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Module structures of source algebras and cohomology of block algebras

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Throughout of this report we let

- k be an algebraically closed field of characteristic $p > 0$
- G a finite group of order divisible by p
- B a block ideal of kG with defect group D.

1 Source algebras of block algebras

Let X be a source module of B: X is an indecomposable $k[G \times D^{\mathrm{op}}]$-direct summand of B with vertex $\Delta(D)$.

Let $A = X^* \otimes_B X$, which is called a source algebra of B.

Theorem 1.1 (Puig [7]). *A and B are Morita equivalent.*

Problem 1. To know module structure of $A = X^* \otimes_B X$.

Because A is a direct summand of kG as (kD, kD)-bimodules, we have

$$ {}_{kD}A_{kD} = {}_{kD}X^* \otimes_B X_{kD} \simeq \text{direct sum of some } k[DgD]\text{s}. $$

- Which $k[DgD]$ appears in the decomposition above?
- How many times does $k[DgD]$ occur?

Let b_D is a unique block of $kDC_G(D)$ with $b_D X(D) \neq 0$, where $X(D)$ is the Brauer construction. Puig [7] showed that the direct summands generated by elements in the inertia group $N_G(D, b_D)$ are well understood.

Theorem 1.2 (Puig [7]). (1) $A \simeq \left(\bigoplus_{gDC_G(D) \in N_G(D,b_D)/DC_G(D)} k[Dg] \right) \oplus N,$

where N is a direct sum of $k[DxD]$s with $x \in G \setminus N_G(D)$.

(2) *No two of $k[Dg]$s, $gDC_G(D) \in N_G(D, b_D)/DC_G(D)$, are isomorphic.*

However we have had few knowledge on the direct summand N above, which is generated by elements outside the $N_G(D, b_D)$; among the the following two facts are important.

Proposition 1.3 (Linckelmann, [4]). *Let Q, $R \leqslant D$ be isomorphic by $\varphi : R \to Q$. If ${}_{\varphi}(kQ)$ is isomorphic to a direct summand of A, where ${}_{\varphi}(kQ)$ is considered as a (kR, kQ)-bimodule via φ, then φ induces a morphism $(R, b_R) \to (Q, b_R)$ in $\mathscr{F}_{(D,b_D)}(B)$. The converse holds if moreover $C_D(Q)$ is a defect group of b_Q.*

Proposition 1.4 (Kulshammer–Okuyama–Watanabe [3]). *If $k[DgD]$ is isomorphic to a direct summand of A, then, being $P = D^g \cap D$ and $Q = D \cap {}^gD$, we have*

$$(Q, b_Q) \subseteq {}^g(D, b_D).$$

In particular

$${}^g(P, b_P) = (Q, b_Q) \subseteq (D, b_D).$$

Here we add two theorems.

Theorem 1.5 (Okuyama-Sasaki [6]). *Let $(Q, b_Q) \leqslant (D, b_D)$. Assume that (Q, b_Q) is an essential B-subpair. Then $N_G(Q, b_Q)$ has a proper subgroup $M \geqslant N_D(Q)C_G(Q)$ such that $M/QC_G(Q)$ is a strongly p-embedded subgroup of $N_G(Q, b_Q)/QC_G(Q)$.*

Let $x \in N_G(Q, b_Q) \smallsetminus M$. Then

(1) $D^x \cap D = Q$,
(2) *the (kD, kD)-bimodule $k[DxD]$ appears in a direct sum decomposition of A into indecomposable (kD, kD)-bimodules with multiplicity congruent to 1 modulo p.*

We include here the very first step of the proof of the theorem. Since $x \in N_G(Q, b_Q) \smallsetminus M$ and $M/QC_G(Q)$ is strongly p-embedded, we see $(N_D(Q) \cap {}^xN_D(Q))\, C_G(Q)/QC_G(Q) \leqslant (M \cap {}^xM)\,/QC_G(Q)$, which is a p'-group; namely $N_D(Q) \cap {}^xN_D(Q) \leqslant C_D(Q) \leqslant Q$. This implies that $N_{D\cap {}^xD}(Q) = N_D(Q) \cap {}^xN_D(Q) = Q$, meaning that $D \cap {}^xD = Q$.

Note that the set $\{\,(D, b_D)\,\} \cup \{\,(Q, b_Q) \subseteq (D, b_D) \mid (Q, b_Q) \text{ is essential}\,\}$ is a conjugation family for the fusion of subpairs contained in (D, b_D). See for example [1].

