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SIMPLE CORRESPONDENCE FUNCTORS AND ESSENTIAL ALGEBRA

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ABSTRACT. This is a report on some recent joint work with Serge Bouc, which appears in [BT1] and [BT2]. It is an expanded version of a talk given at the RIMS workshop *Cohomology of finite groups and related topics*, February 18–20, 2015.

The second part of this joint work is presented in Bouc's report, in these Proceedings.

1. INTRODUCTION

Let $\mathcal{C}$ be the category whose objects are the finite sets, the set of morphisms $\mathcal{C}(Y,X)$ being the set of all correspondences from X to Y (i.e. subsets of the direct product $Y\times X$). Let k be a commutative ring. For convenience, we linearize $\mathcal{C}$ and define the category $k\mathcal{C}$ with the same objects, the set of morphisms from X to Y being the free k-module $k\mathcal{C}(Y,X)$ with basis $\mathcal{C}(Y,X)$. A *correspondence functor* is a k-linear functor from $k\mathcal{C}$ to the category k-Mod of k-modules. We are interested in the classification of all simple correspondence functors, assuming that k is a field.

A correspondence from X to X is usually called a relation on X. The parametrization of simple correspondence functors uses the finite-dimensional algebra $k\mathcal{C}(X,X)$ of all relations on X, which is studied in [BT1]. A relation R on X is called *essential* if it does not factorize through a set of cardinality strictly smaller than $|X|$. The k-submodule generated by set of inessential relations is a two-sided ideal

$$I_X=\sum_{|Y|<|X|}k\mathcal{C}(X,Y)k\mathcal{C}(Y,X)$$

and the quotient $\mathcal{E}_X:=k\mathcal{C}(X,X)/I_X$ is called the *essential algebra*. We shall see that a large part of its structure can be elucidated.

The first parametrization theorem asserts that the set of isomorphism classes of simple correspondence functors S is parametrized by the set of isomorphism classes of pairs (E,W) where E is a finite set and W is a simple $\mathcal{E}_E$-module. Here E is a minimal set for S (i.e. a finite set E of minimal cardinality such that $S(E)\neq 0$) and $W=S(E)$.

This raises the question of finding all simple $\mathcal{E}_E$-modules. To this end, we first have a theorem which says that any essential relation becomes reflexive after a suitable permutation of the columns of $E\times E$. We next define a two-sided ideal N of $\mathcal{E}_E$, generated by all the relations of the form $R,-\overline{R}$, where R is a reflexive relation and $\overline{R}$ is its transitive closure. It turns out that N is a nilpotent ideal of $\mathcal{E}_E$, to the effect that any simple $\mathcal{E}_E$-module is actually a simple $\mathcal{E}_E/N$-module.

Finally, the k-algebra $\mathcal{E}_E/N$ can be described explicitly as a direct product of matrix algebras over suitable group algebras, as follows :

$$\mathcal{E}_E/N \cong \prod_R \mathrm{Mat}_{|\Sigma_E:\mathrm{Aut}(R)|}(k\,\mathrm{Aut}(R))\,,$$

where R runs over the set of all order relations on E, up to conjugation by the group Σ_E of all permutations of E, and where $\mathrm{Aut}(R)$ is the subgroup of the symmetric group Σ_E consisting of all permutations leaving the order relation R invariant. Note that, by an order relation, we always mean a partial order relation.

Since $\mathrm{Mat}_{|\Sigma_E:\mathrm{Aut}(R)|}(k\,\mathrm{Aut}(R))$ is Morita equivalent to $k\,\mathrm{Aut}(R)$, the simple modules for $\mathrm{Mat}_{|\Sigma_E:\mathrm{Aut}(R)|}(k\,\mathrm{Aut}(R))$ correspond to simple $k\,\mathrm{Aut}(R)$-modules. We thus obtain a parametrization of all simple $\mathcal{E}_E/N$-modules W by pairs (R,V), where R is an order relation on E and V is a simple $k\,\mathrm{Aut}(R)$-module, up to conjugation by Σ_E and up to isomorphism.

Putting together both parametrization theorems, we deduce that the simple correspondence functors are parametrized by triples (E,R,V), where E is finite set, R is an order relation on E, and V is a simple $k\,\mathrm{Aut}(R)$-module. Such triples are considered up to isomorphism.

More information about the simple correspondence functors, in particular the dimension of their evaluations, appear in the report by Serge Bouc.

