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Numbers of irreducible Brauer characters of height zero in 2-blocks of finite groups

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1 Introduction

Let p be a prime number and $(\mathcal{K}, \mathcal{O}, k)$ be a p-modular system. Here $\mathcal{O}$ is a complete discrete valuation ring with a unique maximal ideal (π) containing p. $\mathcal{K}$ is the quotient field of $\mathcal{O}$ with characteristic 0. $k=\mathcal{O}/(\pi)$ is the residue class field with characteristic p. We assume that $\mathcal{K}$ and k are splitting fields for G. We can take a finite field as k. Let G be a finite group, and B be a p-block with defect group D. We denote by $\mathrm{IBr}(B)$ and $\mathrm{IBr}_0(B)$ the set of irreducible Brauer characters and of those of height zero in B, respectively. We also denote by $l(B)$ and $l_0(B)$ the number of elements of $\mathrm{IBr}(B)$ and $\mathrm{IBr}_0(B)$, respectively. Let C_B be the Cartan matrix of B and let ρ_B be the Frobenius-Perron eigenvalue (i.e. the unique largest eigenvalue) of C_B. We denote by R_B the set of eigenvalues of C_B. Then we had the following results for any finite groups G and for any p-blocks B.

Lemma 1([12, Proposition 4.5]). Let G be a finite group, let B be a p-block of G with defect group D. If $\rho \in R_B$, then ρ divides $|D|$ in the ring of algebraic integers, which means that $|D|=\rho\times\mu$ for some algebraic integer μ. In particular, if $\rho \in R_B$ is a rational integer then it is a power of p which divides $|D|$.

Lemma 2 ([15, Proposition 3.5]). Let $|G|_p=p^a$ and $|D|=p^d$. If the condition

$$(*) \quad \sum_{\varphi\in\mathrm{IBr}(B)}\left(\frac{\varphi(1)}{p^{a-d}}\right)^2 \not\equiv 0 \pmod p$$

holds, then there exists an eigenvalue $\rho\in R_B$ such that $|D|/\rho$ is a unit of $\mathcal{O}$ (i.e. $|D|/\rho \notin (\pi)$).

In particular, if $(*)$ holds and if all eigenvalues of C_B are rational integers, then the ρ above equals $|D|$ and $\rho=\rho_B=|D|$ by Lemma 1.

Proposition 1 ([15, Corollary 3.6]). Let G be a p-solvable group and B be a p-block of G. Then B satisfies the condition $(*)$.

Example 1 ([11, Examples 7.2 and 7.3]). Let $G=\mathrm{SL}(2,p)$ for $p>3$ and B be a full-defect p-block of G, or let $G=\mathfrak{S}_p$ (the symmetric group on p letters) for $p>3$ and B be the principal

p-block of G. Then B does not satisfy the condition $(*)$.

The conditon $(*)$ is related to the integrality of the Frobenius-Perron eigenvalue ρ_B of the Cartan matrix of B. As Example 1 shows that there exists a counter example of $(*)$ for $p > 3$, we will pay our attention to $p = 2$. It is difficult to find a counter example of $(*)$ for $p = 2$, while we find the following conter example of $(*)$ for $p = 3$.

Example 2. We denote an irreducible Brauer character by its degree. Let $d(B)$ be the defect of B.
(1) Let $G = M_{11}$ be the Mathieu simple group of degree 11, $p = 3$, and B_1 be the principal 3-block of G. Then $l(B_1) = 7, d(B_1) = 2$ and $\mathrm{IBr}(B_1) = \{1, 5, 5, 10, 10, 10, 24\}$. So $l_0(B_1) = 6 \equiv 0 \pmod 3$.
(2) Let $G = He$ be the Held simple group, $p = 3$, and B_1 be the principal 3-block of G. Then $l(B_1) = 7, d(B_1) = 3$ and $\mathrm{IBr}(B_1) = \{1, 7 \cdot 97, 3 \cdot 5^2 \cdot 17, 3673, 2^2 \cdot 1543, 2^7 \cdot 7^2 \cdot 11, 11 \cdot 23 \cdot 43\}$. So $l_0(B_1) = 6 \equiv 0 \pmod 3$.