Example 1.1. Let $D = \langle x, y \mid x^{2^{n-1}} = y^2 = 1, yxy = x^{-1+2^{n-2}} \rangle$, $n \geqslant 4$, be a semidihedral 2-group. Let

$$E = \langle x^{2^{n-2}}, y \rangle \simeq \text{four-group}, \quad Q = \langle x^{2^{n-3}}, xy \rangle \simeq \text{quaternion group}.$$

Let $(E, b_E), (Q, b_Q) \subseteq (D, b_D)$. Assume that

$$N_G(E, b_E)/C_G(E) \simeq \operatorname{Aut} E,\ N_G(Q, b_Q)/QC_G(Q) \simeq \operatorname{Out} Q.$$

Then the set $\{\,(E, b_E), (Q, b_Q)\,\}$ is the set of essential subpairs in (D, b_D) so that there exist elements $g_0 \in N_G(E, b_E)$ and $g_1 \in N_G(Q, b_Q)$ with $D \cap {}^{g_0}D = E$ and $D \cap {}^{g_1}D = R$ for which we have

$$A \simeq kD \oplus m_E k[Dg_0D] \oplus m_Q k[Dg_1D] \oplus \text{(others)},$$

where m_E and m_Q are odd numbers. Similar things hold for other blocks of tame representation type.

Example 1.2. Let $D = \langle a, b, t \mid a^{2^n} = b^{2^n} = t^2 = 1,\ ab = ba,\ tat = b \rangle, n \geqslant 2$, be a wreathed 2-group; let $c = ab$ and $d = ab^{-1}$. Then $Z(D) = \langle c \rangle$ and $D' = \langle d \rangle$. We let moreover

$$x = a^{2^{n-1}},\ y = b^{2^{n-1}},\ z = c^{2^{n-1}} = xy,$$
$$e = xt,\ f = d^{2^{n-2}} (= (ab^{-1})^{2^{n-2}}),$$
$$U = \langle a, b \rangle,\ Q = \langle e, f \rangle (\simeq Q_8),\ V = \langle e, f, c \rangle (= \langle x, t, c \rangle = Q * Z(D)).$$

Let $(U, b_U), (V, b_V) \subseteq (D, b_D)$; assume, concerning these inertia quotients, that

$$N_G(U, b_U)/C_G(U) \simeq \mathrm{GL}(2,2),\ N_G(V, b_V)/VC_G(V) \simeq \mathrm{GL}(2,2).$$

Then the set $\{ (D, b_D), (V, b_V) \}$ is the set of essential subpairs in (D, b_D) so that there exist elements $g_0 \in N_G(U, b_U)$ and $g_1 \in N_G(V, b_V)$ with $D \cap {}^{g_0}D = U, D \cap {}^{g_1}D = V$ for which we have

$$A \simeq kD \oplus m_E k[Dg_0D] \oplus m_Q k[Dg_1D] \oplus \text{(others)},$$

where m_U and m_V are odd numbers.

In this case there exist direct summands interesting from the point of view of cohomology theory of block ideals.

The following theorem explains such direct summands.

Theorem 1.6 (Sasaki [9]). *Let $(P, b_P), (Q, b_Q) \subseteq (D, b_D)$; assume that $PC_D(P)$ is a defect group of b_P or $QC_D(Q)$ is a defect group of b_Q. For $g \in G$ with ${}^g(P, b_P) = (Q, b_Q)$, if the map*

$$t_g : H^*(D, k) \to H^*(D, k); \zeta \mapsto \mathrm{tr}^D \mathrm{res}_Q {}^g\zeta$$

does not vanish, then the following hold:

(1) $Q = D \cap {}^gD$,
(2) *the (kD, kD)-bimodule $k[DgD]$ is isomorphic to a direct summand of the source algebra A,*

Unfortunately Theorem above says nothing about the multiplicity.

The reason why the map t_g above appears will be explained in the next section.

2 Trace maps for cohomology rings of blocks

Definition 2.1 (Linckelmann [5]). The cohomology ring of B w.r.t D and X is defined to be the $\mathscr{F}_{(D,b_D)}(B, X)$-stable subring of $H^*(D, k)$, where $\mathscr{F}_{(D,b_D)}(B, X)$ is the Brauer category (the fusion system) :

$$H^*(G, B; X) = \{ \zeta \in H^*(D, k) \mid \mathrm{res}_Q \zeta = {}^g\mathrm{res}_Q \zeta \ \forall Q \leqslant D \ \forall g \in N_G(Q, b_Q) \}$$

Theorem 2.1 (Linckelmann [5]). *We have*

$$\begin{array}{ccccc} H^*(D,k) & \xrightarrow{\delta_D} & HH^*(kD) & \xrightarrow{T_X} & HH^*(B) \\ \uparrow & \circlearrowright & \uparrow & \circlearrowright & \uparrow \\ H^*(G,B;X) & \longrightarrow & \{{}_{kD}A_{kD}\text{-stables}\} & \rightarrowtail\!\!\twoheadrightarrow & \{X\text{-stables}\} \end{array},$$

where T_X is the normalized transfer map defined by X.

