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Kyoto University
On SCT automorphism groups
of divisible designs

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In this talk we consider automorphism groups SCTs of divisible designs acting regularly on the set of point classes and determine the relations among SCTs, RDSs and $\lambda$-planar functions.

§1 Divisible Designs and class regularity

A divisible design $(m, u, k, \lambda)$-DD is an incidence structure $(\mathbb{P}, \mathbb{B})$, where

(i) $\mathbb{P}$ is a set of $mu$ points partitioned into $m$ classes $\mathcal{C}$ (called point classes), each of size $u$,

(ii) $\mathbb{B}$ is a collection of $k$-subsets of $\mathbb{P}$ (called blocks),

(iii) Any two distinct points in the same point class are incident with no blocks and any two points in distinct point classes are incident with exactly $\lambda$ blocks.

We can show the following: $|\mathbb{P}| = mu$, $|\mathbb{B}| = u^2m(m-1)\lambda/k(k-1)$

An $(m, u, k, \lambda)$-DD with $k = m$ is called a transversal design and denoted by $\text{TD}_\lambda(k, u)$. A $\text{TD}_\lambda(k, u)$ is called a symmetric transversal design and denoted by $\text{STD}_\lambda(k, u)$ with $k = u\lambda$ if its dual is also a $\text{TD}_\lambda(k, u)$. We note that an $(m, 1, k, \lambda)$-DD is just a $2-(m, k, \lambda)$ design.

Partial difference matrices

Definition. (Jungnickel [2]) Let $U$ be a group of order $u$. An $m \times t$ matrix $D = [d_{ij}]$ with entries from $U \cup \{0\}$ is called an $(m, u, k, \lambda)$-partial difference matrix (PDM) over $U$ if the following conditions are satisfied:

(i) Each column of $D$ has exactly $k$ nonzero entries.

(ii) $\sum_{1 \leq j \leq t} d_{ij}d_{\ell j}^{-1} = \lambda U$, $\forall i \neq \ell$, where $0^{-1} = 0$, $0 \cdot g = g \cdot 0 = 0 \forall g \in U$ and $t = |\mathbb{B}|/|G| = m(m-1)u\lambda/k(k-1)$.
An \((m, u, k, \lambda)\)-PDM with \(m = k\) over a group \(U\) of order \(u\) is called a \((u, k, \lambda)\)-difference matrix (DM). Moreover, a \((u, u, \lambda)\)-DM, denoted by \(GH(u, \lambda)\), is called a generalized Hadamard matrix.

**Example.** Set \(U = \langle a \rangle \simeq \mathbb{Z}_3\).

\[
\begin{array}{llllll}
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & a & 0 & a^2 & 0 \\
a & 1 & 0 & a^2 & a & a \\
1 & 0 & a^2 & 1 & a & a \\
\end{array}
\]

\((5, 3, 4, 1)\)-PDM

\[
\begin{array}{llllll}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & a & a & a^2 & a^2 \\
1 & 1 & a^2 & a^2 & a & a \\
1 & a^2 & a & a & a^2 & a \\
\end{array}
\]

\((3, 3, 2)\)-DM

\(GH(3, 1)\)

**Class regularity**

Following results are known.

**Result.** (D. Jungnickel [3]) The existence of an \((m, u, k, \lambda)\)-DD admitting a class regular automorphism group \(U\)

\[\iff\] The existence of a \((m, u, k, \lambda)\)-partial difference matrix over \(U\)

**Result.** (D.A. Drake [2]) Assume \(U\) is a group of even order \(u\) and \(2 \nmid \lambda\). If a Sylow 2-subgroup of \(U\) is cyclic then there exists no \((u, k, \lambda)\)-DM over \(U\) for \(k \geq 3\).

We now consider the regular action of a subgroup \(G\) of \(\text{Aut}(\mathcal{D})\) on the set of point classes \(\mathscr{C} = \{C_i | i \in I_m\}\), where \(I_m = \{1, 2, \cdots, m\}\).

### §2 SCT groups and SCT matrices

Let \((\mathbb{P}, \mathbb{B})\) be a \((m, u, k, \lambda)\)-DD and \(G \leq \text{Aut}(\mathbb{P}, \mathbb{B})\). We say \(G\) is an \(SCT(m, u, k, \lambda)\) group if \(G\) is semiregular on \(\mathbb{P} \cup \mathbb{B}\) and regular on the set of point classes \(\mathscr{C} = \{C_1, \cdots, C_m\}\). (Note that \(|G| = m\).)