Do any 2-blocks satisfy $(*)$? Are there 2-blocks which do not satisfy $(*)$? What kind of 2-blocks satify $(*)$? This is our motiation to consider $l_0(B)$ for 2-blocks B.

2 Good blocks

The following lemmas and propositions are easy to see. So we omit a proof.

Lemma 3. For $p = 2$, $(*)$ holds if and only if $l_0(B)$ is odd.

Remark. As is alredy mentioned in Example 2, Lemma 3 holds for $p = 3$,too. i.e., For $p = 3$, $(*)$ holds if and only if $l_0(B) \not\equiv 0 \pmod 3$. But this result does not hold for $p > 3$ anymore.

We call a 2−block B ***good*** if $l_0(B)$ is odd. What kind of 2-blocks are good? The following is by the theorem of Fong [8].

Proposition 2. The principal 2-block of G is good.

Corollary 1. If G is 2-solvable and B is a 2-block of G, then B is good.

The following is a corollary of the result of Kiyota-Okuyama-Wada in [11] and from Theorems 3 and 4 later and [17].

Proposition 3 ([11, Theorem 1.4]). If $G = \mathfrak{S}_n$ (the symmetric group of n letters) or $\mathfrak{A}_n$ (the alternating group of n letters) and B be a 2-block of G, then B is good. Also any 2-blocks of their automorphism groups and covering groups are good.

Proposition 4. If B is a cylic 2-block (i.e., D is cyclic), then B is a nilpotent block. Hence $l(B) = 1$ and B is good.

We define some notation in order to prove the following theorems which is defined for any prime p and for any p-block B of G. For $\varphi \in \mathrm{IBr}(B)$, we denote by Φ_φ the projective indecomposable character of G corresponding to φ. Let us denote by $X_B := \{\varphi \in \mathrm{IBr}_0(B) \mid \Phi_\varphi(1)_p = p^a\}$, $Y_B := \{\varphi \in \mathrm{IBr}_0(B) \mid \Phi_\varphi(1)_p > p^a\}$ and $Z_B := \{\varphi \in \mathrm{IBr}(B) \mid \varphi(1)_p > p^{a-d}\}$. Then $\mathrm{IBr}(B) = X_B \sqcup Y_B \sqcup Z_B$.

The following is by the theorem of Brauer's [4] which shows $(\dim B)_p = p^{2a-d}$. It is easy to see but very useful.

Lemma 4. Suppose $p = 2$, then $|X_B|$ is odd. In particular, $X_B \neq \emptyset$, and $l_0(B)$ is odd if and only if $|Y_B|$ is even.

3 Tame blocks

We consider the tame block as a typical well known 2-block that is a 2-block with defect group isomorphic to a dihedral, a generalized quaternion or a semidihedral 2-group. In Theorem 1 we will deal with 2-blocks of dihedral or generalized quaternion defect group. In Theorem 2 we will deal with 2-blocks of semidihedral defect group. We note that if B is tame, then $l(B)$ is $1, 2$ or 3. If $l(B) = 1$, then B is good. So we consider when $l(B) = 2$ or 3.

We proceed our argument based on the work of Erdmann [6], where the Morita equivalence classes of B are classified, and the Cartan matrix of B is given for each equivalence class of B. Then the following cases can be 2-blocks of finite groups: $l(B) = 1$, $D(2A), D(2B), D(3A)_1, D(3B)_1, D(3K)$ for dihedral defect groups, $Q(2A), Q(2B)_1, Q(3A)_2, Q(3B), Q(3K)$ for generalized quaternion defect groups, and $SD(2A)_1, SD(2A)_2, SD(2B)_1, SD(2B)_2, SD(3A)_1, SD(3B)_1, SD(3B)_2, SD(3C)_2(1)$, $SD(3C)_2(2), SD(3D), SD(3H)$ for semidihedral defect groups. We note that $Q(2B)_2 = SD(2B)_3$ cannot be a 2-block of a finite group which is pointed out in the beginning of 4.1 of [10].

