Title: Correspondence functors (Cohomology theory of finite groups and related topics)

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Citation: 数理解析研究所講究録 (2015), 1967: 59-67

Issue Date: 2015-10

URL: http://hdl.handle.net/2433/224259

Type: Departmental Bulletin Paper

Publisher: Kyoto University
Correspondence functors

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Abstract: This is a report on some recent joint work with Jacques Thévenaz, which appears in [1] and [2]. It is an expanded version of a talk given at the RIMS workshop Cohomology of finite groups and related topics, February 18-20, 2015.

The first part of this joint work is presented in Thévenaz’s report, in these proceedings.

1. Introduction

1.1. This is an exposition of a joint work in progress with Jacques Thévenaz¹, on the representation theory of finite sets, by which we mean the following: let \( C \) denote the category in which objects are finite sets. For any two finite sets \( X \) and \( Y \), the set of morphisms from \( X \) to \( Y \) in \( C \) is the set of all correspondences from \( X \) to \( Y \), i.e. the set of subsets of \( Y \times X \). We denote² this set by \( \mathcal{C}(Y, X) \). A correspondence from \( X \) to itself is called a relation on \( X \). The composition of correspondences is defined as follows: for finite sets \( X, Y, Z \), for \( R \subseteq Y \times X \) and \( S \subseteq Z \times Y \)

\[
S \circ R = \{(z, x) \in Z \times X \mid \exists y \in Y, (z, y) \in S \text{ and } (y, x) \in R\}
\]

The identity morphism of the finite set \( X \) is the diagonal

\[
\Delta_X = \{(x, x) \mid x \in X\} \subseteq X \times X
\]

We now fix a commutative ring \( k \) (with identity element 1), and we consider functors from \( C \) to the category \( k\text{-Mod} \) of \( k \)-modules. Equivalently, we first introduce the \( k \)-linearization \( kC \) of \( C \), i.e. the category with the same objects as \( C \), but in which the set of morphisms from \( X \) to \( Y \) is the free \( k \)-module \( kC(Y, X) \) on the set \( C(Y, X) \), and composition is \( k \)-linearly extended from composition in \( C \). Then we consider correspondence functors over \( k \), i.e. \( k \)-linear functors from \( kC \) to \( k\text{-Mod} \). These functors are the objects of a category \( \mathcal{F}_k \), in which morphisms are natural transformations of functors. The category \( \mathcal{F}_k \) is an abelian \( k \)-linear category.

¹cf. Jacques Thévenaz’s report in these Proceedings.
²We emphasize that our notation is opposite to the usual notation \( \mathcal{C}(X, Y) \) of category theory.
1.2. Examples: For any finite set $E$, the representable functor $Y_{E,k}$ sending a finite set $X$ to the set $\text{Hom}_{kC}(E, X) = kC(X, E)$ is a projective object of $\mathcal{F}_k$, by the Yoneda Lemma. In particular:

- When $E = \emptyset$, then $Y_{E,k}(X) \cong k$ for any finite set $X$, and for any correspondence $U \subseteq Y \times X$ from $X$ to a finite set $Y$, the map $Y_{E,k}(U) : Y_{E,k}(X) \to Y_{E,k}(Y)$ is the identity map of $k$. In other words, the functor $Y_{\emptyset,k}$ is the constant functor equal to $k$ everywhere.

- When $E = \bullet$ is a set of cardinality one, then for any finite set $X$, the module $Y_{E,k}(X)$ is the free $k$-module with basis the set $2^X$ of subsets of $X$. Hence $Y_{*,k}$ is the functor of subsets.

- The Yoneda Lemma implies that $\text{End}_{\mathcal{F}_k}(Y_{E,k})$ is isomorphic to the algebra $kC(E, E)$ of all relations on $E$. In particular, when $R$ is a preorder on $E$, i.e. $R$ is a reflexive and transitive relation on $E$, or equivalently $\Delta_E \subseteq R = R^2$, then we get a direct summand $Y_{E,k}R$ of $Y_{E,k}$ defined on a finite set $X$ by $Y_{E,k}R(X) = kC(X, E)R$. The functor $Y_{E,k}R$ is a projective object of $\mathcal{F}_k$.

