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Note on the space of polynomials with roots of bounded multiplicity

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Abstract

We study the homotopy type of the space $SP_{n}^{d}(X)$ consisting of all $d$ particles in $X$ with multiplicity less than $n$. When $X = \mathbb{C}$, this space may be identified with the space $SP_{n}^{d}$ of all monic complex coefficient polynomials $f(z) \in \mathbb{C}[z]$ of degree $d$ without roots of multiplicity $\geq n$. In this paper we announce the main result given in [8] concerning to the homotopy stability dimension of this space which improves that obtained in the previous paper [3].

1 Introduction.

Basic definitions and notations. For spaces $X$ and $Y$, let $\text{Map}^{*}(X, Y)$ denote the space consisting of all continuous base-point preserving maps from $X$ to $Y$ with the compact-open topology. When $X$ and $Y$ are complex manifolds, we denote by $\text{Hol}^{*}(X, Y)$ the subspace of $\text{Map}^{*}(X, Y)$ consisting of all base-point preserving holomorphic maps.

For each integer $d \geq 1$, let $\text{Map}_{d}^{*}(S^{2}, \mathbb{C}P^{n-1}) = \Omega_{d}^{2}\mathbb{C}P^{n-1}$ denote the space of all based continuous maps $f : (S^{2}, \infty) \rightarrow (\mathbb{C}P^{n-1}, [1:1: \cdots : 1])$ such that $[f] = d \in \mathbb{Z} = \pi_{2}(\mathbb{C}P^{n-1})$, where we identify $S^{2} = \mathbb{C} \cup \{\infty\}$ and choose $\infty \in S^{2}$ and $[1:1: \cdots : 1] \in \mathbb{C}P^{n-1}$ as the base points of $S^{2}$ and $\mathbb{C}P^{n-1}$, respectively. Let $\text{Hol}_{d}^{*}(S^{2}, \mathbb{C}P^{n-1})$ denote the subspace of $\Omega_{d}^{2}\mathbb{C}P^{n-1}$ consisting of all based holomorphic maps.

Let $S_{d}$ denote the symmetric group of $d$ letters. Then the group $S_{d}$ acts on the space $X^{d} = X \times \cdots \times X$ ($d$-times) by the coordinate permutation and let $SP^{d}(X)$ denote the $d$-th symmetric product of $X$ given by the orbit space $SP^{d}(X) = X^{d}/S_{d}$.

Let $F(X, d) \subset X^{d}$ denote the subspace consisting of all $(x_{1}, \cdots, x_{n}) \in X^{d}$ such that $x_{i} \neq x_{j}$ if $i \neq j$. Since $F(X, d)$ is $S_{d}$-invariant, we define the orbit space $C_{d}(X)$ by $C_{d}(X) = F(X, d)/S_{d}$. The space $C_{n}(X)$ is usually called the configuration space of unordered $n$-distinct points in $X$. Note that there is an inclusion $C_{d}(X) \subset SP^{d}(X)$.

Let $P^{d}(\mathbb{C})$ denote the space consisting of all monic polynomials

$$f(z) = z^{d} + a_{1}z^{d-1} + \cdots + a_{d} \in \mathbb{C}[z]$$
of the degree \( d \). Similarly, let \( \text{SP}_{n}^{d} \) denote the subspace of \( \mathbb{P}^{d}(\mathbb{C}) \) consisting of all monic polynomials \( f(z) \in \mathbb{P}^{d}(\mathbb{C}) \) without root of multiplicity \( \geq n \).

**Definition 1.1.** Note that each element \( \alpha \in \text{SP}^{d}(X) \) can be represented as the formal sum \( \alpha = \sum_{k=1}^{r} n_{k}x_{k} \), where \( \{x_{k}\}_{k=1}^{r} \) are mutually distinct points in \( X \) and each \( n_{k} \) is a positive integer such that \( \sum_{k=1}^{r} n_{k} = d \).

Then by using the notation, we define the subspace \( \text{SP}_{n}^{d}(X) \subset \text{SP}^{d}(X) \) by

\[
\text{SP}_{n}^{d}(X) = \left\{ \sum_{k=1}^{r} n_{k}x_{k} \in \text{SP}^{d}(X) : n_{k} < n \text{ for any } 1 \leq k \leq r \right\}.
\]

Note that there is an increasing filtration

\( \emptyset = \text{SP}_{1}^{d}(X) \subset C_{d}(X) = \text{SP}_{2}^{d}(X) \subset \text{SP}_{3}^{d}(X) \subset \cdots \subset \text{SP}_{d}^{d}(X) \subset \text{SP}_{d+1}^{d}(X) = \text{SP}^{d}(X) \).