In [6], Erdmann classified all blocks of tame representation type up to Morita equivalence. She begins with the basic algebra B_0. So all simple B_0-modules are 1-dimensional ([6, 1.2.4] or [2, p.23]). She classifies Morita equivalence classes of B_0 in terms of the stable Auslander-Reiten quivers. Then she obtains the Cartan matrix of each type of B_0. Since B and B_0 are Morita equivalent, the Cartan matrices of B and B_0 are the same. But the dimensions and the heights of simple B-modules and B_0-modules are different. However, once the Cartan matrix of B is obtained, then we can determine the heights of irreducible Brauer characters in B. Historically, at first, Brauer [4] and Olsson [16] determined $k(B)$ and $l(B)$ and the decomposition numbers of the tame block B with respect to a suitable basic set. They also obtained some informations about the heights of irreducible ordinary characters of B and the Cartan matrix C_B with respect to a suitable basic set. But it seems to be no description about the heights of irreducible Brauer characters in B in their works and also in Erdmann's work.

We use the following notation for the both Theorems 1 and 2 below. We set $|G|_2 = 2^a$. Let D be a defect group of B of order 2^d, and let $\varphi_i \in \mathrm{IBr}(B)$. We arrange so that φ_i is corresponding to the i th row of the Cartan matrix of B. We set the degree $\varphi_i(1) = f_i$. Then $f_i = 2^{a-d+e_i} f_i'$, where e_i is the height of φ_i and f_i' is the odd part of f_i. It is well known that there is at least one φ_i is of height 0. But the height of φ_i can exceed 2^d, which is different from the height of irreducible ordinary characters of B. We also set the degree of the projective indecomposable character $\Phi_i(1) = u_i$ corresponding to φ_i. Then the 2-part of u_i is larger than or equal to 2^a.

In this report we do not write a proof in all cases of the tame block. But we give a proof of some typical cases.

Theorem 1. Let G be a finite group. If B is a 2-block of G with defect group isomorphic to a dihedral 2-group or a generalized quaternion 2-group, then B is good. Furthermore, we have the heights of irreducible Brauer characters in B.

Proof. (D.1) $l(B) = 2$.

(1) $D(2A)$. In this case, the Cartan matrix is of the form $\begin{pmatrix} 2^d & 2^{d-1} \\ 2^{d-1} & 2^{d-2}+1 \end{pmatrix}$, where $d \geq 2$. Then

$$u_1 = 2^{a+e_1} f_1' + 2^{a-1+e_2} f_2' = 2^{a-1}(2^{e_1+1} f_1' + 2^{e_2} f_2'),$$
$$u_2 = 2^{a-1+e_1} f_1' + 2^{a-2+e_2} f_2' + 2^{a-d+e_2} f_2' = 2^{a-d}(2^{d-1+e_1} f_1' + 2^{d-2+e_2} f_2' + 2^{e_2} f_2').$$

If $e_1 = e_2 = 0$, then $u_1 = 2^{a-1}(2f_1' + f_2')$. Since $2f_1' + f_2'$ is odd, this contradicts that $(u_1)_2 \geq 2^a$. If $e_1 > 0$ and $e_2 = 0$, then $u_1 = 2^{a-1}(2^{e_1+1} f_1' + f_2')$, which also contradicts that $(u_1)_2 \geq 2^a$. Then we have $e_1 = 0$ and $e_2 > 0$. Hence $X_B = \{\varphi_1\}, Y_B = \emptyset, Z_B = \{\varphi_2\}$ holds in this case. Furthermore, we have that $u_1 = 2^a(f_1' + 2^{e_2-1} f_2')$. Since $|X_B|$ is odd by Lemma 4, then $(u_1)_2 = 2^a$, hence $f_1' + 2^{e_2-1} f_2'$ must be odd. Therefore, $e_1 = 0, e_2 > 1$. If $d = 2$, then $u_2 = 2^{a-1}(f_1' + 2^{e_2} f_2')$, which is a contradiction. We may assume $d > 2$. Then $u_2 = 2^{a-d}(2^{d-1} f_1' + 2^{d-2+e_2} f_2' + 2^{e_2} f_2')$. If $e_2 < d-1$, then $u_2 = 2^{a-d+e_2}(2^{d-1-e_2} f_1' + 2^{d-2} f_2' + f_2')$, which is a contradiction, because $a - d + e_2 < a - d + d - 1 = a - 1$. If $e_2 > d - 1$, then $u_2 = 2^{a-d+(d-1)}(f_1' + 2^{d-2+e_2-(d-1)} f_2' + 2^{e_2-(d-1)} f_2')$, which is also a contradiction, because $d - 2 + e_2 - (d-1) = e_2 - 1 > 0$ now. Hence $e_2 = d - 1$.

