{ij} \subset G\) \((i \in I_u, j \in I_s)\) we call a \(u \times s\) matrix \[
\begin{bmatrix}
D_{11} & \cdots & D_{1s} \\
\vdots & \ddots & \vdots \\
D_{u1} & \cdots & D_{us}
\end{bmatrix}
\] an SCT\((m, u, k, \lambda)\)-
matrix over $G$ if it satisfies the following for some $\rho \in \mathbb{N}$.

$$
\sum_{j \in I_s} D_{ij} D_{\ell j}^{(-1)} = \begin{cases} 
\rho + \lambda(G-1) & \text{if } i = \ell, \\
\lambda(G-1) & \text{otherwise},
\end{cases}
$$

$$
\sum_{i \in I_u} |D_{ij}| = k \quad \forall j \in I_s
$$

**Remark.**

(i) $s = (m-1)u^2\lambda/k(k-1)$, $\rho = (m-1)u\lambda/(k-1)$

(ii) An SCT$(m, 1, k, \lambda)$-matrix is just an $(m, k, \lambda)$-difference family.

* An incidence structure $\mathcal{D}(\mathbb{P}, \mathbb{B})$ defined by the following is an $(m, u, k, \lambda)$-DD admitting $G$ as an SCT group under the action $(i, w)g = (i, wg)$ for $i \in \{1, \cdots, u\}$ and $w, g \in G$.

$\mathbb{P} = \{1, 2, \cdots, u\} \times G$

$\mathbb{B} = \{B_{j,g} \mid j \in I_s, g \in G\}$, where $B_{j,g} = \bigcup_{i \in I_u} (i, D_{ij}g)$

* $(m, u, k, \lambda)$-DD with SCT-group $\iff$ SCT$(m, u, k, \lambda)$-matrix

**Example.**

(i) The following is an SCT$(9, 2, 9, 9)$ matrix over $G := (a, b) \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$:

$$
\begin{bmatrix}
\langle a \rangle & \langle b \rangle & G - \langle ab \rangle & G - \langle ab^2 \rangle \\
G - \langle a \rangle & G - \langle b \rangle & \langle ab \rangle & \langle ab^2 \rangle
\end{bmatrix}
$$

This matrix gives a TD$_9(9, 2)$, which is not obtained from any difference matrix by Drake's result.

(ii) The following is an SCT$(12, 5, 11, 2)$ matrix over $\text{Alt}(4) = N \rtimes H$, $N = \{1, a, b, c\} \simeq E_4$, $H = \{1, d, d^2\} \simeq \mathbb{Z}_3$:

$$
\begin{bmatrix}
0 & \alpha & \beta & \gamma & \delta \\
\alpha & \beta & \gamma & \delta & 0 \\
\beta & \gamma & \delta & 0 & \alpha \\
\gamma & \delta & 0 & \alpha & \beta \\
\delta & 0 & \alpha & \beta & \gamma
\end{bmatrix}
$$

where $\begin{cases} 
\alpha = ad + cd^2 \\
\beta = d + bd^2 + d^2 + cd \\
\gamma = b + c \\
\delta = ad^2 + bd + a
\end{cases}$

From this we obtain a $(12, 5, 11, 2)$-DD with the full automorphism group isomorphic to $\text{Alt}(5) \simeq \text{Alt}(4) \rtimes \mathbb{Z}_3$. This DD is not class regular, hence not obtained from any partial difference matrix.

**Relations among SCT aut., Class regular aut. and RDS**

$\exists$ SCT aut. $\iff \exists$ partial DM $\supset$ DM $\supset$ GH mat.