2. Functors associated to lattices

2.1. The previous examples are special cases of a more general construction that we now introduce. Recall that a lattice $T = (T, \vee, \wedge)$ is a poset in which any pair $\{x, y\}$ of elements has at least upper bound $x \vee y$ (called the join of $x$ and $y$) and a greatest lower bound $x \wedge y$ (called the meet of $x$ and $y$). A finite lattice $T$ admits a smallest element $0_T$ (the meet of all elements of $T$) and a largest element $1_T$ (the join of all elements of $T$).

2.2. Definition: Let $T$ be a finite lattice.

- When $X$ is a finite set, let $F_T(X) = k(T^X)$ denote the free $k$-module with basis the set $T^X$ of all maps from $X$ to $T$.

- When $U \subseteq Y \times X$ is a correspondence from $X$ to a finite set $Y$, let $F_T(U) : F_T(X) \to F_T(Y)$ be the $k$-linear map sending $\varphi : X \to T$ to the map $F_T(U)(\varphi) : Y \to T$, also denoted by $U \varphi$, defined by

$$\forall y \in Y, \ (U \varphi)(y) = \bigvee_{(y,x) \in U} \varphi(x) \ .$$
Recall that a lattice $T$ is called *distributive* if $\lor$ is distributive with respect to $\land$ or, equivalently, if $\land$ is distributive with respect to $\lor$.

**2.3. Theorem:** Let $T$ be a finite lattice. Then $F_T$ is a correspondence functor. Moreover $F_T$ is projective in $\mathcal{F}_k$ if and only if $T$ is distributive.

This result motivates the following definition:

**2.4. Definition:** Let $k\mathcal{L}$ denote the following category:

- The objects of $k\mathcal{L}$ are the finite lattices.
- For two finite lattices $T$ and $T'$, the set of morphisms from $T$ to $T'$ in $k\mathcal{L}$ is the free $k$-module with basis the set of all maps $f : T \to T'$ which respect the join operation, i.e. such that

$$\forall A \subseteq T, \quad f(\lor_t t) = \lor_{t \in A} f(t).$$

- The composition of morphisms in $k\mathcal{L}$ is the $k$-linear extension of the composition of maps.

**2.5. Remark:** Note that a map from a finite lattice $T$ to a finite lattice $T'$ which respects the join operation need not respect the meet operation. On the other hand, it has to send the smallest element $0_T$ of $T$ (which is equal to the join $\lor_t t$) to the smallest element $0_{T'}$ of $T'$.

**2.6. Theorem:** The assignment $T \mapsto F_T$ is a fully faithful $k$-linear functor from $k\mathcal{L}$ to $\mathcal{F}_k$.

**2.7.** We will conclude this section by introducing a canonical subfunctor $H_T$ of $F_T$, for any finite lattice $T$, which will be fundamental in the explicit description of simple correspondence functors.

First recall that an element $e$ of a finite lattice $T$ is called *irreducible* if for any subset $A$ of $T$, the equality $e = \lor_{t \in A} t$ implies that $e \in A$. In other words $e \neq 0_T$, and if $e = x \lor y$ for $x, y \in T$, then $e = x$ or $e = y$. We denote by $\text{Irr}(T)$ the set of irreducible elements of $T$, viewed as a full subposet of $T$.

**2.8. Definition:** Let $T$ be a finite lattice. For a finite set $X$, let $H_T(X)$ denote the $k$-submodule of $F_T(X) = k(T^X)$ generated by all maps $\varphi : X \to T$ such that $\varphi(X) \not\subseteq \text{Irr}(T)$. 
2.9. Lemma:

1. Let $Y, X$ be finite sets, let $U \in C(Y, X)$, and let $\varphi : X \to T$. Then $(U \varphi)(Y) \cap \text{Irr}(T) \subseteq \varphi(X) \cap \text{Irr}(T)$.