(Q.2). $l(B) = 3$.

(1) $Q(3A)_2$. In this case, the Cartan matrix is of the form $\begin{pmatrix} 2^d & 2^{d-1} & 2^{d-1} \\ 2^{d-1} & 2^{d-2}+2 & 2^{d-2} \\ 2^{d-1} & 2^{d-2} & 2^{d-2}+2 \end{pmatrix}$,

where $d \geq 3$. Then

$$\begin{aligned}
u_1 &= 2^{a+e_1}f_1' + 2^{a-1+e_2}f_2' + 2^{a-1+e_3}f_3' = 2^{a-1}(2^{e_1+1}f_1' + 2^{e_2}f_2' + 2^{e_3}f_3'),\\
u_2 &= 2^{a-1+e_1}f_1' + 2^{a-2+e_2}f_2' + 2^{a-d+e_2+1}f_2' + 2^{a-2+e_3}f_3'\\
&= 2^{a-d+1}(2^{d-2+e_1}f_1' + 2^{d-3+e_2}f_2' + 2^{e_2}f_2' + 2^{d-3+e_3}f_3'),\\
u_3 &= 2^{a-1+e_1}f_1' + 2^{a-2+e_2}f_2' + 2^{a-2+e_3}f_3' + 2^{a-d+e_3+1}f_3'\\
&= 2^{a-d+1}(2^{d-2+e_1}f_1' + 2^{d-3+e_2}f_2' + 2^{d-3+e_3}f_3' + 2^{e_3}f_3').
\end{aligned}$$

(1.1) If $e_1 = e_2 = 0$ and $e_3 > 0$, then $u_1 = 2^{a-1}(2f_1' + f_2' + 2^{e_3}f_3')$. Since $2f_1' + f_2' + 2^{e_3}f_3'$ is odd, this contradicts that $(u_1)_2 \geq 2^a$.

(1.2) If $e_1 = e_3 = 0$ and $e_2 > 0$, then we have the same contradiction for $(u_1)_2$.

(1.3) If $e_2 = e_3 = 0$ and $e_1 > 0$, then we have the same contradiction for $(u_2)_2$, because $d \geq 3$.

(1.4) If $e_1 = 0$ and $e_2 > 0, e_3 > 0$, then there is a possibility to occur.

(1.5) If $e_2 = 0$ and $e_1 > 0, e_3 > 0$, then $(u_1)_2 = 2^{a-1}$ which is a contradiction.

(1.6) If $e_3 = 0$ and $e_1 > 0, e_2 > 0$, then we have the same contradiction as (1.5).

(1.7) If $e_1 = e_2 = e_3 = 0$, then since $u_2 = 2^{a-d+1}(2^{d-2}f_1' + 2^{d-3}f_2' + f_2' + 2^{d-3}f_3')$, we have the same contradiction, because $d \geq 3$.