Assume that \(G\) is an \(SCT(m, u, k, \lambda)\) group of a \((m, u, k, \lambda)\)-DD \(\mathcal{D}(= (\mathbb{P}, \mathbb{B}))\).

Choose a point class \(C = \{p_1, \cdots, p_u\} \in \mathscr{C}\). Then \(\mathbb{P} = \bigcup_{i \in I_u} p_i^G\) and \(\mathbb{B} = \bigcup_{j \in I_s} B_j^G\), where \(s = |\mathbb{B}|/|G|\).

A \(u \times s\) matrix \(M_D = [D_{ij}]\) \((D_{ij} \subset G)\) over \(G\) is defined by \(D_{ij} = \{g \in G | p_i^g \in B_j\}\) \((i \in I_u, j \in I_s)\)

**Theorem 1.** The following holds.

\[\sum_{j \in I_s} D_{ij} D_{ij}^{(-1)} = \begin{cases} \rho + \lambda(G-1) & \text{if } i = \ell, \\ \lambda(G-1) & \text{otherwise,} \end{cases}\]

where \(\rho = (m-1)u\lambda/(k-1)\).

\[\sum_{i \in I_u} |D_{ij}| = k \quad \forall j \in I_s\]

**Definition.** Let \(G\) be a group of order \(m\). Let \(u, s \in \mathbb{N}\). For subsets \(D_{ij} \subset G\) \((i \in I_u, j \in I_s)\) we call a \(u \times s\) matrix \[
\begin{bmatrix}
D_{i1} & \cdots & D_{is} \\
\vdots & \ddots & \vdots \\
D_{u1} & \cdots & D_{us}
\end{bmatrix}
\]
an \(SCT(m, u, k, \lambda)\)-
matrix over $G$ if it satisfies the following for some $\rho \in \mathbb{N}$.

$$
\sum_{j \in I_s} D_{ij} D_{\ell j}^{(-1)} = \begin{cases}
\rho + \lambda (G-1) & \text{if } i = \ell, \\
\lambda (G-1) & \text{otherwise,}
\end{cases}
$$

$$
\sum_{i \in I_u} |D_{ij}| = k \quad \forall j \in I_s
$$

**Remark.**

(i) $s = (m-1)u^2 \lambda / k(k-1)$, $\rho = (m-1)u \lambda / (k-1)$

(ii) An SCT$(m, 1, k, \lambda)$-matrix is just an $(m, k, \lambda)$-difference family.

* An incidence structure $D(\mathbb{P}, \mathbb{B})$ defined by the following is an $(m, u, k, \lambda)$-DD admitting $G$ as an SCT group under the action $(i, w)g = (i, wg)$ for $i \in \{1, \ldots, u\}$ and $w, g \in G$.

$\mathbb{P} = \{1, 2, \ldots, u\} \times G$

$\mathbb{B} = \{B_{j,g} \mid j \in I_s, g \in G\}$, where $B_{j,g} = \bigcup_{i \in I_u} (i, D_{ij}g)$

* $(m, u, k, \lambda)$-DD with SCT-group $\iff$ SCT$(m, u, k, \lambda)$-matrix

**Example.**

(i) The following is an SCT$(9, 2, 9, 9)$ matrix over $G := \langle a, b \rangle \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$:

$$
\begin{bmatrix}
\langle a \rangle & \langle b \rangle & G - \langle ab \rangle & G - \langle ab^2 \rangle \\
G - \langle a \rangle & G - \langle b \rangle & \langle ab \rangle & \langle ab^2 \rangle
\end{bmatrix}
$$

This matrix gives a $TD_{9}(9, 2)$, which is not obtained from any difference matrix by Drake's result.