2. The assignment $X \mapsto H_T(X)$ is a subfunctor of $F_T$.

Proof: Let $U \in C(Y, X)$, let $\varphi : X \to T$, let $e \in (U \varphi)(Y) \cap \text{Irr}(T)$, and $y \in Y$ such that $e = (U \varphi)(y)$. Then $e = \bigvee_{(y, x) \in U} \varphi(x)$, so there exists $x$ such that $(y, x) \in U$ and $e = \varphi(x)$. Hence $e \in \varphi(X) \cap \text{Irr}(T)$, proving Assertion 1. Assertion 2 follows trivially. \qed

3. Simple functors

3.1. Let $S$ be a simple object of $\mathcal{F}_k$, that is, a correspondence functor admitting exactly two subfunctors. Then $S$ is non zero, so there is a set $E$ of minimal cardinality such that $S(E) \neq \{0\}$. As explained in Jacques Thévenaz's report in these proceedings, the evaluation $S(E)$ is a simple module for the algebra $\mathcal{E}_E$ of essential relations on $E$, defined by

$$\mathcal{E}_E = kC(E, E)/ \sum_{|F|<|E|} kC(E, F)C(F, E).$$

It follows from [1] that the simple $\mathcal{E}_E$ modules (up to isomorphism) are parametrized by pairs $(R, W)$ of a partial order $R$ on $E$ and a simple $k\text{Aut}(E, R)$-module $W$ (up to permutation of $E$), where $\text{Aut}(E, R)$ is the automorphism group of the pair $(E, R)$, i.e. the group of permutations of $E$ which preserve $R$.

Conversely, if $E$ is a finite set, if $R$ is a partial order on $E$, and if $W$ is a simple $k\text{Aut}(E, R)$-module, then there is a unique simple correspondence functor $S = S_{E, R, W}$ such that $E$ is minimal with $S(E) \neq \{0\}$ and $S(E) \cong W$ as $\mathcal{E}_E$-modules. This gives the following:

3.2. Theorem: The simple correspondence functors over $k$ (up to isomorphism) are parametrized by triples $(E, R, W)$ consisting of a finite set $E$, a partial order $R$ on $E$, and a simple $k\text{Aut}(E, R)$-module $W$ (up to identification of triples $(E, R, W)$ and $(E', R', W')$ for which there exists an isomorphism of posets $\varphi : (E, R) \to (E', R')$ sending $W$ to $W'$).
3.3. **Examples**: Assume that $k$ is a field.

- The representable functor $\mathcal{Y}_\emptyset,k$ (see 1.2) is simple, projective, and injective in $\mathcal{F}_k$. The corresponding triple is $(\emptyset, \text{tot}, k)$, where \text{tot} is the unique (order) relation on $\emptyset$, and $k$ is the unique simple module for $k\text{Aut}(\emptyset, \text{tot}) \cong k$.

- The representable functor $\mathcal{Y}_{\bullet,k}$ is not simple, but one can show that it is isomorphic to the direct sum of the previous one $\mathcal{Y}_\emptyset,k$ and the simple functor $\mathcal{S}_{\text{tot},k}$, where \text{tot} is the unique order relation on the set $\bullet$, and $k$ is the unique simple module for $k\text{Aut}(\bullet, \text{tot}) \cong k$. This functor $\mathcal{S}_{\bullet,\text{tot},k}$ is also simple, projective, and injective in $\mathcal{F}_k$.

3.4. The two previous examples deal with a total order on a set of cardinality 0 and 1. We now consider the general case of a total order.

For this, we choose a non-negative integer $n$, and we denote by $\underline{n}$ the totally ordered set $\{0, 1, \ldots, n\}$. Then $\underline{n}$ is a lattice, in which $x \vee y = \text{Max}(x, y)$ and $x \wedge y = \text{Min}(x, y)$. We denote by $[n]$ the set $\text{Irr}(T)$. Clearly $[n] = \underline{n} - \{0\} = \{1, 2, \ldots, n\}$.

3.5. **Theorem**: For $n \in \mathbb{N}$, set $\mathcal{S}_{[n]} = F_n/H_n$. Then:

1. The surjection $F_n \rightarrow \mathcal{S}_{[n]}$ splits. The functor $\mathcal{S}_{[n]}$ is projective.

2. If $X$ is a finite set, then $\mathcal{S}_{[n]}(X)$ is a free $k$-module of rank $\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}(i + 1)^{|X|}$.

3. $F_n \cong \bigoplus_{A \subseteq [n]} \mathcal{S}_{|A|} \cong \bigoplus_{j=0}^{n} \mathcal{S}_{[j]}^{\oplus \binom{n}{j}}$.

4. $\text{End}_{k\mathcal{L}}(\underline{n}) \cong \text{End}_{\mathcal{F}_k}(F_n) \cong \prod_{j=0}^{n} M_{\binom{n}{j}}(k)$.

