<table>
<thead>
<tr>
<th>項目</th>
<th>内容</th>
</tr>
</thead>
<tbody>
<tr>
<td>トピック</td>
<td>三つの変成群の構造とその応用における新分野の開拓</td>
</tr>
<tr>
<td>作者</td>
<td>永見 慎二</td>
</tr>
<tr>
<td>発行日</td>
<td>2015-11</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/224269">http://hdl.handle.net/2433/224269</a></td>
</tr>
<tr>
<td>タイプ</td>
<td>学術発表論文</td>
</tr>
</tbody>
</table>
| タイテル版 | 京都大学 }
On orientations of fixed point sets of spin structure preserving involutions on manifolds.

Seiji Nagami
Academic Support Center
Setsunan University

1 Introduction

Let $X$ be an oriented connected closed smooth manifold of dimension $n$ with $n \geq 4$, and $F$ an embedded closed submanifold of codimension 2 with $[F]_2 = 0 \in H_{n-2}(X; \mathbb{Z}_2)$, where $[F]_2$ denote the homology class represented by $F$ in $X$ with coefficients in $\mathbb{Z}_2$. Then we have a double branched covering map $\tilde{X} \to X$ branched along $F$. In [2] and [3], we have obtained the following result;

**Theorem 1.1** Suppose that $H_1(X; \mathbb{Z}_2) = 0$. Then $\tilde{X}$ admits a spin structure if and only if $F$ admits an orientation such that $[F]^{\#} \in H^2(X; \mathbb{Z})$ is twice a cohomology class of which reduction modulo 2 coincides with the second Stifel-Whitney class. Here, $[F]^{\#}$ denote the Poincaré-dual of $[F]$.

The assumption that $H_1(X; \mathbb{Z}_2) = 0$ is essential. As a generalization of the above theorem, we first obtain;

**Theorem 1.2** Let $H$ be a connected closed surface smoothly embedded in $X$. Suppose that $H_1(X, \mathbb{Z}_2) = 0$, that $n = 4$, and that $[H]_m = 0 \in H_2(X; \mathbb{Z}_m)$, where $[H]_m$ denote the homology class represented by the oriented $H$. Then $\tilde{X}$ is spin if and only if $[F]^{\#} \in H^2(X; \mathbb{Z})$ is $m$ times a cohomology class of which reduction modulo 2 coincides with the second Stifel-Whitney class.

Although Theorem 1.2 is the case for $n = 4$, it should hold for all positive integer $n$.

Next we have obtained the following ([3]) as an another generalization of Theorem 1.1;

**Theorem 1.3** $\tilde{X}$ admits a spin structure that is preserved by the covering transformation map $T : \tilde{X} \to \tilde{X}$ if and only if $[F]^{\#} \in H^2(X; \mathbb{Z})$ is twice a cohomology class of which reduction modulo 2 coincides with the second Stifel-Whitney class.

Suppose that $F$ admits an orientation such that $[F]^{\#} = 2w \in H^2(X; \mathbb{Z})$ with $(w)_{2} = w_2(X)$, where $(w)_2 \in H^2(X; \mathbb{Z}_2)$ denotes the reduction modulo 2. Then we have

$$H_1(X; \mathbb{Z}) \cong \oplus_{i=1}^{n} \mathbb{Z}_2 \oplus_{i=1}^{N_0} \mathbb{Z}(p_i) \oplus_{i=1}^{N_1} \mathbb{Z}_{2^r_i}(q_i) \oplus_{i=1}^{N_2} \mathbb{Z}_{k_i},$$

where $r_i \geq 2$ and $k_i$ odd. Therefore we obtain

$$H_1(X; \mathbb{Z}_2) \cong \oplus_{i=1}^{n} \mathbb{Z}_2 \oplus_{i=1}^{N_0} \mathbb{Z}_2(p_i) \oplus_{i=1}^{N_1} \mathbb{Z}_2(q_i).$$
Then the 1-st homology group $H_1(X-F;\mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2(\mu_1,\ldots,\mu_s) \oplus H_1(X;\mathbb{Z}_2)$, where $\mu_i$ is homology class represented by a meridian circle to $F$.