Since only the case (1.4) occurs, we have that $X_B = \{\varphi_1\}, Y_B = \emptyset, Z_B = \{\varphi_2, \varphi_3\}$ in this case. Then $u_1 = 2^a(f_1' + 2^{e_2-1}f_2' + 2^{e_3-1}f_3')$. Since $|X_B|$ is odd by Lemma 4, then $(u_1)_2 = 2^a$, hence $f_1' + 2^{e_2-1}f_2' + 2^{e_3-1}f_3'$ must be odd. Therefore, $e_2 = e_3 = 1$, or $e_2 > 1$ and $e_3 > 1$. Assume $e_1 = 0, e_2 = e_3 = 1$. Then $u_2 = 2^{a-d+2}(2^{d-3}f_1' + 2^{d-3}f_2' + f_2' + 2^{d-3}f_3')$. If $d > 3$, we have a contradiction. So $d = 3$, in this case. Assume $e_1 = 0, e_2 > 1, e_3 > 1$. Then $d > 3$, because if $d = 3$, then $u_2 = 2^{a-1}(f_1' + 2^{e_2}f_2' + 2^{e_3-1}f_3')$, which is a contradiction. So now we have $d > 3$ and $u_2 = 2^{a-d+1}(2^{d-2}f_1' + 2^{d-3+e_2}f_2' + 2^{e_2}f_2' + 2^{d-3+e_3}f_3')$. If $e_2 < d-2$, then we have $u_2 = 2^{a-d+1+e_2}(2^{d-2-e_2}f_1' + 2^{d-3}f_2' + f_2' + 2^{d-3+e_3-e_2}f_3')$, which is a contradiction, because $a - d + 1 + e_2 < a - d + 1 + d - 2 = a - 1$, and $d > 3, d - 3 + e_3 - e_2 > 0$. If $e_2 > d - 2$, then we have $u_2 = 2^{a-d+1+d-2}(f_1' + 2^{e_2-1}f_2' + 2^{e_2-(d-2)}f_2' + 2^{e_3-1}f_3')$,which is also a contradiction. Hence $e_2 = d - 2$. The same augument for u_3 yields that $e_3 = d - 2$. Therefore, we have $d > 3$ and $e_2 = e_3 = d - 2$, in this case. We can unify these two cases into $d \geq 3, e_1 = 0, e_2 = e_3 = d - 2$.

We have the following:

Table 1

type	defect	height	$l_0(B)$
$D(2A)$	$d \geq 3$	$e_1 = 0, e_2 = d-1$	1
$D(2B)$	$d \geq 3$	$e_1 = 0, e_2 = 1$	1
$D(3A)_1$	$d \geq 2$	$e_1 = 0, e_2 = e_3 = d-1$	1
$D(3B)_1$	$d \geq 2$	$e_1 = 0, e_2 = 1, e_3 = d-1$	1
$D(3K)$	$d \geq 2$	$e_1 = e_2 = e_3 = 0$	3
$Q(2A)$	$d \geq 3$	$e_1 = 0, e_2 = d-2$	1
$Q(2B)_1$	$d > 3$	$e_1 = 0, e_2 = 1$	1
$Q(3A)_2$	$d \geq 3$	$e_1 = 0, e_2 = e_3 = d-2$	1
$Q(3B)$	$d \geq 3$	$e_1 = 0, e_2 = 1, e_3 = d-2$	1
$Q(3K)$	$d \geq 3$	$e_1 = e_2 = e_3 = 0$	3

Theorem 2. Let G be a finite group. Suppose that B is a 2-block of G with defect group isomorphic to a semidihedral 2-group. Then B is good except the two cases of $SD(3C)_2$. Furthermore, we have the heights of irreducible Brauer characters in B.

Proof. (5) $SD(3C)_2(2)$. In this case, the Cartan matrix is of the form $\begin{pmatrix} 2^{d-2}+2 & 2 & 2 \\ 2 & 3 & 1 \\ 2 & 1 & 3 \end{pmatrix}$, where $d \geq 4$. Then

$u_1 = 2^{a-2+e_1} f_1{}' + 2^{a-d+e_1+1} f_1{}' + 2^{a-d+e_2+1} f_2{}' + 2^{a-d+e_3+1} f_3{}' = 2^{a-d+1}(2^{d-3+e_1} f_1{}' + 2^{e_1} f_1{}' + 2^{e_2} f_2{}' + 2^{e_3} f_3{}')$,
$u_2 = 2^{a-d+e_1+1} f_1{}' + 3 \cdot 2^{a-d+e_2} f_2{}' + 2^{a-d+e_3} f_3{}' = 2^{a-d}(2^{e_1+1} f_1{}' + 3 \cdot 2^{e_2} f_2{}' + 2^{e_3} f_3{}')$,
$u_3 = 2^{a-d+e_1+1} f_1{}' + 2^{a-d+e_2} f_2{}' + 3 \cdot 2^{a-d+e_3} f_3{}' = 2^{a-d}(2^{e_1+1} f_1{}' + 2^{e_2} f_2{}' + 3 \cdot 2^{e_3} f_3{}')$.