5. If $k$ is a field, then $\mathcal{S}_{[n]}$ is simple (and projective, and injective), isomorphic to $\mathcal{S}_{[n],\text{tot},k}$.

3.6. In order to deal with the general case of simple functors, we need to introduce some notation. We start with a finite poset $(E, R)$, and we first choose a finite lattice $T$ with the following two properties:

1. The poset $\text{Irr}(T)$ is isomorphic to $(E, R)$.

2. The natural restriction map $\text{Aut}(T) \rightarrow \text{Aut}(E, R)$ is an isomorphism.

Using Condition (1), we will identify $(E, R)$ with the subposet $\text{Irr}(T)$ of $T$. In Condition (2), we denote by $\text{Aut}(T)$ the group of automorphisms of the
poset $T$ (one can show that this is equal to the group of bijections of $T$ which respect the join operation - see Definition 2.4). An automorphism of $T$ clearly maps an irreducible element to an irreducible element, so we have a restriction map $\text{Aut}(T) \to \text{Aut}(\text{Irr}(T))$. This map is injective, because any element $t$ of $T$ is equal to the join $\bigvee_{e \leq t} e$ of those irreducible elements smaller that $t$

in $T$, thus any automorphism of $T$ is determined by its restriction to $\text{Irr}(T)$. So Condition (2) above amounts to requiring that any automorphism of the poset $(E, R)$ can be extended to an automorphism of $T$.

The poset $(E, R)$ being given, it is always possible to choose a finite lattice $T$ with the above two properties, e.g. the lattice $I_\downarrow(E, R)$ consisting of lower ideals of $(E, R)$ (i.e. subsets $A$ of $E$ such that $(x, y) \in R$ and $y \in A$ implies $x \in A$, for any $x, y \in E$), ordered by inclusion of subsets (the join operation on $I_\downarrow(E, R)$ is union of subsets, and the meet operation is intersection of subsets).

3.7. When $T$ is a finite poset, and $t \in T$, we set

$$r(t) = \bigvee_{x \in T} x \quad \text{s.t.} \quad x < t$$

Thus $r(t) = t$ if $t \notin \text{Irr}(T)$, and if $t \in \text{Irr}(T)$, then $r(t)$ is the largest element of $T$ strictly smaller than $t$.

When $A \subseteq T$, we denote by $\gamma_A : E \to T$ the map defined by

$$\forall e \in E, \quad \gamma_A(e) = \begin{cases} e & \text{if } e \notin A \\ r(e) & \text{if } e \in A \end{cases}$$

We define moreover an element $\gamma$ of $k(T^E)$ by

$$\gamma = \sum_{A \subseteq E} (-1)^{|A|} \gamma_A$$

and we view $k(T^E)$ as the evaluation at $E$ of the functor $F_{T^{op}}$, where $T^{op}$ is the opposite lattice to $T$ (i.e. the lattice obtained by replacing the order relation on $T$ by its opposite, or equivalently, by switching the join and meet operations of $T$).

Finally we denote by $S_{E,R}$ the subfunctor of $F_{T^{op}}$ generated by the element $\gamma$ of $F_{T^{op}}(E)$, i.e. the intersection of all subfunctors $M$ of $F_{T^{op}}$ such that $\gamma \in M(E)$. 

3.8. Theorem:

1. The functor $S_{E,R}$ doesn’t depend on the choice of $T$, up to isomorphism.
2. There exists a positive integer $f = f_{E,R}$ (explicitly computable) such that, for any finite set $X$, the $k$-module $S_{E,R}(X)$ is free of rank

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} (i+f)^{|X|}.$$ 

Moreover $S_{E,R}(X)$ is a free right $k\text{Aut}(E,R)$-module.
3. Let $W$ be a $k\text{Aut}(E,R)$-module. For a finite set $X$, define

$$S_{E,R,W}(X) = S_{E,R}(X) \otimes_{k\text{Aut}(E,R)} W.$$ 

Then the assignment $X \mapsto S_{E,R,W}(X)$ is a correspondence functor.
4. If $k$ is a field and $W$ is simple, then $S_{E,R,W} \cong S_{E,R}.$