2. Correspondence functors

Let X and Y be finite sets. A *correspondence* from X to Y is a subset of the cartesian product $Y \times X$. Note that we reverse the order of X and Y for reasons mentioned below. When $X = Y$, a correspondence is often called a *relation on* X. Correspondences can be composed as follows. If $R \subseteq Z \times Y$ and $S \subseteq Y \times X$, then RS is the correspondence from X and Z defined by

$$RS = \{(z,x) \in Z \times X \mid \exists y \in Y \text{ such that } (z,y) \in R \text{ and } (y,x) \in S\}\,.$$

In particular the set of all relations on X is a monoid.

We consider the category $\mathcal{C}$ whose objects are the finite sets and, for any two finite sets X and Y, the set of morphisms $\mathcal{C}(Y,X)$ is the set of all correspondences from X to Y. We adopt a slightly unusual notation by writing $\mathcal{C}(Y,X)$ for the set of all morphisms from X to Y. We reverse the order of X and Y in view of having a left action of morphisms behaving nicely under composition. The identity morphism Id_X is the diagonal subset $\Delta_X \subseteq X \times X$ (in other words the equality relation on X).

If k is any commutative ring, the k-*linearization* of the category $\mathcal{C}$ is the category whose objects are the objects of $\mathcal{C}$ and the set of morphisms from X to Y is the free k-module $k\mathcal{C}(Y,X)$ with basis $\mathcal{C}(Y,X)$. The composition of morphisms in $k\mathcal{C}$ is the k-bilinear extension of the composition in $\mathcal{C}$.

A *correspondence functor* is a k-representation of the category $k\mathcal{C}$, that is, a k-linear functor from $k\mathcal{C}$ to the category k-Mod of k-modules. A *minimal set* for a correspondence functor F is a finite set X of minimal cardinality such that $F(X) \neq 0$.

Clearly, for any nonzero functor, such a minimal set always exists and is unique up to bijection.

In order to describe the parametrization of simple correspondence functors, we use the algebra $k\mathcal{C}(X,X)$ of all relations on X, which is studied in [BT1]. A relation R on X is called *essential* if it does not factorize through a set of cardinality strictly smaller than $|X|$. The k-submodule generated by set of inessential relations is a two-sided ideal

$$I_X = \sum_{|Y|<|X|} k\mathcal{C}(X,Y)k\mathcal{C}(Y,X)$$

and the quotient $\mathcal{E}_X := k\mathcal{C}(X,X)/I_X$ is called the *essential algebra* (for X).

The following parametrization theorem is similar to the result proved in Theorem 4.3.10 in [Bo] for biset functors. The context here is different, but the proof is essentially the same. Actually, a general parametrization result of this kind holds for k-linear functors $k\mathcal{D} \to k$-Mod whenever $\mathcal{D}$ is a pre-additive category in which every object is 'measured' by an integer (e.g. its cardinality), so that it makes sense to talk about a minimal object.

Theorem 2.1. *Assume that k is a field.*

(1) *Let S be a simple correspondence functor, let E be a minimal set for S, and let $W = S(E)$. Then W is a simple module for the essential algebra $\mathcal{E}_E$ (with I_E acting by zero).*
(2) *The set of isomorphism classes of simple correspondence functors is parametrized, via the procedure in (1), by the set of isomorphism classes of pairs (E,W) where E is a finite set and W is a simple $\mathcal{E}_E$-module.*

We write $S \cong S_{E,W}$ for the simple correspondence functor parametrized by the pair (E,W). This parametrization will be improved in Section 6.

3. The essential algebra

Theorem 2.1 shows that we need to understand the essential algebra and its simple modules. In this section, we fix a finite set E of cardinality n and we consider the essential algebra $\mathcal{E}_E$. We work over a fixed commutative ring k, which will be assumed later to be a field when we consider simple modules.

Our first lemma says that we can characterize essential relations in a useful way. A *block* in $E \times E$ is a subset of the form $U \times V$, where U and V are subsets of E.