(ii) The following is an SCT$(12, 5, 11, 2)$ matrix over $\text{Alt}(4) = N \rtimes H$, $N = \{1, a, b, c\} \simeq E_4$, $H = \{1, d, d^2\} \simeq \mathbb{Z}_3$:

$$
\begin{bmatrix}
0 & \alpha & \beta & \gamma & \delta \\
\alpha & \beta & \gamma & \delta & 0 \\
\beta & \gamma & \delta & 0 & \alpha \\
\gamma & \delta & 0 & \alpha & \beta \\
\delta & 0 & \alpha & \beta & \gamma
\end{bmatrix}
$$

where

\[
\begin{align*}
\alpha &= ad + cd^2 \\
\beta &= d + bd^2 + d^2 + cd \\
\gamma &= b + c \\
\delta &= ad^2 + bd + a
\end{align*}
\]

From this we obtain a $(12, 5, 11, 2)$-DD with the full automorphism group isomorphic to $\text{Alt}(5) \geq \text{Alt}(4) \simeq N \rtimes H$. This DD is not class regular, hence not obtained from any partial difference matrix.

**Relations among SCT aut. , Class regular aut. and RDS**

\[\exists \text{SCT aut. } \iff \exists \text{SCT mat.}\]

\[
\text{Divisible design } \supset \text{Transversal design}
\]

\[\exists \text{class regular aut. } \iff \exists \text{partial DM } \supset \text{DM } \supset \text{GH mat.}\]

\[\exists \text{SCT aut. } \& \exists \text{class regular aut. } \iff \exists \text{splitting relative difference set}\]
Difference families and SCT matrices

A family of $k$-subsets $\{D_1, \cdots, D_n\}$ of a group $G$ of order $v$ is called an $n-(v, k, \lambda)$ difference family if

$$D_1D_1^{(-1)} + \cdots + D_nD_n^{(-1)} = kn + \lambda(G-1).$$

From an $n-(v, k, \lambda)$ difference family in a group $G$ we obtain a $2-(v, k, \lambda)$ design $(\mathbb{P}, \mathbb{B})$:

$$\mathbb{P} = G,$$ $$\mathbb{B} = \{D_i x | i \in I_n, x \in G\}.$$  

In the following we give a relation between difference families and SCT matrices with $u=2$.

**Theorem 2.** Let $\{D_1, \cdots, D_{4d}\}$ be a $4d-(m, k, d(4k-m))$ difference family in a group $G$ of order $m$. Set $C_i = G - D_i$ for $i \in I_{4d}$. Then the following is an $SCT(m, 2, m, dm)$ matrix corresponding to a $TD_{dm}(m, 2)$.

$$M = \begin{bmatrix} D_1 \cdots D_{2d} & C_{2d+1} \cdots & C_{4d} \\ C_1 \cdots C_{2d} & D_{2d+1} \cdots & D_{4d} \end{bmatrix}$$

$$C_iC_i^{(-1)} = D_iD_i^{(-1)} + (m-2k)G$$

$$D_iC_i^{(-1)} = C_iD_i^{(-1)} = kG - D_iD_i^{(-1)}$$

Some theorems on difference families

The following results on difference families are known.

**Result.** (Leung-Ma-Schmidt [4]) Let $q$ be a prime power and $d > 0$ an integer. Suppose, either (i) $q \equiv 2d-1 \pmod{4d}$ and $2 \nmid d$ or (ii) $q \equiv 4d-1 \pmod{8d}$. Then there exists a $4d-(q^2, (q^2-q)/2, dq^2-2dq)$ difference family in $(GF(q^2), +)$.

**Result.** (Q. Xiang [6]) Let $q$ be a power of a prime and $b, c$ positive integers such that $q+1 = 2^cb$ and $c \geq 2$ with $2 \nmid b$. Then there exists a $2^c-(q^2, (q^2-q)/2, 2^{c-2}(q^2-2q))$ difference family in $(GF(q^2), +)$.

**Remark.** Set $d = 2^{c-2}$ in the above result. Then $2^c-(q^2, (q^2-q)/2, 2^{c-2}(q^2-2q))$ is identical with $4d-(q^2, (q^2-q)/2, dq^2-2dq)$.

We now apply Theorem 2 to the above results for $m = q^2, k = (q^2-q)/2$.

**Proposition.** Let $q$ be a power of a prime and $d$ a positive integer satisfying one of the following:

(i) $q \equiv 2d-1 \pmod{4d}$.  

(ii) $q \equiv 4d-1 \pmod{8d}$.  

(iii) $4d \mid q + 1, 8d \nmid q + 1$ with $d$ a power of $2$.

Then, there exists an $SCT(q^2, 2, q^2, dq^2)$ matrix over $(GF(q^2), +)$ and the resulting $TD_{dq^2}(q^2, 2)$ admits an SCT automorphism group of order $q^2$.