(5.1) If $e_1 = e_2 = 0$ and $e_3 > 0$, then $u_2 = 2^{a-d}(2f_1{}' + 3f_2{}' + 2^{e_3} f_3{}')$ which contradicts that $(u_1)_2 \geq 2^a$, because $d \geq 4$.
(5.2) If $e_1 = e_3 = 0$ and $e_2 > 0$, then we have the same contradiction for $(u_2)_2$.
(5.3) If $e_2 = e_3 = 0$ and $e_1 > 0$, then there is a possibility to occur.
(5.4) If $e_1 = 0$ and $e_2 > 0, e_3 > 0$, then we have a contradiction for $(u_1)_2$.
(5.5) If $e_2 = 0$ and $e_1 > 0, e_3 > 0$, then we have the same contradiction as (5.4).
(5.6) If $e_3 = 0$ and $e_1 > 0, e_2 > 0$, then we have the same contradiction as (5.4).
(5.7) If $e_1 = e_2 = e_3 = 0$, then we have the same contradiction as (5.4).
In (5.3), $e_1 = 1, e_2 = e_3 = 0$ can possibly occur. Then if this case occurs, B is bad, and $X_B = \{\varphi_i\}, Y_B = \{\varphi_j\}, Z_B = \{\varphi_1\}$ for $\{i,j\} = \{2,3\}$.
In this case, since one of the 2-parts of u_2 and u_3 is 2^a by Lemma 4, we have $(u_2 + u_3)_2 = 2^a$. Since $u_2 + u_3 = 2^{a-d+2}(2^{e_1} f_1{}' + f_2{}' + f_3{}')$, so we have $2^{e_1} f_1{}' + f_2{}' + f_3{}' = 2^{d-2}x$ for some odd x. On the other hand, $u_1 = 2^{a-d+1}(2^{d-3+e_1} f_1{}' + 2^{e_1} f_1{}' + f_2{}' + f_3{}') = 2^{a-d+1}(2^{d-3+e_1} f_1{}' + 2^{d-2}x) = 2^{a-1}(2^{e_1-1} f_1{}' + x)$. Hence $e_1 = 1$.

Remark. In this case, calculating elementary divisors of C_B using $\mathbb{Z}$-elementary operations, the set E_B of elementary divisors of C_B is also $\{2^d, 2, 1\}$ and we have no contradiction for elementary divisors.

We have the following:

Table 2

type	defect	height	$l_0(B)$
$SD(2A)_1 = Q(2A)$			
$SD(2A)_2 = D(2A)$			
$SD(2B)_1 = Q(2B)_1$			
$SD(2B)_2 = D(2B)$			
$SD(3A)_1$	$d \geq 4$	$e_1 = 0, e_2 = d-1, e_3 = d-2$	1
$SD(3B)_1$	$d \geq 4$	$e_1 = 0, e_2 = 1, e_3 = d-2$	1
		$e_1 = e_2 = e_3 = 0$	3
$SD(3B)_2$	$d \geq 4$	$e_1 = 0, e_2 = 1, e_3 = d-1$	1
$SD(3C)_2(1)$	$d \geq 4$	$e_1 = d-2, e_2 = e_3 = 0$	**2**
$SD(3C)_2(2)$	$d \geq 4$	$e_1 = 1, e_2 = e_3 = 0$	**2**
$SD(3D) = SD(3B)_1$			
$SD(3H)$	$d \geq 4$	$e_1 = e_2 = e_3 = 0$	3

Remark. At present, no example of a block of a finite group which is one of $SD(3C)_2$ seems to be known yet as is seen in [6, p.301].

By the Tables 1,2 we have the following corollary. It might be worth mentioning, because it is known that there exists $\varphi \in \mathrm{IBr}(B)$ with $\varphi(1)_2 > |G|_2$. For example, $G =$ McL (McLaughlin's simple group), $p = 2, \varphi$ is in the principal 2-block and $\varphi(1)_2 = 2^9 > |G|_2 = 2^7$ which is shown by J. G. Thackrey [7, p.166], and $G = \mathfrak{S}_{15}, p = 2, \varphi$ is in the principal 2-block and $\varphi(1) = 2^{12} > |G|_2 = 2^{11}$ which is shown by D. J. Benson [1]. We also have $G = \mathrm{J}_1, p = 2$ and $\varphi(1)_2 = 2^3 = |G|_2$, and φ is in the principal 2-block, not in a 2-block of defect 0 in [8]. A simiar case is seen in [17] for $G = PSp(4,5), p = 2, \varphi(1)_2 = 2^6 = |G|_2$.

