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The splitting of cohomology of metacyclic $p$-groups

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Abstract

Let $BP$ be the $p$-complete classifying space of a metacyclic $p$-group $P$. By using stable homotopy splitting of $BP$, we study the decomposition of $H^{\text{even}}(P; \mathbb{Z})/p$ and $CH^*(BP)/p$.

1 Introduction

Let $P$ be a $p$-group and $BP$ be its $p$-completed classifying space of $P$. We study the stable splitting and splitting of cohomology

\[(*) \quad BP \cong X_1 \lor \ldots \lor X_i,\]
\[(**) \quad H^*(P) \cong H^*(X_1) \oplus \ldots \oplus H^*(X_i) \quad (\text{for } * > 0)\]

where $X_i$ are irreducible spaces in the stable homotopy category. Using the answer of the Segal conjecture by Carlsson, the splitting (*) is given by only using modular representation theory by Nishida [Ni], Benson-Feshbach [Be-Fe] and Martino-Priddy [Ma-Pr]. These theorems do not use splittings of cohomology.

In particular, Dietz and Dietz-Priddy [Di], [Di-Pr] gave the stable splitting (*) for groups $P$ with $\text{rank}_p(P) = 2$ for $p \geq 5$. However it was not used splittings (**) of the cohomology $H^*(P)$, and the cohomologies $H^*(X_i)$ were not given there.

In [Hi-Ya 1,2], we gives the cohomology of $H^*(X_i)$ (and hence (**)) for $P = (\mathbb{Z}/p)^2$ and $P = p_1^{1+2}$ the extraspecial $p$ group of order $p^3$ and exponent $p$. Their cohomology $H^*(X_i)$ have very complicated but rich structures, in fact $p_1^{1+2}$ is a $p$-Sylow subgroup of many interesting groups, e.g., $GL_3(\mathbb{F}_p)$ and many simple groups e.g. $J_4$ for $p = 3$.

In this paper, we give the decomposition of

$$H^*(P) = H^*(P; \mathbb{Z})/(p, \sqrt{0}) \quad (\text{and} \quad H^{\text{even}}(P) = H^{\text{even}}(P; \mathbb{Z})/p$$

for metacyclic $p$-groups for odd primes $p$, while in most cases, $H^*(X_i)$ are easily got and seemed not to have so rich structure as $p_1^{1+2}$, because they are not $p$-Sylow subgroups of so interesting groups. Indeed, metacyclic $p$-groups $P$ are
Swan groups, i.e. for all groups $G$ which have a Sylow $p$-subgroup isomorphic to $P$, we have the isomorphism

$$H^*(G) \cong H^*(P)^W \text{ for some } W \subset \text{Out}(P).$$

However, we believe that it becomes quite clear the relations among splittings of different types of metacyclic $p$-groups. (We compute the coarse splitting of $H^*(X_i)$ at first, and next more fine splitting $H^*(X'_i)$, in the case $H^*(P) \cong H^*(P')$).

In the last section, we note the relation to the Chow ring $CH^*(BP)/p$ and $H^{even}(P; \mathbb{Z})/p$, and note that the Chow group of the direct summand $X_i$ is represented by some motive.

## 2 The double Burnside algebra and stable splitting

Let us fix an odd prime $p$ and $k = \mathbb{F}_p$. For finite groups $G_1$, $G_2$, let $A_{\mathbb{Z}}(G_1, G_2)$ be the double Burnside group defined by the Grothendieck group generated by $(G_1, G_2)$-biset. Each element $\Phi$ in $A_{\mathbb{Z}}(G_1, G_2)$ is generated by elements $[Q, \phi] = (G_1 \times (Q, \phi), G_2)$ for some subgroup $Q \leq G_1$ and a homomorphism $\phi : Q \to G_2$. In this paper, we use the notation

$$[Q, \phi] = \Phi : G_1 \geq Q \xrightarrow{\phi} G_2.$$

For each element $\Phi = [Q, \phi] \in A_{\mathbb{Z}}(G_1, G_2)$, we can define a map from $H^*(G_2; k)$ to $H^*(G_1; k)$ by

$$x \cdot \Phi = x \cdot [Q, \phi] = Tr^G_Q \phi^*(x) \text{ for } x \in H^*(G_2; k).$$

When $G_1 = G_2$, the group $A_{\mathbb{Z}}(G_1, G_2)$ has the natural ring structure, and call it the (integral) double Burnside algebra. In particular, for a finite group $G$, we have an $A_{\mathbb{Z}}(G, G)$-module structure on $H^*(G; k)$ (and $H^*(G; \mathbb{Z})/p$).

The following lemma is an easy consequence of Quillen's theorem such that the restriction map

$$H^*(G; \mathbb{Z}/p) \to \lim_V H^*(V; \mathbb{Z}/p)$$

is an $F$-isomorphism (i.e. the kernel and cokernel are nilpotent) where $V$ ranges elementary abelian $p$-subgroups of $G$.