**Remark.** If $2 \nmid dq$, then no $TD_{dq^2}(q^2, 2)$s are obtained from difference matrices by Drake’s result.
§3 Direct product RDSs and SCTs

Let \( \mathcal{G} \) be a group of order \( um \) and \( U \) its (not necessarily normal) subgroup of order \( u \). A \( k \)-subset \( D \) of \( \mathcal{G} \) is called an \((m, u, k, \lambda)\)-relative difference set (or, RDS for short) relative to \( U \) if \( DD^{(-1)} = k + \lambda(\mathcal{G} - U) \). Usually \( U \) is called the forbidden subgroup.

An \((m, u, k, \lambda)\)-divisible design \( D = (\mathbb{P}, \mathbb{B}) \) is obtained from \((m, u, k, \lambda)\)-RDS in the following way: the set \( \mathbb{P} \) of points are elements of \( \mathcal{G} \) and the set of blocks \( \mathbb{B} \) are subsets \( Dx(x \in \mathcal{G}) \). We note that the set of point classes are \( \{Ug \mid g \in \mathcal{G}\} \).

We say \( \mathcal{G} \) is splitting (over \( U \)) if there exists a subgroup \( G \) of \( \mathcal{G} \) of order \( m \) such that \( \mathcal{G} = GU \) and \( G \cap U = 1 \). In this case \( G \) is an SCT\((m, u, k, \lambda)\) group of \( D \).

From now on we consider an SCT matrix obtained from a splitting abelian RDS ; \( \mathcal{G} = G \times U \).

**Hypothesis 3.** Let \( G = \{g_1, \ldots, g_m\} \) and \( U = \{w_1, \ldots, w_u\} \) be abelian groups of order \( m \) and \( u \), respectively. Suppose \( D \) is an \((m, u, k, \lambda)\)-RDS in the group \( \mathcal{G} = G \times U \) relative to \( U \). Set \( \mathbb{P} = \mathcal{G} = \{w_ig_j \mid i \in I_u, j \in I_m\} \) and \( \mathbb{B} = \{Dw_ig_j \mid i \in I_u, j \in I_m\} \). Then \( \mathcal{D}_{D,G} := (\mathbb{P}, \mathbb{B}) \) is a \((m, u, k, \lambda)\)-DD with the set \( \mathcal{C} := \{Ug_1, \ldots, Ug_m\} \) of point classes.

We now consider the action of \( G \) on \((\mathbb{P}, \mathbb{B}) \) as an SCT group.

\[ \{w_ig \mid i \in I_u\} : \text{the set of } G \text{-orbits on } \mathbb{P}, \]
\[ \{Dw_ig \mid i \in I_u\} : \text{the set of } G \text{-orbits on } \mathbb{B}, \]
\[ D = G_{w_1}w_1 \cup \cdots \cup G_{w_u}w_u \quad (\exists G_{w_1}, \ldots, \exists G_{w_u} \subset G). \]

We choose a point class \( \mathcal{C} = \{w_1, \ldots, w_u\}(\in \mathcal{C}) \) as a set of representatives of \( G \)-orbits on \( \mathbb{P} \) and \( \{Dw_1, \ldots, Dw_u\}(\subset \mathbb{B}) \) as a set of representatives of \( G \)-orbits on \( \mathbb{B} \).

**Direct product RDSs and SCTs**

Under Hypothesis 3, the corresponding \( u \times u \) SCT matrix \( [D_{ij}] \) is given by

\[ D_{ij} = \{g \in G \mid (w_i)g \in Dw_j\} = G \cap Dw_i^{-1}w_j. \]

As \( D = G_{w_1}w_1 \cup \cdots \cup G_{w_u}w_u \quad (G_{w_1}, \ldots, G_{w_u} \subset G) \),

we have \( [D_{ij}] = [G_{w_i}^{-1}g_{n_j}] \), which we call an SCT matrix of standard form with respect to \( D, G \times U \).

Similarly, if we choose a point class \( \mathcal{C} = \{w_1g, \ldots, w_ug\} \in \mathcal{C} \left( g \in G \right) \) and \( \{Dw_1g_{n_1}, \ldots, Dw_ug_{n_u}\} \subset \mathbb{B} \left( n_1, \ldots, n_u \in I_m \right) \) as sets of representatives of \( G \)-orbits on \( \mathbb{P} \) and \( \mathbb{B} \), respectively, then we have the following.