Corollary 2. If B is a block of tame type with positive defect and $\varphi \in \mathrm{IBr}(B)$, then $\varphi(1)_2 < |G|_2$.

Remark. Let $l(B) = 2$ and let $C_B = (c_{ij})$ for $1 \leq i,j \leq 2$ be the Cartan matrix of B. Assume that there exist at least different two 2-parts of c_{ij} for $1 \leq i,j \leq 2$. Then $l_0(B) = 1$ holds. So, if we want to show only the consequence $l_0(B) = 1$ for tame blocks with $l(B) = 2$, it follows from this more general result.

4 Nearly simple groups

It is not easy to determine the irreducible Brauer character table of 2-blocks of simple groups of large order. But we see many examples of 2-blocks of the simple groups in the Modular Atlas Homepage [17]. However, we cannot find a bad 2-block of the "simple groups" in the Modular Atlas. Hence we consider 2-blocks of almost simple groups and quasi-simple groups.

We omit the proof of the following theorems.

At frst, we must notice that the modular version of the Alperin-McKay conjecture does not hold.

Example 3. Let $G = \mathfrak{A}_5$ (the alternating group on 5 letters) and B be the principal 2-block of G. Let $D \in \mathrm{Syl}_2(G)$. Then $D \simeq E_4$ and $N_G(D) \simeq \mathfrak{A}_4$ and $\mathrm{IBr}(B) = \{1_1, 2_1, 2_2\}$. So, $l_0(B) = 1$. But, for its Brauer correspondent b, $\mathrm{IBr}(b) = \{1_1, 1_2, 1_3\}$ and $l_0(b) = 3$.

We consider the case that there exists a normal 2-subgroup $R \neq \{1\}$ of G. Let B be a p-block of G with defect group D. Then $\varphi \in \mathrm{IBr}(B)$ can be regarded as an irreducible Brauer character of $\overline{G} = G/R$. Let π be the canonical algebra homomorphism from kG onto $k\overline{G}$ by $\pi(\sum_{g\in G} \gamma_g g) := \sum_{\overline{g}\in\overline{G}} \gamma_g \overline{g}$. Let $\overline{B}$ be the image $\pi(B)$ for a p-block B of G. Then $\overline{B}$ is decomposed into p-blocks of $k\overline{G}$; $\overline{B} = \overline{B}_1 + \cdots + \overline{B}_n$. Let $\overline{D}_i$ be a defect group of $\overline{B}_i$ for $1 \leq i \leq n$. Then $\overline{D}_i \subseteq_{\overline{G}} D/R$ for $1 \leq i \leq n$. Furthermore, if $\overline{B}_i$ contains an irreducible Brauer character of height 0 in B, then $\overline{D}_i =_{\overline{G}} D/R$ [7, Lemmas V.4.2, V.4.4]. Consequently we have the following.

Theorem 3. Let R be a normal 2-subgroup of G and $\overline{G} = G/R$. Let B be a 2-block of G. If 2-block $\overline{B}_i$ of $\overline{G}$ is good for all $1 \leq i \leq n$, then B is good.

Theorem 4. Let H is a normal subgroup of G such that G/H is 2-solvable. Let B be a 2-block of G which covers a 2-block b of H. Then B is good if and only if b is good.

We must consider when $O(G) = O_{2'}(G) \neq \{1\}$ (the maximal normal subgroup of G of odd order). This case can be reduced to a cyclic central $2'$-extension of $G/O(G)$ (see [7, Lemma X.1.1 and Theorem X.1.2 (Fong)] or [13, Theorem 7.4 (Fong)]). In this case, we cannot obtain any effective results for reduction. Then we finally find the following example in The Modular Atlas [17].