**Lemma 2.1.** Let $\sqrt{0}$ be the nilpotent ideal in $H^*(G; k)$ (or $H^*(G; \mathbb{Z})/p$). Then $\sqrt{0}$ itself is an $A_{\mathbb{Z}}(G, G)$-module.

In this paper we consider, at first, the cohomology modulo nilpotents elements, since it is not so complicated from the above Quillen's theorem. Hence we write it simply

$$H^*(G) = H^*(G; \mathbb{Z})/(p, \sqrt{0}).$$
However we also compute $H^{even}(G;\mathbb{Z})/p$ in §4 below.

By the preceding lemma, we see that $H^*(G)$ has the $A_{\mathbb{Z}}(G, G)$-module structure. (Here note that $A_{\mathbb{Z}}(G, G)$ acts on unstable cohomology.) Throughout this paper, we assume that degree $* > 0$ (or we consider $H^*(-)$ as the reduced theory $\tilde{H}^*$). (We consider $H^*(G)$ as an element in $K_0(\text{Mod}(A_{\mathbb{Z}}(G, G)))$.)

Let $BG = BG_p$ be the $p$-completion of the classifying space of $G$. Recall that \{BG, BG\}_p is the (p-completed) group generated by stable homotopy self maps. It is well known from the Segal conjecture (Carlsson's theorem) that this group is isomorphic to the double Burnside group $A_{\mathbb{Z}}(G_1, G_2)^\wedge$ completed by the augmentation ideal.

Since the transfer is represented as a stable homotopy map $Tr$, an element $\Phi = [Q, \phi] \in A(G_1, G_2)$ is represented as a map $\Phi \in \{BG_1, BG_2\}_p$

$$\Phi : BG_1 \xrightarrow{Tr} BQ \xrightarrow{B\phi} BG_2.$$ 

(Of course, the action for $x \in H^*(G_2)$ is given by $Tr_Q^{G_1} \phi^*(x)$ as stated.)

Let us write

$$A(G_1, G_2) = A_{\mathbb{Z}}(G_1, G_2) \otimes k \ (k = \mathbb{Z}/p).$$

Hereafter we consider the cases $G_i = P$; $p$-groups. Given a primitive idempotents decomposition of the unity of $A(P, P)$

$$1 = e_1 + ... + e_n,$$

we have an indecomposable stable splitting

$$BP \cong X_1 \vee ... \vee X_n \text{ with } e_i BP = X_i.$$ 

In this paper, an isomorphism $X \cong Y$ for spaces means that it is a stable homotopy equivalence.

Recall that

$$M_i = A(P, P)e_i/(\text{rad}(A(P, P)e_i))$$

is a simple $A(P, P)$-module where $\text{rad}(-)$ is the Jacobson ideal. By Wedderburn's theorem, the above decomposition is also written as

$$BP \cong \vee_j (\vee_k X_{jk}) = \vee_j m_j X_{j1} \text{ where } m_j = \dim(M_j)$$

for $A(P, P)e_{jk}/\text{rad}(A(P, P)e_{jk}) \cong M_j$. Therefore the stable splitting of $BP$ is completely determined by the idempotent decomposition of the unity in the double Burnside algebra $A(P, P)$.

For a simple $A(P, P)$-module $M$, define a stable summand $X(M)$ by

$$e_M = \sum_{M_i \cong M} e_i, \quad X(M) = \vee_{M_i \cong M} X_{jk} = e_M BP.$$ 

Here $X(M)$ is only defined in the stable homotopy category. (So strictly, the cohomology ring $H^*(X(M))$ is not defined.) However we define $H^*(X(M))$ by

$$H^*(X(M)) = H^*(P) \cdot e_M \quad (= e_M^* H^*(P) \text{ stably})$$

where we think $e_M \in A(P, P)$ (rather than $e_M \in \{BP, BP\}$).
Lemma 2.2. Given a simple $A(P, P)$-module $M$, there is a filtration of $H^*(X(M))$ such that the associated graded ring $grH^*(X(M))$ is isomorphic to a sum of $M$, i.e., (for $* > 0$)

$$grH^*(X(M)) \cong \bigoplus_{i=1} M[k_i], \quad 0 \leq k_1 \leq ... \leq k_s \leq ...$$

where $[k_s]$ is the operation ascending degree $k_s$.

From Benson-Feshbach [Be-Fe] and Martino-Priddy [Ma-Pr], it is known that each simple $A(P, P)$-module is written as

$$S(P, Q, V) \text{ for } Q \leq P, \text{ and } V : \text{ simple } k[Out(Q)] - \text{ module.}$$

(In fact $S(P, Q, V)$ is simple or zero. ) Thus we have the main theorem of stable splitting of $BP$.