We choose a homomorphism $v : H_1(X-F;\mathbb{Z}_2) \to \mathbb{Z}_2$ so that $v(\mu_i) = 1 \in \mathbb{Z}_2$ holds for all $1 \leq i \leq s$. Let $\Omega \subset X$ be an oriented closed $n-2$-submanifold of $X$ such that $|\Omega|^2 = \omega \in H^2(X;\mathbb{Z})$. Let $L_i \subset X$ be an oriented loop such that $[L_i] = l_i \in H_1(X)$. Then fix an embedding $f_i : S^1 \times D^{n-1} \to X$ so that

\[
\begin{cases}
f_i(S^1 \times 0) = L_i \\
f_i(S^1 \times D^{n-1}) \cap (F \cup \Omega) = \emptyset.
\end{cases}
\]

Since $2l_i = 0$, we can choose an embedded surface $G$ such that $\partial G = f_i(S^1 \times \{a, b\})$, where $a, b \in \partial D^{n-1}$. Then by setting $X' = X - f_i(S^1 \times D^{n-1})$, we have that $(G_i, \partial G_i) \subset (X', \partial X')$. Then define $v \in H^1(X-F;\mathbb{Z}_2)$ as

\[v(l_i) = [\Omega] \cdot [G_i, \partial G_i] - \frac{1}{2}([F] \cdot [G_i, \partial G_i]).\]

Here, $\cdot : H_{n-2}(X';\mathbb{Z}) \times H_2(X',\partial X');\mathbb{Z} \to \mathbb{Z}$ denote the intersection pairing. Then the covering transformation map $T : \tilde{X} \to X$ of the double branched covering $\tilde{X} \to X$ determined by $v$ is a spin structure preserving.

**Remark 1.1** One gives semi-orientation of fixed point set for each spin structure on $\tilde{X}$ that is preserved by $T : \tilde{X} \to X$ as follows([6]): let $SO(\tilde{X}) \to \tilde{X}$ denote the orthonormal frame bundle of $\tilde{X}$ together with a spin structure $Spin(\tilde{X}) \to SO(\tilde{X})$ that is preserved by $T$. Then the differential $dT : SO(\tilde{X}) \to SO(\tilde{X})$ has a lift $\tilde{dT} : Spin(\tilde{X}) \to Spin(\tilde{X})$.

Since the restriction $\tilde{dT}|_{\tilde{F}}$ is a bundle automorphism, it is a section of the adjoint bundle $Ad(Spin(\tilde{X})) \to \tilde{X}$. Because $Ad(Spin(\tilde{X}))$ is a subbundle of the Clifford algebra bundle $Cl(\tilde{X}) \to \tilde{X}$, and $Cl(\tilde{X}) \to \tilde{X}$ is isomorphic to the exterior bundle $\wedge^*T\tilde{X}$, $\tilde{dT}$ is a section of $\wedge^*T\tilde{X}$. Moreover we can see that $\wedge^*T\tilde{X}$ is a section of a exterior bundle of a normal bundle $\nu$ of $\tilde{F}$ in $\tilde{X}$. Thus $\tilde{dT}$ determines an orientation of $\nu$, and given $\tilde{X}$ determines an orientation of $\tilde{F}$. In [4], we have shown that the homology class $[F]^2 \subset H^2(X;\mathbb{Z})$, represented by this orientation on $F \approx \tilde{F}$ is twice a characteristic cohomology class.

**Example 1.1** Let $A \approx S^1 \times (0, 1)$ be an annulus embedded in $\mathbb{R}^2$ and $t : A \to A$ the involution given by $t(x, y) = (-x, y)$. Let the tangent bundle $TA|_{S^1 \times 0} \approx S^1 \times \mathbb{R}^2 \to TA|_{S^1 \times 0}$ of $t$ has the following form;

\[dt : \begin{pmatrix} e^{\theta i} \end{pmatrix} \left( \begin{array}{c} a \\ b \end{array} \right) \to \begin{pmatrix} -e^{-\theta i} \end{pmatrix} \left( -R(-2\theta) \begin{array}{c} a \\ b \end{array} \right),\]

where $R(\theta)$ denote the matrix $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$. Then the lift $\tilde{dt} : Spin(A) \approx A \times Spin(2) \to Spin(A)$ of $dt : SO(A) \to SO(A)$ to the spin structure $Spin(A) \to SO(A)$ with respect to the given framing has the form;

\[\left( e^{\theta i}, \xi \right) \to \left( -e^{-\theta i}, (\cos \theta - \sin \theta \frac{\partial}{\partial x} \frac{\partial}{\partial y}) \xi \right).\]