**Lemma 4.** Under Hypothesis 3, set \( D = G_{w_1}w_1 \cup \cdots \cup G_{w_u}w_u \), where \( G_{w_1}, \ldots, G_{w_u} \subset G \). Then a \( u \times u \) matrix \( [G_{w_i}^{-1}g_{n_j}^{-1}g_{n_j}] \) is an SCT\((m, u, k, \lambda)\) matrix.
Let notations be as in Lemma 4. Then we have the following.

**Proposition 5.** Set \( M = [G_{w_{i}w_{j}^{-1}}] \), the SCT matrix of standard form with respect to \( \{D, G \times U\} \). Then, we have the following.

(i) any SCT matrix is obtained from \( M \) by multiplication of any column by an element of \( G \) and any permutation of rows and columns;

(ii) \( M \) is circulant if \( u \) is a prime and \( w_{i} = w^{i-1} \) for \( i \in I_{u} \), where \( U = \langle w \rangle \).

§4 Spreads and SCTs

**Theorem 6.** Let \( q = p^{e} \) be a power of a prime \( p \) and let \( G \) be an elementary abelian \( p \)-group of order \( q^{2} \). Let \( \{H_{1}, \cdots, H_{q+1}\} \) be a spread of \( G \) (i.e., \(|H_{i}| = q, |H_{i} \cap H_{j}| = 1, \forall i \neq j\) ). Set \( q_{0} = q/p^{m} (= p^{e-m}) \) and

\[
A_{i} = H_{(i+1)q_{0}+1}^{*} + H_{(i+1)q_{0}+2}^{*} + \cdots + H_{iq_{0}+1}^{*} (0 \leq i \leq p^{m} - 2),
\]

\[
A_{p^{m}-1} = H_{1}^{*} + H_{2}^{*} + \cdots + H_{p^{m}q_{0}+1}^{*} + H_{p^{m}q_{0}+2}^{*} + \cdots + H_{p^{m}q_{0}+p^{m}}^{*}.
\]

Let \( L = [n_{ij}] \) be a Latin square of order \( p^{m} \) with entries from \( \{0, 1, \cdots, p^{m}-1\} \). Then the following is an \( STD_{q^{2}/p^{m}}(p^{2e}, p^{m}) \) matrix, which gives an \( SCT(p^{2e}, p^{m}, p^{2e}, p^{2e-m}) \) matrix.

\[
\begin{bmatrix}
A_{n_{1,1}} & A_{n_{1,2}} & \cdots & A_{n_{1,p^{m}}} \\
A_{n_{2,1}} & A_{n_{2,2}} & \cdots & A_{n_{2,p^{m}}} \\
\vdots & \vdots & \ddots & \vdots \\
A_{n_{p^{m}-1}} & \cdots & A_{n_{p^{m},p^{m}-1}} & A_{n_{p^{m},p^{m}}}
\end{bmatrix}
\]

Sketch of proof:

(1) \( \sum_{i \in I_{p^{m}}} A_{i}A_{i}^{-1} = q^{2} + qq_{0}(G-1) \) (\( \forall i \in I_{p^{m}} \)).

(2) If \( \{n_{i_{1}}, \cdots, n_{i_{p^{m}}}\} = \{n_{\ell_{1}}, \cdots, n_{\ell_{p^{m}}}\} = I_{p^{m}} \) and \( n_{i_{1}} \neq n_{\ell_{1}}, \cdots, n_{i_{p^{m}}} \neq n_{\ell_{p^{m}}} \), then

\[
A_{i_{1}}A_{\ell_{1}}^{-1} + \cdots + A_{i_{p^{m}}}A_{\ell_{p^{m}}}^{-1} = q_{0}q(G-1)
\]

**An equivalence class in Latin squares of order \( n \)**

We show that some of the STDs obtained in Theorem 6 admit no class regular automorphism groups. This implies that these STDs are never obtained from generalized Hadamard matrices. In order to prove this we need a lemma on the set of Latin squares.