Theorem 2.3. (Benson-Feshbach [Be-Fe], Martino-Priddy [Ma-Pr]) There are indecomposable stable spaces $X_{S(P, Q, V)}$ for $S(P, Q, V) \neq 0$ such that

$$BP \cong \vee X(S(P, Q, V)) \cong \vee(dimS(P, Q, V))X_{S(P, Q, V)}.$$ 

3 Metacyclic groups for $p \geq 3$

In this section, we consider metacyclic $p$ groups $P$ for $p \geq 3$

$$0 \rightarrow \mathbb{Z}/p^m \rightarrow P \rightarrow \mathbb{Z}/p^n \rightarrow 0.$$ 

These groups are represented as

$$(*) \quad P = \langle a, b | a^{p^m} = 1, a^{p^{m'}} = b^{p^n}, [a, b] = a^{r^f} \rangle \quad r \neq 0 \mod(p).$$

It is known by Thomas [Th], Huebuschmann [Hu] that $H^{even}(P; \mathbb{Z})$ is generated by Chern classes of complex representations. Let us write

$$\begin{cases}
y = c_1(\rho), \quad \rho : P \rightarrow P/(a) \rightarrow \mathbb{C}^* \\
v = c_{p^{m-\ell}}(\eta), \quad \eta = Ind_H^P(\xi), \quad \xi : H = \langle a, b^{p^{m-\ell}} \rangle \rightarrow \langle a \rangle \rightarrow \mathbb{C}^*
\end{cases}$$

where $\rho, \xi$ are nonzero linear representations. Then $H^{even}(P; \mathbb{Z})$ is generated by

$$y, c_1(\eta), c_2(\xi), ..., c_{p^{m-\ell}}(\eta) = v.$$ 

(Lemma 3.5 and the explanation just before this lemma in [Ya1].) We can see

$$c_1(\eta) = 0, ..., c_{p^{m-\ell-1}}(\eta) = 0 \quad \text{in } H^*(P) = H^*(P; \mathbb{Z})/(p, \sqrt{0}).$$

By using Quillen's theorem and the fact that $P$ has just one conjugacy class of maximal abelian $p$-subgroups, we can prove
Theorem 3.1. (Theorem 5.45 in [Ya1]) For any metacyclic $p$-group $P$ in (*) with $p \geq 3$, we have a ring isomorphism

$$H^*(P) \cong k[y, v], \quad |v| = 2p^{m-\ell}.$$ 

We now consider the stable splitting.

(I) Non split cases. For a nonsplit metacyclic groups, it is proved that $BP$ itself is irreducible [Di].

(II) Split cases with $(\ell, m, n) \neq (1, 2, 1)$. We consider a split metacyclic group.

it is written as $P = M(\ell, m, n) = \langle a, b | a^{p^{m}} = b^{p^{n}} = 1, [a, b] = a^{p^{\ell}} \rangle$ for $m > \ell \geq \max(m-n, 1)$.

The outer automorphism is the semidirect product

$$Out(P) \cong (p-group) : \mathbb{Z}/(p-1).$$

The $p$-group acts trivially on $H^*(P)$, and $j \in \mathbb{Z}/(p-1)$ acts on $a \mapsto a^{j}$ and so acts on $H^*(P)$ as $j^*: v \mapsto jv$. There are $p-1$ simple $\mathbb{Z}/(p-1)$-modules $S_i \cong k\{v^i\}$. We consider the decomposition by idempotents for $Out(P)$. Let us write $Y_i = e_{S_i}BP$ and

$$H^*(P) = H^*(S_i) \cong (dim(S_i))H^*(Y_i) \subset H^*(P).$$

Hence we have the decomposition for $Out(P)$-idempotents

$$H^*(Y_i) = H_i(P) \cong k[y, V]\{v^i\}, \quad V = v^{p-1}.$$ 

Here we used the notation such that $A\{a, b, ...,\}$ means the $A$-free module generated by $a, b, ...,\$.

We assume $P \neq M(1, 2, 1)$. By Dietz, we have splitting

$$BP \cong \bigvee_{i=0}^{p-2} X_i \lor \bigvee_{i=0}^{p-2} L(1, i).$$

Here we write $X_i = e_{S(P, P, S_i}BP$ identifying $S_i$ as the $A(P, P)$ simple module (but not the simple $Out(P)$-module).