$\exists$ SCT aut. $\& \exists$ class regular aut. $\iff \exists$ splitting relative difference set
Difference families and SCT matrices

A family of \(k\)-subsets \(\{D_{1}, \cdots, D_{n}\}\) of a group \(G\) of order \(v\) is called an \(n-(v, k, \lambda)\) difference family if

\[D_{1}D_{1}^{-1} + \cdots + D_{n}D_{n}^{-1} = kn + \lambda(G-1).\]

From an \(n-(v, k, \lambda)\) difference family in a group \(G\) we obtain a \(2-(v, k, \lambda)\) design \((\mathbb{P}, \mathbb{B})\):

\[
\mathbb{P} = G, \quad \mathbb{B} = \{D_{i}x \mid i \in I_{n}, x \in G\}.
\]

In the following we give a relation between difference families and SCT matrices with \(u=2\).

**Theorem 2.** Let \(\{D_{1}, \cdots, D_{4d}\}\) be a \(4d-(m, k, d(4k-m))\) difference family in a group \(G\) of order \(m\). Set \(C_{i} = G - D_{i}\) for \(i \in I_{4d}\). Then the following is an \(SCT(m, 2, m, dm)\) matrix corresponding to a \(TD_{dm}(m, 2)\).

\[
M = \begin{bmatrix} D_{1} & \cdots & D_{2d} & C_{2d+1} & \cdots & C_{4d} \\ C_{1} & \cdots & C_{2d} & D_{2d+1} & \cdots & D_{4d} \end{bmatrix}
\]

\[
C_{i}C_{i}^{-1} = D_{i}D_{i}^{-1} + (m-2k)G, \quad D_{i}C_{i}^{-1} = C_{i}D_{i}^{-1} = kG - D_{i}D_{i}^{-1}.
\]

Some theorems on difference families

The following results on difference families are known.

**Result.** (Leung-Ma-Schmidt [4]) Let \(q\) be a prime power and \(d > 0\) an integer. Suppose, either (i) \(q \equiv 2d-1\) (mod \(4d\)) and \(2 \nmid d\) or (ii) \(q \equiv 4d-1\) (mod \(8d\)). Then there exists a \(4d-(q^2, (q^2-q)/2, dq^2-2dq)\) difference family in \((GF(q^2), +)\).

**Result.** (Q. Xiang [6]) Let \(q\) be a power of a prime and \(b, c\) positive integers such that \(q+1 = 2^cb\) and \(c \geq 2\) with \(2 \nmid b\). Then there exists a \(2^c-(q^2, (q^2-q)/2, 2^{c-2}(q^2-2q))\) difference family in \((GF(q^2), +)\).

**Remark.** Set \(d = 2^{c-2}\) in the above result. Then \(2^c-(q^2, (q^2-q)/2, 2^{c-2}(q^2-2q))\) is identical with \(4d-(q^2, (q^2-q)/2, dq^2-2dq)\).

We now apply Theorem 2 to the above results for \(m = q^2, k = (q^2-q)/2\).

**TD_{dq^2}(q^2, 2)\)s admitting SCT groups

**Proposition.** Let \(q\) be a power of a prime and \(d\) a positive integer satisfying one of the following:

(i) \(q \equiv 2d-1\) (mod \(4d\)).

(ii) \(q \equiv 4d-1\) (mod \(8d\)).

(iii) \(4d \mid q+1, 8d \nmid q+1\) with \(d\) a power of \(2\).

Then, there exists an \(SCT(q^2, 2, q^2, dq^2)\) matrix over \((GF(q^2), +)\) and the resulting \(TD_{dq^2}(q^2, 2)\) admits an SCT automorphism group of order \(q^2\).

**Remark.** If \(2 \nmid dq\), then no \(TD_{dq^2}(q^2, 2)\)s are obtained from difference matrices by Drake’s result.
§3 Direct product RDSs and SCTs

Let \( G \) be a group of order \( um \) and \( U \) its (not necessarily normal) subgroup of order \( u \). A \( k \)-subset \( D \) of \( G \) is called an \((m, u, k, \lambda)\)-relative difference set (or, RDS for short) relative to \( U \) if \( DD^{(-1)} = k + \lambda(G - U) \). Usually \( U \) is called the forbidden subgroup.