Lemma 3.1. *Let R be a relation on E. Then E is inessential if and only if R is a union of at most $n-1$ blocks (where $n = |E|$).*

Proof. If R factorizes through a set Y with $|Y| < n$, then $R = ST$ with $S \subseteq E \times Y$ and $T \subseteq Y \times E$. Then $R = \cup_{y \in Y}(U_y \times V_y)$ where $U_y = \{e \in E \mid (e,y) \in S\}$ and $V_y = \{f \in E \mid (y,f) \in T\}$. Thus R is a union of at most $n-1$ blocks. The converse is proved in a similar fashion. Details can be found in Lemma 2.1 of [BT1]. □

Corollary 3.2. *Let R be a preorder relation on E (i.e. reflexive and transitive). If R is not an order relation (i.e. not antisymmetric), then R is inessential.*

Proof. As a subset of $E \times E$, the relation R is a union of n columns. Since R is not antisymmetric, there exists $a \neq b \in E$ such that $(a,b) \in R$ and $(b,a) \in R$. By transitivity, a and b are in relation with exactly the same set V of elements of E. Therefore, we can construct a block $\{a,b\} \times V$ with two columns. Every other column is a block and it follows that R is a union of $n-1$ blocks. Thus R is inessential, by Lemma 3.1. □

If σ is a permutation of the set E, we define the relation

$$\Delta_\sigma = \{(\sigma(e), e) \in E \times E \mid e \in E\},$$

which we still call a permutation.

Theorem 3.3. *Any essential relation contains a permutation.*

In other words, any essential relation R can be written $R = S\Delta_\sigma$ where S is reflexive and σ is some permutation (which can be viewed as permutation of the columns in $E \times E$). The theorem can be proved directly by showing that if R does not contain any permutation, then it can be decomposed as a union of at most $n-1$ blocks. Otherwise, it can also be proved by applying a theorem of Philip Hall, proved in 1935 (see Theorem 5.1.1 in [HaM], or [HaP] for the original version). For both proofs, details can be found in Theorem 3.2 of [BT1].

We now define N to be the k-submodule of $\mathcal{E}_E$ generated by all elements of the form $(S-\overline{S})\Delta_\sigma$, where S is a reflexive relation, $\overline{S}$ is its transitive closure, and σ is some permutation of E.

Theorem 3.4. (1) *N is a nilpotent ideal of $\mathcal{E}_E$.*

(2) *The k-algebra $\mathcal{P}_E = \mathcal{E}_E/N$ has a k-basis consisting of all elements of the form $S\Delta_\sigma$, where S is an order relation on E and σ is a permutation of E.*

Proof. We sketch some ideas of the proof. The detailed proof can be found in Theorem 5.3 of [BT1].

The transitive closure of a reflexive relation S is some power S^n of S, that is, $\overline{S} = S^n = S^{n+k}$ for all $k \geq 0$. Then

$$(S-\overline{S})^n = (S-S^n)^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^i S^{n-i} S^{ni} = \Big(\sum_{i=0}^{n} \binom{n}{i} (-1)^i\Big) S^n = (1-1)^n S^n = 0\,.$$

This shows that $S-\overline{S}$ is nilpotent. This is one of the main ideas, but of course further arguments are needed to prove that N is a nilpotent ideal.

For part (2), we write any essential relation R as a product $R = S\Delta_\sigma$ where S is reflexive and σ is a permutation. The reflexive relation S becomes equal to its transitive closure $\overline{S}$ in the quotient $\mathcal{P}_E = \mathcal{E}_E/N$. But $\overline{S}$ is a preorder relation (i.e. reflexive and transitive). If $\overline{S}$ is not an order relation, then it is inessential by Corollary 3.2, hence zero in $\mathcal{E}_E$. This explains why we end up with relations of the form $S\Delta_\sigma$, where S is an order relation on E and σ is a permutation of E. □

The description of the basis of $\mathcal{P}_E = \mathcal{E}_E/N$ makes it clear that the k-algebra $\mathcal{P}_E$ is graded by the group Σ_E of all permutations of E. More precisely, if we let $\mathcal{P}^1_E$ be the subalgebra spanned by the set $\mathcal{O}$ of all order relations, we obtain

$$\mathcal{P}_E = \bigoplus_{\sigma\in\Sigma_E} \mathcal{P}^1_E \Delta_\sigma \,.$$

The product in $\mathcal{P}_E$ is completely determined by the product in the subalgebra $\mathcal{P}^1_E$, the product in the symmetric group Σ_E, and the conjugation action of Σ_E on $\mathcal{P}^1_E$. Hence we first need to understand the subalgebra $\mathcal{P}^1_E$, which we call the algebra of orders. The full algebra $\mathcal{P}_E$ is called the algebra of permuted orders.