The summand $L(1, i)$ is defined as follows. Recall that $H^*(\langle b \rangle) \cong k[y]$. The outer automorphism group is $Out(\langle b \rangle) \cong (\mathbb{Z}/p^n)^*$ and its simple $k$ modules are $S_i' = k\{y^i\}$ for $0 \leq i \leq p-2$. Hence we can decompose

$$B\langle b \rangle \cong \bigvee_{i=0}^{p-2} L(1, i), \quad H^*(L(1, i)) \cong k[Y]\{y^i\} \quad \text{with} \quad Y = y^{p-1}.$$ 

Next we consider $L(1, i)$ as a split summand in $BP$ as follows. (Consider the $A(P, P)$-simple module $S(P, \langle b \rangle, S_i'$. Let $\Phi \in A(P, P)$ be the element defined by the map $\Phi : P \rightarrow \langle b \rangle \subset P$ which induced the isomorphism

$$H^*(P)\Phi \cong H^*(\bigvee_{i=0}^{p-2} L(1, i)) \cong k[y].$$

Thus we can show (since $k[y]$ is invariant under elements in $Out(P)$)

$$Y_i \cong \left\{ \begin{array}{ll} X_i & i \neq 0 \\ X_0 \lor \bigvee_{j=0}^{p-2} L(1, j) & i = 0. \end{array} \right.$$
Theorem 3.2. Let $P$ be a split metacyclic group with $(\ell, m, n) \neq (1, 2, 1)$. Then we have

$$H^*(X_i) \cong \begin{cases} k[y, V]\{v^i\} & i \neq 0 \\ k[y, V]\{V\} & i = 0. \end{cases}$$

Proof. For $i \neq 0$, we have $H^*_i(P) = H^*(Y_i) \cong H^*(X_i)$. Let us use the notation that $A \oplus B = C$ means $A \cong B \oplus C$. Then we see

$$H^*(X_0) \cong H^*(Y_0) \ominus H^*(\fbox{Error::0x0000}L(1,j)) \cong k[y, V] \ominus k[y] \cong k[y, V]\{V\}.$$

$$\square$$

(III) Split metacyle group with $(\ell, m, n) = (1, 2, 1)$.

This case $P = p^{1+2}$ and its cohomology is the same as (II). But the splitting is given

$$BP \cong \bigvee_{i=0}^{p-2} L(2, i) \bigvee \bigvee_{i=0}^{p-2} L(1, i).$$

Detailed explanation for $L(2, i)$ see [M-P], [Hi-Ya1]. Let $H = \langle b, a^p \rangle$ the maximal elementary abelian subgroup. The space $L(2, i)$ is the transfer $(Tr : BH \to BG)$ image of the same named summand of $BH$. By using the double coset formula

$$Tr_H^{P}(u^{p-1})|_H = \sum_{i=0}^{p-1} (u + iy)^{p-1} = -y^{p-1}$$

taking the generator $u$ in $H^*([b, a^p]) \cong k[y, u]$.

The group $P$ has just one conjugacy class $H$ of the maximal abelian $p$-groups. Hence by Quillen's theorem, we have

$$Tr_H^{P}(u^{p-1}) = -y^{p-1} \text{ in } H^*(P) = H^*(P; \mathbb{Z}))/(p, \sqrt{0}).$$

We consider an element $\Phi \in A(P, P)$ defined by $\Phi : P \geq H \subset P$. Then we see

$$Im(Tr_H^{P}H^*(H)) \supset H^*(P)\Phi = H^*(\bigvee_{i=0}^{p-2} L(2, i)).$$

Thus we have the isomorphism

$$Y_i \cong \begin{cases} X_i \bigvee L(2, i) & i \neq 0 \\ X_0 \bigvee L(2, 0) \bigvee \bigvee_{j=0}^{p-2} L(1, j) & i = 0. \end{cases}$$

To compute cohomology of irreducible components $X_i$ and $L(2, j)$, we recall the Dickson algebra

$$\mathbb{D}A = k[y, u]^{GL_2(\mathbb{Z}/p)} \cong k[D_1, D_2] \text{ with } D_1 = Y^p + V, \ D_2 = YV.$$
\[ \mathbb{CB} = k[Y, D_2] \cong \mathbb{DA}\{1, Y, ..., Y^{p-1}\}. \]

Hence \( \mathbb{CA} \cong \mathbb{DA} \oplus \mathbb{CB}\{Y\} \). Then it is known (see [Hi-Ya1] for details)

\[ H^*(L(2, i)) \cong \begin{cases} \mathbb{CB}\{Yd_2^i\} & i \neq 0 \\ \mathbb{CB}\{YD_2\} & i = 0. \end{cases} \]

**Theorem 3.3.** Let \( P = M(1, 2, 1) \cong p_1^{1+2} \). Then we have

\[ H^*(X_i) \cong \begin{cases} \mathbb{CA}\{1, \hat{y}^i, y^{p-2}\}\{v^i\} \oplus \mathbb{DA}\{d_2^i\} & i > 0 \\ \mathbb{CA}\{y, ..., y^{p-2}\}\{V\} \oplus \mathbb{DA} & i = 0. \end{cases} \]

**Proof.** Let \( i \neq 0 \). We see

\[ H^*(Y_i) \cong k[y, V]\{v^i\} \cong \mathbb{CA}\{1, y, y^{p-2}\}\{v^i\}. \]

The cohomology of the summand \( X_i \) is

\[ H^*(X_i) \cong H^*(Y_i) \oplus H^*(L(2, i)) \cong (\mathbb{DA} \oplus \mathbb{CB}\{Y\})\{v^i\}\{1, ..., y^{p-2}\} \oplus \mathbb{CB}\{Yd_2^i\}. \]

Here \( v^i y^i = d_2^i \) we have the isomorphism in the theorem for \( i \neq 0 \).