An \((m, u, k, \lambda)\)-divisible design \( D = (\mathcal{P}, \mathcal{B}) \) is obtained from \((m, u, k, \lambda)\)-RDS in the following way: the set \( \mathcal{P} \) of points are elements of \( G \) and the set of blocks \( \mathcal{B} \) are subsets \( Dx(x \in G) \). We note that the set of point classes are \( \{ Ug \mid g \in G \} \).

We say \( G \) is splitting (over \( U \)) if there exists a subgroup \( G \) of \( G \) of order \( u \) such that \( G = GU \) and \( G \cap U = 1 \). In this case \( G \) is an \( SCT(m, u, k, \lambda) \) group of \( D \).

From now on we consider an SCT matrix obtained from a splitting abelian RDS; \( G = G \times U \).

**Hypothesis 3.** Let \( G = \{g_1, \ldots, g_m\} \) and \( U = \{w_1, \ldots, w_u\} \) be abelian groups of order \( m \) and \( u \), respectively. Suppose \( D \) is an \((m, u, k, \lambda)\)-RDS in the group \( G = G \times U \) relative to \( U \). Set \( \mathcal{P} = G = \{w_ig_j \mid i \in I_u, j \in I_m\} \) and \( \mathcal{B} = \{Dw_iw_j \mid i \in I_u, j \in I_m\} \). Then \( \mathcal{D}_{D,G} := (\mathcal{P}, \mathcal{B}) \) is a \((m, u, k, \lambda)\)-DD with the set \( \mathcal{G} = \{Ug_1, \ldots, Ug_m\} \) of point classes.

We now consider the action of \( G \) on \((\mathcal{P}, \mathcal{B})\) as an SCT group.

\[
\{w_iG \mid i \in I_u\} : \text{the set of } G\text{-orbits on } \mathcal{P}, \\
\{Dw_iG \mid i \in I_u\} : \text{the set of } G\text{-orbits on } \mathcal{B}, \\
D = G_{w_1}w_1 \cup \cdots \cup G_{w_u}w_u \quad (\exists G_{w_1}, \ldots, \exists G_{w_u} \subset G).
\]

We choose a point class \( C = \{w_1, \ldots, w_u\} \in \mathcal{C} \) as a set of representatives of \( G\)-orbits on \( \mathcal{P} \) and \( \{Dw_1, \ldots, Dw_u\} \subset \mathcal{B} \) as a set of representatives of \( G\)-orbits on \( \mathcal{B} \).

**Direct product RDSs and SCTs**

Under Hypothesis 3, the corresponding \( u \times u \) SCT matrix \([D_{ij}]\) is given by

\[
D_{ij} = \{g \in G \mid (w_i)g \in Dw_j\} = G \cap Dw_i^{-1}w_j.
\]

As \( D = G_{w_1}w_1 \cup \cdots \cup G_{w_u}w_u \quad (G_{w_1}, \ldots, G_{w_u} \subset G) \), we have \([D_{ij}] = [G_{w_iw_j}^{-1}]\), which we call an SCT matrix of standard form with respect to \( D, G \times U \).

Similarly, if we choose a point class \( C = \{w_1g, \ldots, w_ug\} \in \mathcal{C} \) \( (g \in G) \) and \( \{Dw_1g_{n_1}, \ldots, Dw_ug_{n_u}\} \subset \mathcal{B} \) \( (n_1, \ldots, n_u \in I_m) \) as sets of representatives of \( G\)-orbits on \( \mathcal{P} \) and \( \mathcal{B} \), respectively, then we have the following.

**Lemma 4.** Under Hypothesis 3, set \( D = G_{w_1}w_1 \cup \cdots \cup G_{w_u}w_u \), where \( G_{w_1}, \ldots, G_{w_u} \subset G \). Then a \( u \times u \) matrix \([G_{w_iw_j}^{-1}g_{n_j}]\) is an SCT\((m, u, k, \lambda)\) matrix.
Let notations be as in Lemma 4. Then we have the following.