**Definition.** Let \( e_{1} = (1, 0, 0, \cdots, 0), e_{2} = (0, 1, 0, \cdots, 0), \cdots \) be vectors of \( V(n, \mathbb{C}) \). For a permutation \( \sigma = (r_{1}, 1 \ 2 \ \cdots \ n) \) of \( \Omega := \{1, 2, \cdots, n\} \), a permutation matrix \( P_{\sigma} \) is defined by \( e_{i}P_{\sigma} = e_{r_{i}} \) for each \( i \in I_{n} \). Let \( N \) be the group of permutation matrices of order \( n \) and \( \mathcal{L} \) the set of Latin squares on \( \Omega \). We say Latin squares \( L_{1} \) and \( L_{2} \) in \( \mathcal{L} \) are equivalent if \( L_{2} = PL_{1}Q \) for some \( P,Q \in N \). Let \( H := N \times N \) be the direct product and define the action of \( H \) on \( \mathcal{L} \) by \( (P, Q) = P^{T}LQ \) for \( L \in \mathcal{L} \). Then \( H \) is a permutation group on \( \mathcal{L} \).
The number of Latin squares of order $n$

Let $\mathcal{L}_{n}$ be the set of Latin squares of order $n$ on $\{1, \ldots, n\}$.

By Theorem III.1.19 of [1],

$$|\mathcal{L}_{n}| > f(n) := (n!)^{2n}/n^{n^{2}} \text{ for } n > 1.$$  

$|\mathcal{L}_{2}| = (2-1)!2! > \lceil f(2) \rceil = 1,$  

$|\mathcal{L}_{3}| = (3-1)!3! > \lceil f(3) \rceil = 2,$  

$|\mathcal{L}_{4}| = 4(4-1)!4! > \lceil f(4) \rceil = 25,$  

$|\mathcal{L}_{5}| = 56(5-1)!5! = 161280 > \lceil f(5) \rceil = 2077$.

Latin squares equivalent to a circulant one

$\mathcal{L} = \text{the set of Latin squares on } \Omega := \{1, 2, \ldots, n\}$  

$N = \text{the group of permutation matrices of order } n$  

$N \times N = \text{the permutation group on } \mathcal{L} \text{ defined by } L(P, Q) = P^{T}LQ$

Lemma. Let $C$ be a circulant matrix of order $n$ whose first row is $(a_1, a_2, \ldots, a_n)$ with $\{a_1, a_2, \ldots, a_n\} = \Omega$. Let $T \in N$ be a circulant permutation matrix whose first row is $(0, 1, 0, \cdots, 0)$. If $Q, R \in N$ and $QC = CR$ then $Q = R$ and $Q \in \langle T \rangle$.

Lemma 7. Assume $C \in \mathcal{L}$ and $C$ is circulant. Then,

(i) The number of Latin squares in $\mathcal{L}$ equivalent to $C$ is $(n!)^{2}/n$;

(ii) If $n \geq 4$, then there exists a Latin square of $\mathcal{L}$ not equivalent to circulant one.

\[\vdots\]

By Theorem III.1.19 of [1], $|\mathcal{L}_{n}| > (n!)^{2n}/n^{n^{2}}$.

As $(n!)^{2n}/n^{n^{2}} > (n-1)!(n!)^{2}/n$, $(n \geq 4)$, the lemma holds.

Non class regular STDs

Theorem. Let $p > 3$ be a prime and $A_L$ the $\text{SCT}(p^{2e-1},p^{2e},p,p^{2e})$ matrix defined in Theorem 6. Then the $\text{STD}_{p^{2e-1}}(p^{2e},p)$ obtained from $A_L$ is not class regular.

Proof. By Lemma 7, there exists a Latin square $L$ not equivalent to a circulant one. Let $(\mathbb{P}, \mathbb{B})$ be the $\text{STD}_{p^{2e-1}}(p^{2e},p)$ obtained from $A_L$ and let $G$ be the $\text{SCT}(p^{2e-1},p^{2e},p,p^{2e})$ automorphism group of order $p^{2e}$. Suppose false and let $U$ be a class regular automorphism group of $(\mathbb{P}, \mathbb{B})$. Then, as $G$ normalizes $U$ and $|U| = p$, $G$ centralizes $U$. The direct product $G := G \times U$ contains a $(p^{2e},p,p^{2e},p^{2e-1})$-RDS corresponding to $(\mathbb{P}, \mathbb{B})$. By Proposition 5, $L$ must be equivalent to a circulant Latin square, a contradiction.
In this section we define a \( \lambda \)-planar function as a generalization of planar functions.