Next we consider in the case \( i = 0 \). We have

\[ H^*(X_0) \cong H^*(Y_0) \oplus H^*(\vee_j L(1, j)) \oplus H^*(L(2, 0)) \cong \mathbb{CA}\{1, y, ..., y^{p-2}\}\{V\} \oplus \mathbb{CB}\{YD_2\} \cong \mathbb{CA}\{y, ..., y^{p-2}\}\{V\} \oplus B \]

where

\[ B = \mathbb{CA}\{V\} \oplus \mathbb{CB}\{YD_2\} \cong \mathbb{CA} \oplus H^*(L(1, 0)) \oplus H^*(L(2, 0)). \]

We can see \( B \cong \mathbb{DA} \) by Lemma 3.4 below. \hfill \square

**Lemma 3.4.** Let \( M(2) = L(2, 0) \vee L(1, 0) \) (as the usual notation of the homotopy theory). Then we have

\[ H^*(M(2)) \cong \mathbb{CB}\{Y\}, \quad \mathbb{CA} \cong \mathbb{DA} \oplus H^*(M(2)). \]

**Proof.** We can compute

\[ H^*(M(2)) \cong k[Y] \oplus \mathbb{CB}\{YD_2\} \cong k[Y] \oplus k[Y, D_2]\{YD_2\} \cong (k[Y] \oplus k[Y, D_2]\{D_2\})\{Y\} \cong \mathbb{CB}\{Y\} \quad (\text{assumed } * > 0). \]

Since \( \mathbb{CA} \cong \mathbb{DA} \oplus \mathbb{CB}\{Y\} \), we have the last isomorphism in this lemma. \hfill \square
4 Nilpotent elements

Let us write $H^{even}(X;\mathbb{Z})/p$ by simply $H^{even}(X)$ so that

$$H^{even}(G) = H^{*}(G) \oplus N(G)$$

where $N(G)$ is the nilpotent ideal in $H^{even}(G)$.

Since $BP$ is irreducible in nonsplit cases, we only consider in split cases,

$$P = M(\ell, m, n) = \langle a, b|a^{p^{m}} = b^{p^{n}} = 1, [a, b] = a^{p^{\ell}} \rangle$$

for $m > \ell \geq \max(m - n, 1)$.

(I) Split metacyclic groups with $\ell > m - n$.

By Diethelm [Di], its mod $p$-cohomology is

$$H^{*}(P;\mathbb{Z}/p) \cong k[y, u] \otimes \Lambda(x, z) |y| = |u| = 2, |x| = |z| = 1.$$ 

Of course all elements in $H^{*}(P;\mathbb{Z})$ are (higher) $p$-torsion. The additive structure of $H^{*}(P;\mathbb{Z}/p)$ is decided by that of $H^{*}(P;\mathbb{Z}/p)$ by the universal coefficient theorem. Hence we have additively (but not as rings)

$$H^{*}(P;\mathbb{Z}/p) \cong H^{*}(\mathbb{Z}/p \times \mathbb{Z}/p;\mathbb{Z}) \cong k[y, u]\{1, \beta(xz) = yz - ux\}.$$ 

Since $H^{*}(P)$ is multiplicatively generated by $y$ and $v$ with $|v| \geq 2p$ from Theorem 4.1, the element $u$ is not integral class (i.e. $u \notin Im(\rho)$ for $\rho : H^{*}(P;\mathbb{Z}) \to H^{*}(P;\mathbb{Z}/p)$). Therefore $xz$ is an integral class since

$$H^{even}(P;\mathbb{Z}/p) \cong k[y, u]\{1, xz\}.$$ 

In $H^{4}(P;\mathbb{Z}/p)$, the elements $y^{2}, yxz$ are integral but $u^{2}$ is not. Note that $dim(H^{4}(P;\mathbb{Z})/p) = 3$ and so $xz$ must be integral. Inductively, we see that

$$x_{1} = xz, x_{2} = xzu, ..., x_{p^{m-\ell}-1} = xzu^{p^{m-\ell}-2}$$

are integral classes.