**Proposition 5.** Set $M = [G_{w_{i}w_{j}^{-1}}]$, the SCT matrix of standard form with respect to $\{D, G \times U\}$. Then,

(i) any SCT matrix is obtained from $M$ by multiplication of any column by an element of $G$ and any permutation of rows and columns;

(ii) $M$ is circulant if $u$ is a prime and $w_i = w^{i-1}$ for $i \in I_u$, where $U = \langle w \rangle$.

### §4 Spreads and SCTs

**Theorem 6.** Let $q = p^e$ be a power of a prime $p$ and let $G$ be an elementary abelian $p$-group of order $q^2$. Let $\{H_1, \ldots, H_{q+1}\}$ be a spread of $G$ (i.e. $|H_i| = q, |H_i \cap H_j| = 1, \forall i \neq j$). Set $q_0 = q/p^m (= p^{e-m})$

$$A_i = H_{q_0-i}^* + H_{q_0-i+1}^* + \cdots + H_{q_0}^*$$

(iii) $A_i = H_{(p^m-1)q_0-i}^* + H_{(p^m-1)q_0-i+1}^* + \cdots + H_{p^mq_0-i}^* + H_{p^mq_0-i+1}^*$

Let $L = [n_{ij}]$ be a Latin square of order $p^m$ with entries from $\{0, 1, \ldots, p^m - 1\}$. Then the following is an $SCT(p^2, p^m, p^2, p^2-m)$ matrix, which gives an $STD_{q^2/p^m}(p^{2e}, p^{m})$.

$$[A_{n_{p^m,1}}A_{n_{2,1}}A_{n_{1,1}} A_{n_{2,2}}A_{n_{1,2}}A_{n_{p^m,2}}A_{n_{p^m,1}}^* \cdots A_{n_{p^m,p^m-1}}A_{n_{2,p^m}}A_{n_{1,p^m}}]$$

**Sketch of proof:**

1. $\sum_{i \in I_{p^m}} A_i A_i^{(-1)} = q^2 + qq_0(G - 1)$ (\forall $i \in I_{p^m}$).

2. If $\{n_1, \ldots, n_{ipm}\} = \{n_{\ell 1}, \ldots, n_{\ell p^m}\} = I_{p^m}$ and $n_{i1} \neq n_{\ell 1}, \ldots, n_{ipm} \neq n_{\ell p^m}$, then

$$A_{i1} A_{\ell 1}^{(-1)} + \cdots + A_{ipm} A_{\ell p^m}^{(-1)} = q_0 q(G - 1)$$

### An equivalence class in Latin squares of order $n$

We show that some of the STDs obtained in Theorem 6 admit no class regular automorphism groups. This implies that these STDs are never obtained from generalized Hadamard matrices. In order to prove this we need a lemma on the set of Latin squares.

**Definition.** Let $e_1 = (1, 0, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \cdots$ be vectors of $V(n, \mathbb{C})$. For a permutation $\sigma = \left( \begin{array}{cccc} r_1 & r_2 & \cdots & r_n \end{array} \right)$ of $\Omega := \{1, 2, \ldots, n\}$, a permutation matrix $P_\sigma$ is defined by $e_i P_\sigma = e_{r_i}$ for each $i \in I_n$. Let $N$ be the group of permutation matrices of order $n$ and $\mathcal{L}$ the set of Latin squares on $\Omega$. We say Latin squares $L_1$ and $L_2$ in $\mathcal{L}$ are equivalent if $L_2 = PL_1Q$ for some $P, Q \in N$. Let $H := N \times N$ be the direct product and define the action of $H$ on $\mathcal{L}$ by $L(P, Q) = P^T LQ$ for $L \in \mathcal{L}$. Then $H$ is a permutation group on $\mathcal{L}$. 
The number of Latin squares of order $n$

Let $\mathcal{L}_n$ be the set of Latin squares of order $n$ on $\{1, \cdots, n\}$.