Conversely

Theorem 2.2 (Sasaki [8]). *For $\zeta \in H^*(D,k)$*

$$\delta_D\zeta \in HH^*(kD) \text{ is } {}_{kD}A_{kD}\text{-stable} \Longrightarrow \zeta \in H^*(G,B;X).$$

Example 2.1 (Kawai–Sasaki [2]). In [2] we calculated cohomology rings of some 2-blocks of rank 2. Here let D be isomorphic to a wreathed 2-group again. Keeping the notation and the assumption on the inertia quotients $N_G(U,b_U)/C_G(U)$ and $N_G(V,b_V)/VC_G(V)$ in Example 1.2, we can define a map $\mathrm{Tr}_D^B : H^*(D,k) \to H^*(D,k)$ such that

$$\mathrm{Im}\,\mathrm{Tr}_D^B = H^*(G,B;X)$$

of the following form

$$\mathrm{Tr}_D^B : \zeta \mapsto \zeta + \mathrm{tr}^D \mathrm{res}_U{}^{g_0}\zeta + \mathrm{tr}^D \mathrm{res}_V{}^{g_1}\zeta + \mathrm{tr}^D \mathrm{res}_T{}^{g_1g_0}\zeta + \mathrm{tr}^D \mathrm{res}_W{}^{g_0g_1}\zeta + \mathrm{tr}^D \mathrm{res}_F{}^{g_1g_0g_1}\zeta,$$

where $g_0 \in N_G(U,b_U)$, $g_1 \in N_G(V,b_V)$ and $T = U\cap{}^{g_0}V$, $W = V\cap{}^{g_1}U$, and$F = V\cap{}^{g_1}U\cap{}^{g_1g_0}V$.

We know that $k[Dg_1g_0D]$ and $k[Dg_0g_1D]$ are isomorphic to direct summands of the source algebra A by applying Theorem 1.6 to the fourth and fifth term of Tr_D^B.

As a matter of fact, Theorem 1.6 was found to see the meaning of this formula.

The (kD,kD)-bimodule A induces a transfer map t on $H^*(D,k)$:

$$\begin{array}{ccc} H^*(D,k) & \xrightarrow{\delta_D} & HH^*(kD) \\ {\scriptstyle t}\downarrow & \circlearrowright & \downarrow{\scriptstyle t_A} \\ H^*(D,k) & \xrightarrow[\delta_D]{} & HH^*(kD) \end{array}.$$

The following would be so natural.

Conjecture.

$$H^*(G,B;X) = t(H^*(D,k)).$$

Example 2.2. If $N_G(D, b_D)$ controls the fusion of subpairs in (D, b_D), then the above does hold. For example

- D is abelian,
- D is normal in G, and so on.

The transfer map t is described as follows:

$$t : H^*(D, k) \to H^*(D, k); \zeta \mapsto \sum_{A \simeq \bigoplus_{DgD} k[DgD]} \operatorname{tr}^D \operatorname{res}_{D \cap {}^g D} {}^g\zeta.$$

Example 2.3. Let D be semidihedral again. Keeping the notation and assumption in Example 1.1 we can describe the trace map induced by the source algebra A:

$$t : \zeta \mapsto \zeta + \operatorname{tr}^D \operatorname{res} E^{g_0}\zeta + \operatorname{tr}^D \operatorname{res}_Q {}^{g_1}\zeta.$$

Moreover it holds that $tH^*(D, k) = H^*(G, B; X)$, namely the conjecture holds.

The same thing hold for another blocks of tame representation type.

Example 2.4. Let D be wreathed again. Keeping the notation and assumption in Examples 1.2 and 2.1 we can describe the trace map induced by the source algebra A:

$$t : \zeta \mapsto \zeta + \operatorname{tr}^D \operatorname{res}_U {}^{g_0}\zeta + \operatorname{tr}^D \operatorname{res}_V {}^{g_1}\zeta + m_T \operatorname{tr}^D \operatorname{res}_T {}^{g_1 g_0}\zeta + m_W \operatorname{tr}^D \operatorname{res}_W {}^{g_0 g_1}\zeta + m_F \operatorname{tr}^D \operatorname{res}_F {}^{g_1 g_0 g_1}\zeta,$$

where $m_T, m_W \geqslant 1$ and $m_F \geqslant 0$. Note, however, that we do not know whether m_T and m_W are odd or even; we know nothing about the integer m_F.

On the other hand, analysis of the fusions of subpairs using the conjugation family reveals that the transfer map t is of the following form

$$t : \zeta \mapsto \zeta + \operatorname{tr}^D \operatorname{res}_U {}^{g_0}\zeta + \operatorname{tr}^D \operatorname{res}_V {}^{g_1}\zeta \\ + m_1 \operatorname{tr}^D \operatorname{res}_T {}^{g_1 g_0}\zeta + m_2 \operatorname{tr}^D \operatorname{res}_W {}^{g_0 g_1}\zeta + m_3 \operatorname{tr}^D \operatorname{res}_F {}^{g_1 g_0 g_1}\zeta,$$

where m_1, m_2, m_3 are integers with $m_1, m_2 \geqslant 1$ and $m_3 \geqslant 0$.

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