The element $u \in H^{2}(P;\mathbb{Z}/p)$ is defined [Dim] using the spectral sequence

$$E^{\infty}_{s, t} \cong H^{*}(P/\langle a \rangle; H^{*}(\langle a \rangle;\mathbb{Z}/p)) \Rightarrow H^{*}(P;\mathbb{Z}/p).$$

In fact $u = [u'] \in E^{0,2}_{\infty}$ identifying $H^{2}(\langle a \rangle;\mathbb{Z}/2) \cong k\{u'\}$. Hence $u|\langle a \rangle = u'$. On the other hand $v|\langle a \rangle = (u')^{p^{m-\ell}}$ because $v = c_{p^{m-\ell}}(\eta)$ and the total Chern class is

$$\sum c_{i}(\eta)|\langle a \rangle = (1 + u')^{p^{m-\ell}} = 1 + (u')^{p^{m-\ell}}.$$ 

Therefore we see $v = u^{p^{m-\ell}} \mod(y, xz)$ in $H^{*}(P;\mathbb{Z}/p)$. Thus we get

**Theorem 4.1.** Let $P$ be a split metacyclic group $M(\ell, m, n)$ with $\ell > m - n$. Then we have

$$H^{even}(P) \cong k[y, v]\{1, x_{1}, ..., x_{p^{m-\ell}-1}\} \text{ with } x_{i}x_{j} = 0,$$

that is $N(P) \cong k[y, v]\{x_{1}, ..., x_{p^{m-\ell}-1}\}$. 

These $x_i$ are also defined by Chern classes (from the arguments just before Theorem 4.1), and as $\text{Out}(P)$ modules, $x_i \cong S_j$ when $i = j \mod(p-1)$. Therefore we have

**Corollary 4.2.** Let $P$ be a split metacyclic group $M(\ell, m, n)$ with $\ell > m - n$. Then

$$H^{ev}(X_i) \cong H^*(X_i) \oplus k[y, V]\{v^r x_s| r+s = i \mod(p-1)\}$$

where $1 \leq s \leq p^{m-\ell} - 1$.

(II) Split metacyclic groups $P = M(\ell, m, n)$ with $\ell = m - n$.

By also Diepholm, its mod $p$-cohomology is

$$H^*(P; \mathbb{Z}/p) \cong k[y, v'] \otimes \Lambda(a_1, a_{p-1}, b, w)/(a_i a_j = a_i y = a_i w = 0)$$

where $|a_i| = 2i - 1$, $|b| = 1$, $|y| = 2$, $|w| = 2p - 1$, $|v'| = 2p$. So we see

$$H^*(P; \mathbb{Z}/p)/\sqrt{0} \cong k[y, v'].$$

Note that additively $H^*(P; \mathbb{Z}/p) \cong H^*(_{p+2}; \mathbb{Z}/p)$, which is well known. In particular, we get additively

$$H^{ev}(P) \cong (k[y] \oplus k\{x_1, ..., x_{p-1}\}) \otimes k[v'] \quad (\text{with } x_i = a_i b)$$

$$\cong (k[y] \oplus k\{x_1, ..., x_{p-1}\}) \otimes k[v]{1, v', ..., (v')^{p^{m-\ell-1}-1}}.$$

Therefore $H^{ev}(P)$ is additively isomorphic to

$$H^{ev}(P) \cong \oplus_{i,j} k[v]\{a_i b (v')^j\} \oplus \oplus_{j} k[v, y]\{(v')^j\}$$

where $1 \leq i \leq p-1$ and $0 \leq j \leq p^{m-\ell-1} - 1$. Here $a_i b (v')^j$ is nilpotent and hence integral class and let $x_{jp+i} = a_i b (v')^j$. The element $(v')$ is not nilpotent and we can take as the integral class $wb$ of dimension $2p$. Let us write $x_{pj} = wb(v')^{j-1}$. Thus we have

**Theorem 4.3.** Let $P$ be a split metacyclic group $M(\ell, m, n)$ with $\ell = m - n$. Then

$$H^{ev}(P) \cong k[y, v] \oplus k[y, v]\{x_i| i = 0 \mod(p)\} \oplus k[v]\{x_i| i \neq 0 \mod(p)\}$$

where $i$ ranges $1 \leq i \leq p^{m-\ell} - 1$. Here the multiplications are given by $x_i x_j = 0$, $yx_k = 0$ for $k \neq 0 \mod(p)$.

Hence we have

**Corollary 4.4.** Let $P = M(\ell, m, n)$ for $\ell = m - n$. Then

$$H^{ev}(X_i) = H^*(X_i) \oplus k[y, V]\{v^r x_s| s = 0 \mod(p), r+s = i \mod(p-1)\}$$

$$\oplus k[V]\{v^r x_s| s \neq 0 \mod(p), r+s = i \mod(p-1)\}.$$
Let $\text{CH}^*(BG)$ be the Chow ring of the classifying space $BG$ (see §5 below for the definition). The following theorem is proved by Totaro, with the assumption $p \geq 5$ but without the assumption of transferred Euler classes (since it holds when $p \geq 5$).