4. The algebra of permuted orders

The subalgebra $\mathcal{P}^1_E$ defined above has as a k-basis the set $\mathcal{O}$ of all order relations on E. We now describe the product of basis elements.

Lemma 4.1. *Let $S,T \in \mathcal{O}$. The product $S\cdot T$ in $\mathcal{P}^1_E$ is equal to the transitive closure of $S\cup T$ if this closure is an order, and zero otherwise. In particular, the product in $\mathcal{P}^1_E$ is commutative.*

Proof. By definition of $\mathcal{P}_E$ as a quotient, any reflexive relation R becomes equal to its transitive closure $\overline{R}$ in the quotient $\mathcal{P}_E$. For any $S,T\in\mathcal{O}$, the transitive closure $\overline{ST}$ is also the transitive closure of $S\cup T$. If this is not an order relation, then it is inessential by Corollary 3.2, hence zero. The commutativity follows because $S\cup T = T\cup S$. □

The structure of $\mathcal{P}^1_E$ is given by the following result (Theorem 6.2 in [BT1]).

Theorem 4.2. *$\mathcal{P}^1_E$ is isomorphic to a product of copies of k, indexed by $\mathcal{O}$:*

$$\mathcal{P}^1_E \cong \prod_{R\in\mathcal{O}} k \,.$$

Let $\{f_R \mid R\in\mathcal{O}\}$ be the k-basis of $\mathcal{P}^1_E$ corresponding, under this isomorphism, to the canonical basis of $\prod_{R\in\mathcal{O}} k$. Then the set $\{f_R \mid R\in\mathcal{O}\}$ consists of mutually orthogonal idempotents whose sum is 1. They are obtained by Möbius inversion from the set of idempotents $\{R \mid R\in\mathcal{O}\}$:

$$f_R = \sum_{\substack{S\in\mathcal{O}\\ R\subseteq S}} \mu(R,S)S \qquad\text{and}\qquad R = \sum_{\substack{S\in\mathcal{O}\\ R\subseteq S}} f_S \,.$$

Here $\mu(S,T)$ denotes the Möbius function of the poset $\mathcal{O}$ (ordered by inclusion), so the change of basis is unitriangular. Details appear in Theorem 6.2 of [BT1].

Having elucidated the structure of $\mathcal{P}^1_E$, we then take into account permutations to obtain the structure of $\mathcal{P}_E$. Under the action of the symmetric group Σ_E, the orbit sum of one idempotent f_R is :

$$e_R = \sum_{\sigma\in[\Sigma_E/\operatorname{Aut}(R)]} \Delta_\sigma f_R \Delta_{\sigma^{-1}} \,,$$

where $[\Sigma_E/\operatorname{Aut}(R)]$ is a set of representatives of cosets $\sigma \operatorname{Aut}(R)$. It is rather easy to see that these idempotents e_R are central in $\mathcal{P}_E$, allowing for a direct product decomposition of $\mathcal{P}_E$:

$$\mathcal{P}_E \cong \prod_{R\in[\Sigma_E\backslash\mathcal{O}]} \mathcal{P}_E\, e_R\,.$$

Since e_R is an orbit sum, conjugates relations give the same idempotent, so e_R runs over a set of representatives of the Σ_E-orbits in $\mathcal{O}$, written $[\Sigma_E\backslash\mathcal{O}]$.

Moreover, the factor $\mathcal{P}_E\, e_R$ of the direct product corresponding to e_R turns out to be a matrix algebra with entries in the group algebra $k\operatorname{Aut}(R)$. This is the following main theorem, which appears as Theorem 8.1 in [BT1].

Theorem 4.3. *Let R be an order relation on E and let $\operatorname{Aut}(R)$ be its stabilizer in the symmetric group Σ_E. Then*

$$\mathcal{P}_E\, e_R \cong \operatorname{Mat}_{|\Sigma_E:\operatorname{Aut}(R)|}(k\operatorname{Aut}(R))\,,$$

a matrix algebra of size $|\Sigma_E : \operatorname{Aut}(R)|$ with entries in the group algebra $k\operatorname{Aut}(R)$. In other words

$$\mathcal{P}_E \cong \prod_{R\in[\Sigma_E\backslash\mathcal{O}]} \operatorname{Mat}_{|\Sigma_E:\operatorname{Aut}(R)|}(k\operatorname{Aut}(R))\,,$$

where $[\Sigma_E\backslash\mathcal{O}]$ denotes a set of representatives of the Σ_E-orbits in $\mathcal{O}$.