**Remark 1.2.** (i) If \( X = \mathbb{C} \) we can easily see that there is a natural homeomorphism \( \mathbb{P}^{d}(\mathbb{C}) \cong \text{SP}^{d}(\mathbb{C}) \) by identifying \( \mathbb{P}^{d}(\mathbb{C}) \ni \prod_{k=1}^{r} (z - \alpha_{k})^{n_{k}} \mapsto \sum_{k=1}^{r} n_{k}\alpha_{k} \in \text{SP}^{d}(\mathbb{C}) \), where \( (\alpha_{1}, \cdots, \alpha_{r}) \in F(\mathbb{C}, r) \) and \( \sum_{k=1}^{r} n_{k} = d \).

(ii) It is easy to see that the space \( \text{Hol}^{d}_{*}(S^{2}, \mathbb{C}P^{n-1}) \) can be identified with the space consisting of all \( n \)-tuples \( (f_{1}(z), \cdots, f_{n}(z)) \in \mathbb{P}^{d}(\mathbb{C})^{n} \) of monic polynomials of the same degree \( d \) such that polynomials \( f_{1}(z), \cdots, f_{n}(z) \) have no common root. \( \square \)

**Definition 1.3.** Define the jet map \( j^{d}_{n} : \text{SP}_{n}^{d} \to \Omega^{2}S^{2n-1} \) by

\[
j^{d}_{n}(f)(x) = \begin{cases} [f(x) : f(x) + f'(x) : \cdots : f(x) + f^{(n-1)}(x)] & \text{if } x \in \mathbb{C} \\ [1 : 1 : \cdots : 1] & \text{if } x = \infty \end{cases}
\]

for \( (f, x) \in \text{SP}_{n}^{d} \times S^{2} \), where we identify \( S^{2} = \mathbb{C} \cup \infty \).

**Remark 1.4.** A map \( f : X \to Y \) is called a homotopy equivalence (resp. a homology equivalence) up to dimension \( D \) if the induced homomorphism \( f_{*} : \pi_{k}(X) \to \pi_{k}(Y) \) (resp. \( f_{*} : H_{k}(X, \mathbb{Z}) \to H_{k}(Y, \mathbb{Z}) \)) is an isomorphism for any \( k < D \) and an epimorphism if \( k = D \). Similarly, it is called a homotopy equivalence (resp. a homology equivalence) through dimension \( D \) if \( f_{*} : \pi_{k}(X) \to \pi_{k}(Y) \) (resp. \( f_{*} : H_{k}(X, \mathbb{Z}) \to H_{k}(Y, \mathbb{Z}) \)) is an isomorphism for any \( k \leq D \). \( \square \)

## 2 The main result.

**The previous results.** Let \( M_{g} \) denote closed Riemann surface of genus \( g \), and let \( * \in M_{g} \) be its base-point. Note that \( M_{g} = S^{2} \) if \( g = 0 \). Then, recall the following two results given in [12] and [3].
Theorem 2.1 ([12]; the case $g \geq 1$). If $g \geq 1$, there is a map $\text{SP}_n^d(M_g \setminus \{\ast\}) \to \text{Map}_0^*(M_g, \mathbb{C}P^{n-1})$ which is a homology equivalence up to dimension $D(d,n)$, where $\lfloor x \rfloor$ is the integer part of a real number $x$ and $D(d,n)$ denotes the positive integer given by

$$D(d,n) = \begin{cases} \lfloor \frac{d}{2} \rfloor & \text{if } n = 2 \\ \lfloor \frac{d}{n} \rfloor - n + 3 & \text{if } n \geq 3 \end{cases}$$

Remark 2.2. Recently the much better stability dimension for the case $g \geq 1$ was obtained by A. Kupers and J. Miller in [?] (cf. [5], [6], [10]).

Theorem 2.3 ([3]; the case $g = 0$). If $g = 0$, the jet map

$$j_n^d : \text{SP}_n^d \to \Omega_2^0 \mathbb{C}P^{n-1} \simeq \Omega^2 S^{2n-1}$$

is a homotopy equivalence up to dimension $(2n-3) \lfloor \frac{d}{n} \rfloor$ if $n \geq 3$ and it is a homology equivalence up to dimension $\lfloor \frac{d}{2} \rfloor$ if $n = 2$. \hfill \square

Theorem 2.4 ([4], [11]). There is a homotopy equivalence

$$\text{SP}_n^d \simeq \text{Hol}_{\lfloor \frac{d}{n} \rfloor}^\ast (S^2, \mathbb{C}P^{n-1}) \text{ if } n \geq 3$$

and there is a stable homotopy equivalence $\text{SP}_2^d \simeq_{s} \text{Hol}_{\lfloor \frac{d}{2} \rfloor}^\ast (S^2, \mathbb{C}P^1)$ if $n = 2$. \hfill \square

The new result. We can improve the stability dimension of the above result for $n \geq 3$ as follows:

Theorem 2.5 ([8]). If $n \geq 3$ and $g = 0$, the jet map $j_n^d : \text{SP}_n^d \to \Omega_2^0 \mathbb{C}P^{n-1} \simeq \Omega^2 S^{2n-1}$ is a homotopy equivalence through dimension $D(d,n) = (2n-3)(\lfloor \frac{d}{n} \rfloor + 1) - 1$. \hfill \square

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References


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