**Theorem 4.5.** (Theorem 14.3 in [To2]) Suppose $\text{rank}_p P \leq 2$ and $P$ has a faithful complex representation of the form $W \oplus X$ where $\dim(W) \leq p$ and $X$ is a sum of 1 dimensional representation. Moreover $H^{\text{ev}}(P)$ is generated by transferred Euler classes. Then we have $\text{CH}^*(P)/p \cong H^{\text{ev}}(P)$.

**Proof.** (See page 179-180 in [To2].) First note the cycle map is surjective, since $H^{\text{ev}}(P)$ is generated by transferred Euler classes. Using the Riemann-Roch theorem without denominators, we can show

$$CH^*(BP)/p \cong H^{2*}(P;\mathbb{Z})/p \text{ for } * \leq p.$$ 

By the dimensional conditions of representations $W \oplus X$ and Theorem 12.7 in [To2], we see the following map

$$CH^*(BP)/p \to \prod_V CH^*(BV) \otimes_{\mathbb{Z}/p} CH^{\leq p-1}(BC_P(V))$$

$$\to \prod_V H^*(V;\mathbb{Z}/p) \otimes_{\mathbb{Z}/p} H^{\leq 2(p-1)}(C_G(V);\mathbb{Z}/p)$$

is also injective. Here $V$ ranges elementary abelian $p$-subgroups of $P$ and $C_P(V)$ is the centralizer group of $V$ in $P$. So we see that the cycle map is also injective. $\square$

Therefore we have

**Corollary 4.6.** Let $P$ be the metacycle group $M(\ell, m, n)$ with $m - \ell = 1$. Then $\text{CH}^*(BP)/p \cong H^{\text{ev}}(BG)$.

Totaro computed $\text{CH}^*(BP)/p$ for split metacyclic groups with $m - \ell = 1$ in 13.12 in [To]. When $P$ is the extraspecial $p$-groups of order $p^3$, the above result is first proved in [Ya2].

For a cohomology theory $h^*(-)$, define the $h^*(-)$-theory toplogical nilpotence degree $d_0(h^*(BG))$ to be the least nonnegative integer $d$ such that the map

$$h^*(BG)/p \to \prod_V h^*(BG) \otimes h^{\leq d}(BC_G(V))/p$$

is injective. Note that $d_0(H^*(BG;\mathbb{Z})) \leq d_0(H^*(BG;\mathbb{Z}/p))$.

Totaro computed in the many cases of groups $P$ with $\text{rank}_p P = 2$. In particular, if $P$ is a split metacyclic $p$-group for $p \geq 3$, then $d_0(H^*(BP;\mathbb{Z}/p)) = 2$ and $d_0(\text{CH}^*(BP)) = 1$ when $m - \ell = 1$. Hence $d_0(H^*(P;\mathbb{Z})) = 2$ for these split metacyclic groups $P$ (for $p \geq 3$).

This fact also show easily from Theorem 8.1 and 8.2. Consider the restriction map

$$H^{\text{ev}}(P) \to H^{\text{ev}}(V) \otimes H^2(P) \text{ (where } V = \langle a^{p^{m-1}} \rangle \subset Z(P) : \text{center})$$
induced the product map $V \times P \to P$. Then the element defined in Theorem 8.1, 8.3
\[ c_j = xzu^{j-1} - \sum_i xzu^i \otimes u^{j-i-1} \equiv u^{j-1} \otimes x_1 \neq 0 \in H^{ev}(V) \otimes H^2(P) \]
for $\ell > m - n$. For $\ell = m - n$ and $n = 1$, we also see that the nilpotent element $x_j$ maps to $ab \otimes u^{j-1}$ (or $wb \otimes u^{j-p-1}$ for $j = 0 \mod(p)$) in $H^{ev}(V) \otimes H^2(P)$. (From the proof of Theorem 2 in [Dim], we see $w|V = zu^{p-1}$.)

5 Motives and stable splitting

For a smooth projective algebraic variety $X$ over $\mathbb{C}$, let $CH^*(X)$ be the Chow ring generated by algebraic cycles of codimension $*$ modulo rational equivalence. There is a natural (cycle) map
\[ cl : CH^*(X) \to H^{2*}(X(\mathbb{C}); \mathbb{Z}). \]
where $X(\mathbb{C})$ is the complex manifold of $\mathbb{C}$-rational points of $X$.