One may ask which finite groups appear in Theorem 4.3, that is, which finite groups have the the form $\operatorname{Aut}(R)$ for some order relation R. The answer is that all finite groups occur, provided the set E is allowed to be large enough. In other words, for any finite group G, there exists a finite set E and an order relation R on E such that $G \cong \operatorname{Aut}(R)$. This was proved by Birkhoff [Bi] in 1946, but a recent short proof appears in [BM]. However, for a fixed finite set E, it seems to be quite difficult to characterize which finite groups occur as $\operatorname{Aut}(R)$ for some order relation R on E.

5. Simple modules for the essential algebra

Throughout this section, assume that k is a field. As before, E is a fixed finite set and $\mathcal{E}_E$ is the corresponding essential algebra. We can now describe the simple $\mathcal{E}_E$-modules.

Theorem 5.1. *Let E be a finite set. The set of isomorphism classes of simple $\mathcal{E}_E$-modules is parametrized by the set of isomorphism classes of pairs (R, V) where R is an order relation on E and V is a simple $k\operatorname{Aut}(R)$-module. Here $\operatorname{Aut}(R)$ is the stabilizer of R in the symmetric group Σ_E.*

Proof. Since the algebra of permuted orders $\mathcal{P}_E = \mathcal{E}_E/N$ is a quotient by a nilpotent ideal, any simple $\mathcal{E}_E$-module is actually a simple $\mathcal{P}_E$-module (with N acting by zero). Now $\mathcal{P}_E$ decomposes as a direct product, by Theorem 4.3, so any simple $\mathcal{P}_E$-module is a module for one of the factors $\mathcal{P}_E\, e_R$ (the other factors acting by zero). But Theorem 4.3 also says that the factor $\mathcal{P}_E\, e_R$ is isomorphic to the matrix algebra $\operatorname{Mat}_{|\Sigma_E:\operatorname{Aut}(R)|}(k\operatorname{Aut}(R))$, hence is Morita equivalent to the group algebra $k\operatorname{Aut}(R)$.

It follows that the simple $\mathcal{P}_E\, e_R$-modules are parametrized by isomorphism classes of simple $k\,\mathrm{Aut}(R)$-modules. Therefore, the simple $\mathcal{E}_E$-modules are parametrized by the set of isomorphism classes of pairs (R, V) where R is an order relation on E and V is a simple $k\,\mathrm{Aut}(R)$-module. □

This parametrization can be made explicit, with a detailed description of the action of $\mathcal{E}_E$ on simple modules. Details appear in Section 8 of [BT1].

6. The parametrization of simple correspondence functors

We return to correspondence functors and describe now the final parametrization of simple correspondence functors. Throughout this section, assume that k is a field. Theorem 2.1 shows that the simple correspondence functors $S_{E,W}$ are parametrized by isomorphism classes of pairs (E, W), where E is a finite set and W is a simple module for the essential algebra $\mathcal{E}_E$. Now in turn, by Theorem 5.1, the simple $\mathcal{E}_E$-modules W are parametrized by isomorphism classes of pairs (R, V), where R is an order relation on E and V is a simple $k\,\mathrm{Aut}(R)$-module. Putting both theorems together, we obtain the following result.

Theorem 6.1. *The set of isomorphism classes of simple correspondence functors is parametrized by the set of isomorphism classes of triples (E, R, V), where E is a finite set, R is an order relation on E, and V is a simple $k\,\mathrm{Aut}(R)$-module.*

Proof. The proof is an immediate consequence of Theorem 2.1 and Theorem 5.1. □

We write $S_{E,R,V}$ for the simple correspondence functor parametrized by the triple (E, R, V). The next question is to obtain more information about such simple functors, in particular about their evaluations $S_{E,R,V}(X)$ at all finite sets X. Since E is a minimal set for $S_{E,R,V}$, we know that $S_{E,R,V}(X) = 0$ if X has cardinality strictly smaller than $|E|$. We also know that $S_{E,R,V}(E)$ is the simple $\mathcal{E}_E$-module parametrized by (R, V), viewed as a $k\mathcal{C}(E, E)$-module by making I_E act by zero. But the other evaluations are much more difficult to describe. This question is addressed in the report by Serge Bouc in these Proceedings.

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