**Theorem.** Let \( \mathcal{G} = GU \) be a group of order \( mu \) and \( G, U \) its subgroups with \(|G| = m, |U| = u \) and \( G \triangleright U \). Let \( D \) be a \((m, u, k, \lambda)\)-RDS in \( \mathcal{G} \) relative to \( U \). Then there exists a \( k \)-subset \( C \) of \( G \) and a function \( f : C \rightarrow U \) satisfying the following.

(i) \( D = \{xf(x) \mid x \in C\} \)

(ii) \( \#\{x \in C \mid ax \in C, f(ax)^{\varphi(a)}f(x)^{-1} = b\} = \lambda \)

for any \( a \in G \setminus \{1\} \) and \( b \in U \).

**Proposition.** Let \( G, U \) be groups of order \( m, u \), respectively. Let \( \varphi \) be a homomorphism from \( G \) to \( \text{Aut}(U) \) and \( f \) a function from \( C \) to \( U \) for a \( k \)-subset \( C \) of \( G \). Assume that for any \( a \in G \setminus \{1\} \) and \( b \in U \)

\[
\#\{x \in C \mid ax \in C, f(ax)^{\varphi(a)}f(x)^{-1} = b\} = \lambda.
\]

Then \( D = \{xf(x) \mid x \in C\} \) is a \((m, u, k, \lambda)\)-RDS in a semi-direct product \( \mathcal{G} = GU \) of \( G \) by \( U \) with respect to \( \varphi \).

**Definition.** Let \( G \) and \( U \) be groups. Let \( C \) be a subset of \( G \) and \( \varphi \in \text{Hom}(G, \text{Aut}(U)) \). We call a function \( f : C \rightarrow U \) a \( \lambda \)-planar function relative to \( (C, U, \varphi) \) if \( f \) satisfies (*) if \( \varphi \) is a trivial homomorphism, we say \( f \) is a \( \lambda \)-planar function relative to \( (C, U) \). We note that a 1-planar function relative to \( (G, U) \) is just a planar function in the usual sense (see Pott [5]).

**Example.** Let \( q = p^e \) be a power of a prime \( p \) and set \( G = F = (GF(q^2), +) \supset U = K = (GF(q), +) \). Then a function

\[
f(x) = x^{q+1}
\]

from \( G \) to \( U \) is a \( q \)-planar function relative to \( (G, U) \).

\[
\therefore \text{Let } 0 \neq a \in G \text{ and } b \in U. \text{ Then,}
\]

\[
f(a + x) - f(x) = b \iff (a^q + x^q)(a + x) - x^{q+1} = b
\]

\[
\iff ax^q + a^q x = b - a^{q+1} \quad (**)\text{.}
\]

As \( ax^q + a^q x = ax^q + (ax^q)^q = \text{Tr}_{F/K}(ax^q) \), \( (**) \) has exactly \( q \) solutions in \( G \). Thus \( f \) is a \( q \)-planar function relative to \( (G, U) \).
λ-planar functions, SCTs, and RDSs

Theorem 8. Let $G$ be a group of order $m$ and $U$ a group of order $u$. Let $D_y$ be subsets of $G$ for each $y \in U$. If a $u \times u$ matrix $D = [D_{yz^{-1}}]_{y,z \in U}$ over $\mathbb{Z}[G]$ is an $SCT(m, u, k, \lambda)$ matrix, then the following holds.

(i) Set $C = \bigcup_{y \in U} D_y \subset G$. Then $|C| = k, G = \langle C \rangle$ and a function $f : C \to U$ defined by $f(D_y) = y$ ($y \in U$) is a λ-planar function relative to $(C, U)$.

(ii) Set $D = \{(x, f(x) | x \in C\}$. Then $D$ is an $(m, u, k, \lambda)$-RDS in $G \times U$ relative to $1 \times U$.

Remark. A $(u\lambda, u, u\lambda, \lambda)$-RDS is called semiregular. It is conjectured that any forbidden subgroup of a semiregular RDS is a $p$-group for a prime $p$. Concerning this we can show the following as an application of Theorems 6 and 8.

Theorem. Any $p$-group can be a forbidden subgroup of a semiregular RDS.

As a corollary we have the following, which gives another proof of de Launey's result on generalized Hadamard matrices (cf. [1], Theorem 5.9).

Corollary There exists a $GH(p^m, p^{2e-m})$ matrix over any group of order $p^m$ whenever $e \geq m$.

References