Proof: (Sketch) • First we introduce a non-degenerate functorial bilinear pairing $F_T \times F_{T^{op}} \to k$, in the following way: if $X$ is a finite set, if $\varphi : X \to T$ and $\psi : X \to T^{op}$, we set

$$(\varphi, \psi)_X = \begin{cases} 1 & \text{if } \phi(x) \leq_T \psi(x) \forall x \in X, \\ 0 & \text{otherwise.} \end{cases}$$

This pairing is functorial in the sense that for any correspondence $U \subseteq Y \times X$ from $X$ to a finite set $Y$, for any $\varphi : X \to Y$ and any $\psi : Y \to T^{op}$, we have that

$$(U \varphi, \psi)_Y = (\varphi, U^{op} \star \psi)_X,$$

where $U^{op} = \{(x, y) \in X \times Y \mid (y, x) \in U\}$ denotes the opposite correspondence, and $U^{op} \star \psi = F_{T^{op}}(U^{op})(\psi) \in F_{T^{op}}(X)$ is the image of $\psi$ under $U^{op}$.

This pairing is non-degenerate in the strong sense that it induces an isomorphism between $F_T(X)$ and the $k$-dual of $F_{T^{op}}(X)$, for any finite set $X$ (so it induces an isomorphism between $F_{T^{op}}$ and the dual functor $(F_T)^{\natural}$).

• We show that there exists a surjective homomorphism of correspondence functors

$$\Theta_T : F_T / H_T \to S_{E,R^{op}},$$

where $R^{op}$ is the opposite partial order to $R$ on $E$.

• We define a subset $G$ of $T$, containing $E$, and invariant under $\text{Aut}(E,R)$, with the property that for any finite set $X$, the image under $\Theta_{T,X} \circ \pi_{T,X}$ of
the set
\[ \{ \varphi : X \to T \mid E \subseteq \varphi(X) \subseteq G \} \]
of elements of \( F_T(X) \) is a \( k \)-basis of \( S_{E,R}(X) \), where \( \pi_T : F_T \to F_T/H_T \) is the quotient morphism. Then the integer \( f = f_{E,R} \) appearing in Theorem 3.8 is equal to \( |G| - |E| \). \( \square \)

3.9. Corollary: Let \( k \) be a field. Let \( (E, R) \) be a finite poset, and \( W \) be a simple \( k \text{Aut}(E, R) \)-module. Then for any finite set \( X \),

\[
\dim_k S_{E,R,W}(X) = \frac{\dim_k W}{|\text{Aut}(E, R)|} \sum_{i=0}^{|E|} (-1)^{|E| - i} \binom{|E|}{i} (i + f_{E,R})^{|X|}.
\]

4. Examples

4.1. Let \( D \) denote the following lattice:

![Lattice Diagram](image)

where the white dots are the irreducible elements. Then over a field of odd characteristic, the functor \( F_D \) is semisimple: its splits as

\[ F_D \cong S_{[0]} \oplus 4S_{[1]} \oplus 4S_{[2]} \oplus S_{[3]} \oplus 2S_{\bullet} \oplus S_{\bullet i}, \]

where \( S_{\bullet} \) denotes the functor \( S_{E,\Delta} \) for a set \( E \) of cardinality 2, ordered by the equality relation, and \( S_{\bullet i} \) is the functor \( S_{F,R} \) associated to a poset \( (F, R) \) of cardinality 3 with 2 connected components.

Observe that for any \( i \in \mathbb{N} \), the multiplicity of the functor \( S_{[i]} \) as a summand of \( F_D \) is equal to the number of increasing sequences

\[ 0_D = x_0 < x_1 < \ldots < x_i \]

in \( D \). This statement holds more generally for an arbitrary finite lattice \( T \).

4.2. There are 16 posets up to isomorphism on a set of cardinality 4. The
The following table displays the Hasse diagrams of these posets, together with the corresponding value of the integer $f$ appearing in Theorem 3.8:

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The only poset for which $f = 1$ is the total order. This is a general phenomenon: if $(E, R)$ is a finite poset, then $f_{E,R} = 1$ if and only if $R$ is a total order.

**Acknowledgements:** I wish to thank Professor Fumihito Oda for his invitation, and Kinki University, Osaka for support. I also thank Professor Akihiko Hida for the opportunity of giving a talk at the RIMS workshop *Cohomology of finite groups and related topics*, February 18-20, 2015.

**References**


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