By Theorem III.1.19 of [1],

$$|\mathcal{L}_n| > f(n) := (n!)^{2n}/n^{n^2} \text{ for } n > 1.$$ 

$|\mathcal{L}_2| = (2-1)!2! > \lceil f(2) \rceil = 1,$

$|\mathcal{L}_3| = (3-1)!3! > \lceil f(3) \rceil = 2,$

$|\mathcal{L}_4| = 4(4-1)!4! > \lceil f(4) \rceil = 25,$

$|\mathcal{L}_5| = 56(5-1)!5! = 161280 > \lceil f(5) \rceil = 2077$.

Latin squares equivalent to a circulant one

$\mathcal{L}$ = the set of Latin squares on $\Omega := \{1, 2, \cdots, n\}$

$N$ = the group of permutation matrices of order $n$

$N \times N$ = the permutation group on $\mathcal{L}$ defined by $L(P, Q) = P^T LQ$

Lemma. Let $C$ be a circulant matrix of order $n$ whose first row is $(a_1, a_2, \cdots, a_n)$ with $\{a_1, a_2, \cdots, a_n\} = \Omega$. Let $T \in N$ be a circulant permutation matrix whose first row is $(0, 1, 0, \cdots, 0)$. If $Q, R \in N$ and $QC = CR$ then $Q = R$ and $Q \in \langle T \rangle$.

Lemma 7. Assume $C \in \mathcal{L}$ and $C$ is circulant. Then,

(i) The number of Latin squares in $\mathcal{L}$ equivalent to $C$ is $(n!)^2/n$;

(ii) If $n \geq 4$, then there exists a Latin square of $\mathcal{L}$ not equivalent to circulant one.

By Theorem III.1.19 of [1], $|\mathcal{L}_n| > (n!)^{2n}/n^{n^2}$.

As $(n!)^{2n}/n^{n^2} > (n-1)!(n!)^2/n$, $(n \geq 4)$, the lemma holds.

Non class regular STDs

Theorem. Let $p > 3$ be a prime and $A_L$ the $SCT(p^{2e-1}, p^{2e}, p, p^{2e})$ matrix defined in Theorem 6. Then the $STD_{p^{2e-1}}(p^{2e}, p)$ obtained from $A_L$ is not class regular.

Proof. By Lemma 7, there exists a Latin square $L$ not equivalent to a circulant one. Let $\langle (\mathbb{P}, \mathbb{B}) \rangle$ be the $STD_{p^{2e-1}}(p^{2e}, p)$ obtained from $A_L$ and let $G$ be the $SCT(p^{2e-1}, p^{2e}, p, p^{2e})$ automorphism group of order $p^{2e}$. Suppose false and let $U$ be a class regular automorphism group of $\langle (\mathbb{P}, \mathbb{B}) \rangle$. Then, as $G$ normalizes $U$ and $|U| = p$, $G$ centralizes $U$. The direct product $G := G \times U$ contains a $(p^{2e}, p^{2e}, p^{2e-1})$-RDS corresponding to $\langle (\mathbb{P}, \mathbb{B}) \rangle$. By Proposition 5, $L$ must be equivalent to a circulant Latin square, a contradiction.
§5 RDS and $\lambda$-planar functions

In this section we define a $\lambda$-planar function as a generalization of planar functions.

Theorem. Let $\mathcal{G} = GU$ be a group of order $mu$ and $G, U$ its subgroups with $|G| = m, |U| = u$ and $G \triangleright U$. Let $D$ be a $(m, u, k, \lambda)$-RDS in $\mathcal{G}$ relative to $U$. Then there exists a $k$-subset $C$ of $G$ and a function $f : C \rightarrow U$ satisfying the following.