Therefore at the north pole $N = ((1, 0), 0) \in A$ (resp. south pole $S = ((-1, 0), 0) \in A$), the given spin structure determines the orientation $\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ (resp. $-\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$).
The above argument shows that the hyperelliptic involution $t : S^2 \to S^2$ together with the unique spin structure $S^2$ gives the orientation of the branched locus $N \cup S$ of the quotient space $S^2 \approx S^2/(t)$ so that $[N \cup S]^2 = 0 \in \mathbb{Z} \cong H^2(S^2; \mathbb{Z})$, which is twice a characteristic cohomology class.

Next we consider the case for annulus. Set $T = S^1 \times S^1, l = S^1 \times *$ and $m = * \times S^1$. Then the cohomology classes represented by $l$ and $m$ generate the cohomology group $H^1(T; \mathbb{Z}_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. If we give a spin structure on $T$ that restricts to Lie group spin structures on $l$ and $m$, then the induced orientation of the branched locus $F$ in the quotient space $S^2 \approx T/(t)$ by the hyperelliptic involution $t : T \to T$ satisfies $[F]^2 = \pm 4 \in \mathbb{Z} \cong H^2(S^2; \mathbb{Z})$. If we consider a spin structure that restricts to Lie group spin structures on $l$ and to bounding spin structure on $m$, then the induced orientation satisfies $[F]^2 = \pm 0 \in \mathbb{Z}$. These are orientations which are twice a characteristic cohomology class.

By considering the Gysin exact sequence for the double cover $\tilde{X} - \tilde{F} \to X - F$, we obtain the following([5]);

**Theorem 1.4** $\tilde{X}$ is spin if and only there exists a class $w \in H^1(X - F; \mathbb{Z}_2)$ such that $v \cup w = w_2(X - F)$ and that $(\nu, \mu) = 1 \in \mathbb{Z}_2$, where $v \in H^1(X - F; \mathbb{Z}_2)$ determines the double cover $\tilde{X} - \tilde{F} \to X - F$, and $\mu \in H_1(X - F; \mathbb{Z}_2)$ is a homology class represented by a meridian to $F$.

Theorem 1.4 implies another way to state Theorem 1.3([5]);

**Theorem 1.5** $\tilde{X}$ admits a spin structure that is preserved by the covering transformation map $T : \tilde{X} \to \tilde{X}$ if and only if $v \cup v = w_2(X - F)$.

As a corollary, we have([5]);

**Corollary 1.1** Let $\tilde{Y} \to Y$ be an unbranched double cover determined by $\rho \in H^1(Y; \mathbb{Z}_2)$. Then $\tilde{Y}$ admits a spin structure that is preserved by the covering transformation map $t : \tilde{Y} \to \tilde{Y}$ of odd type if and only if $\rho \cup \rho = w_2(Y)$.

**Example 1.2**

Let $\tau : S^n \to S^n$ denote the antipodal map with odd $n$ and $q : S^n \to RP^n$ its quotient map. Then $\tau$ is a spin structure preserving map and $q$ is the double covering map that corresponds to $\rho = 1 \in \mathbb{Z}_2 \cong H^1(RP^n; \mathbb{Z}_2)$. Recall that $\tau$ is odd type with respect to the unique spin structure on $S^n$ if and only if $n \equiv 3 \mod 4$ ([1]). Note that $n \equiv 3 \mod 4$ if and only if $\rho \cup \rho = w_2(RP^n)$ because $w_2(RP^n) = \frac{n(n+1)}{2} \in \mathbb{Z}_2 \cong H^2(RP^n; \mathbb{Z}_2)$.

**References**