Let $V_n$ be a $G - \mathbb{C}$-vector space such that $G$ acts freely on $V_n - S_n$, with $\text{codim}_{V_n} S_n = n$. Then it is known that $(V_n - S_n)/G$ is a smooth quasi-projective algebraic variety. Then Totaro define the Chow ring of $BG$ ([To1]) by
\[ CH^*(BG) = \lim_{n \to \infty} CH^*((V_n - S_n)/G). \]
(Note that $H^*(G, \mathbb{Z}) = \lim_{n \to \infty} H^*((V_n - S_n)/G)$ also.) Moreover we can approximate $\mathbb{P}^\infty \times BG$ by smooth projective varieties from Godeaux-Serre arguments ([To1]).

Let $P$ be a $p - group$. By the Segal conjecture, the $p$-complete automorphism $\{BP, BP\}$ of stable homotopy groups is isomorphic to $A(P, P)_{\mathbb{Z}_p}$, which is generated by transfers and map induced from homomorphisms. Since $CH^*(BP)$ also has the transfer map, we see $CH^*(BP)$ is an $A(P, P)$-module. For an $A(P, P)$-simple module $S$, recall $e_S$ is the corresponding idempotent element and $X_S = e_S BP$ the irreducible stable homotopy summand. Let us define
\[ CH^*(X_S) = e_S CH^*(BP) \]
so that the following diagram commutes.
\[ \begin{array}{ccc}
CH^*(BP)_{(p)} & \xrightarrow{cl} & H^{2*}(BP; \mathbb{Z}_{(p)}) \\
\downarrow & & \downarrow \\
CH^*(X_S)_{(p)} & \xrightarrow{cl} & H^{2*}(X_S; \mathbb{Z}_{(p)}).
\end{array} \]

For smooth schemes $X, Y$ over a field $K$, let $Cor(X, Y)$ be the group of finite correspondences from $X$ to $Y$ (which is a $\mathbb{Z}_p$-module on the set of closed
subvarieties of \( X \times_K Y \) which are finite and surjective over some connected component of \( X \). Let \( Cor(K, \mathbb{Z}_p) \) be the category of smooth schemes whose groups of morphisms \( \text{Hom}(X, Y) = Cor(X, Y) \). Voevodsky constructs the triangulated category \( DM = DM(K, \mathbb{Z}_p) \) which contains the category \( Cor(K, \mathbb{Z}_p) \) (and limit of objects in \( Cor(K, \mathbb{Z}_p) \)).

**Theorem 5.1.** Let \( S \) be a simple \( A(P, P) \)-module. Then there is a motive \( M_S \in DM(\mathbb{C}, \mathbb{Z}_p) \) such that

\[
CH^*(M_S) \cong CH^*(X_S) = e_S CH^*(BP).
\]

**Remark.** Of course \( M_S \) is (in general) not irreducible, while \( X_S \) is irreducible.

The category \( \text{Chow}^{eff}(K, \mathbb{Z}_p) \) of (effective) pure Chow motives is defined as follows. An object is a pair \((X, p)\) where \( X \) is a projective smooth variety over \( K \) and \( p \) is a projector, i.e. \( p \in \text{Mor}(X, X) \) with \( p^2 = p \). Here a morphism \( f \in \text{Mor}(X, Y) \) is defined as an element \( f \in CH^{\dim(Y)}(X \times Y)_{\mathbb{Z}_p} \). We say that each \( M = (X, p) \) is a (pure) motive and define the Chow ring \( CH^*(M) = p^*CH^*(X) \), which is a direct summand of \( CH^*(X) \). We identify that the motive \( M(X) \) of \( X \) means \((X, id.)\). (The category \( DM(K, \mathbb{Z}_p) \) contains the category \( \text{Chow}^{eff}(K, \mathbb{Z}_p) \)).

It is known that we can approximate \( \mathbb{P}^\infty \times BP \) by smooth projective varieties from Godeaux-Serre arguments ([To1]). Hence we can get the following lemma since

\[
CH^*(X \times \mathbb{P}^\infty) \cong CH^*(X)[y] \quad |y|=1.
\]

**Lemma 5.2.** Let \( S \) be a simple \( A(P, P) \)-module. There are pure motives \( M_S(i) \in \text{Chow}^{eff}(\mathbb{C}, \mathbb{Z}_p) \) such that

\[
\lim_{n \to \infty} CH^*(M_S(i)) \cong CH^*(X_S)[y], \quad \deg(y) = 1.
\]

**Corollary 5.3.** Let \( P \) be a split metacycle \( p \)-group \( M(\ell, m, n) \) with \( m - \ell = 1 \). Then for each simple \( A(P, P) \)-module \( S \), there is a motive \( M_S \in DM(\mathbb{C}, \mathbb{Z}_p) \) with

\[
CH^*(M_S)/p \cong H^{ev}(X_S) = H^{even}(X_S; \mathbb{Z})/p.
\]

**References**