(i) $D = \{xf(x) \mid x \in C\}$

(ii) $\#\{x \in C \mid ax \in C, f(ax)^{\varphi(a)}f(x)^{-1} = b\} = \lambda$

for any $a \in G \setminus \{1\}$ and $b \in U$.

Proposition. Let $G, U$ be groups of order $m, u$, respectively. Let $\varphi$ be a homomorphism from $G$ to $\text{Aut}(U)$ and $f$ a function form $C$ to $U$ for a $k$-subset $C$ of $G$. Assume that for any $a \in G \setminus \{1\}$ and $b \in U$

$(\star) \#\{x \in C \mid ax \in C, f(ax)^{\varphi(a)}f(x)^{-1} = b\} = \lambda.$

Then $D = \{xf(x) \mid x \in C\}$ is a $(m, u, k, \lambda)$-RDS in a semi-direct product $\mathcal{G} = GU$ of $G$ by $U$ with respect to $\varphi$.

Definition. Let $G$ and $U$ be groups. Let $C$ be a subset of $G$ and $\varphi \in \text{Hom}(G, \text{Aut}(U))$. We call a function $f : C \rightarrow U$ a $\lambda$-planar function relative to $(C, U, \varphi)$ if $f$ satisfies $(\star)$. If $\varphi$ is a trivial homomorphism, we say $f$ is a $\lambda$-planar function relative to $(C, U)$. We note that a 1-planar function relative to $(G, U)$ is just a planar function in the usual sense (see Pott [5]).

Example. Let $q = p^e$ be a power of a prime $p$ and set $G = F = (GF(q^2), +) \supset U = K = (GF(q), +)$. Then a function $f(x) = x^{q+1}$ from $G$ to $U$ is a $q$-planar function relative to $(G, U)$.

Let $0 \neq a \in G$ and $b \in U$. Then,

$f(a + x) - f(x) = b \iff (a^q + x^q)(a + x) - x^{q+1} = b$

$\iff ax^q + a^q x = b - a^{q+1} \quad (**).$

As $ax^q + a^q x = ax^q + (ax^q)^q = \text{Tr}_{F/K}(a x^q)$, $(**)$ has exactly $q$ solutions in $G$. Thus $f$ is a $q$-planar function relative to $(G, U)$.
\textbf{\textit{\(\lambda\)-planar functions, SCTs, and RDSs}}

\textbf{Theorem 8.} Let \(G\) be a group of order \(m\) and \(U\) a group of order \(u\). Let \(D_y\) be subsets of \(G\) for each \(y \in U\). If a \(u \times u\) matrix \(D = [D_{yz^{-1}}]_{y,z \in U}\) over \(\mathbb{Z}[G]\) is an \(\text{SCT}(m, u, k, \lambda)\) matrix, then the following holds.

(i) Set \(C = \bigcup_{y \in U} D_y (\subset G)\). Then \(|C| = k, G = \langle C \rangle\) and a function \(f : C \rightarrow U\) defined by \(f(D_y) = y\ (y \in U)\) is a \(\lambda\)-planar function relative to \((C, U)\).

(ii) Set \(D = \{(x, f(x) \mid x \in C\}\). Then \(D\) is an \((m, u, k, \lambda)\)-RDS in \(G \times U\) relative to \(1 \times U\).

\textbf{Remark.} A \((u\lambda, u, u\lambda, \lambda)\)-RDS is called semiregular. It is conjectured that any forbidden subgroup of a semiregular RDS is a \(p\)-group for a prime \(p\). Concerning this we can show the following as an application of Theorems 6 and 8.

\textbf{Theorem.} Any \(p\)-group can be a forbidden subgroup of a semiregular RDS.

As a corollary we have the following, which gives another proof of de Launey’s result on generalized Hadamard matrices (cf. [1], Theorem 5.9).

\textbf{Corollary} There exists a \(\text{GH}(p^m, p^{2e-m})\) matrix over any group of order \(p^m\) whenever \(e \geq m\).

\textbf{References}