Example 4. There exists the faithful and full-defect 2-block B_3 (and also $B_4 = \overline{B}_3$) of the 6 fold cover G = 6.Suz of the sporadic simple Suzuki group, where B_3, B_4 are the names in the Modular Atlas Home Page [22]. These blocks satisfy $l_0(B_3) = l_0(B_4) = 4$. Then these blocks are bad. There are 14 irreducible Brauer characters in B_3, in which $\varphi_{20}, \varphi_{22}, \varphi_{23}, \varphi_{28}$ are of height 0 with degrees $3 \cdot 11 \cdot 13$, $3 \cdot 5^2 \cdot 11$, $3 \cdot 5^2 \cdot 11$, $3 \cdot 11 \cdot 509$, respectively. Here the numbering of irreducible Brauer characters is the same as in The Modular Atlas, in which they are ordered so that the degrees are from small to large, and the first one is corresponding to the first row of the following Cartan matrix. We abbreviate φ_i simly to its degree. Then $\mathrm{IBr}(B_3) = \{12_1, 66_1, \mathbf{429_1}, 780_1, \mathbf{825_1}, \mathbf{825_3}, 2100_1, 2100_3, 3456_1, 6720_1, \mathbf{16797_1}, 19722_1, 27456_1, 46488_1\}$. B_4 is the dual (complex conjugate) of B_3 as is well known by [3, Proposition (8A)] and [9, Corollary 1.6]. Then there also exsists a 2-block B_3 of 6.Suz.2 which satisfies $l_0(B_3) = 4$ by Theorem 4.

We notice that if G =Suz itself, then all 2-blocks of G are good, and then so are for 2.Suz, Suz.2 and 2.Suz.2 by Theorem 4.

References

[1] D. J. Benson, *Some remarks on the decomposition numbers for the symmetric groups*, Proceedings of Symposia in Pure Math., 47 (1987), 381–394.

[2] D. J. Benson, Representations and cohomology, 1:Basic representation theory of finite groups and associative algebras, Cambridge Univ. Press, Cambridge, New York, 1991.

[3] R. Brauer, *Some applications of the theory of blocks of characters of finite groups IV*, J. Algebra, 17 (1971), 489–521.

[4] R. Brauer, *On 2-blocks with dihedral defect groups*, Symposia Mathematics, vol.XIII (1974), 200–226.

[5] R. Brauer, *Notes on representations of finite groups, I*, J.London Math. Soc.(2), 13 (1976), 162–166.

[6] K. Erdmann, Blocks of tame representation type and related algebras, Springer-Verlag, Lecture Notes in Math. 1428, Berlin, 1990.

[7] W. Feit, The Representaton Theory of Finite Groups, North-Holland, New York, 1982.

[8] P. Fong, *On decomposition numbers of* J_1 *and* $R(q)$, Symposia Mathematica, vol. XIII, Academic Press, 1974, 415–422 .

[9] R. Gow and W. Willems, *Quadratic geometries, projective modules, and idempotents*, J. Algebra, 160 (1993), 257–272.

[10] T. Holm, *Derived equivalent tame blocks*, J. Algebra, 194 (1997), 178–200.

[11] M. Kiyota and T. Okuyama and T. Wada, *The heights of irreducible Brauer characters in 2-blocks of the symmetric groups*, J. Algebra, 368 (2012), 329–344.

[12] M. Kiyota and T. Wada, *Some remarks on eigenvalues of the Cartan matrix in finite groups*, Comm. Algebra, 21 (1993), 3839–3860.

[13] H. Nagao and Y. Tsushima, Representatons of Finite Groups, Academic Press Inc., Boston, MA, 1989.

[14] G. Navarro, Characters and blocks of Finite Groups, Lecture Notes, 250, London Math. Soc., Cambridge Univ. Press, London, 1998.

[15] T. Okuyama and T. Wada, *Eigenvalues of Cartan matrices of blocks in finite groups*, Contemporary Math., 524, Amer. Math. Soc., (2010), 127–138.

[16] J. B. Olsson, *On 2-blocks with quaternion and quasidihedral defect groups*, J. Algebra, 36 (1975), 212–241.

[17] R. Wilson, R. Parker, J. Muller, K. Lux, F. Lübeck, C. Jansen, G, Hiss and T. Breuer, The Modular Atlas Homepage, http://www.math.rwth-aachen.de/ MOC/

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