数理解析研究所講究録
第 1968 巻 2015 年 76-80

On the finite space with a finite group action

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1 Introduction

The purpose of our presentation was to study actions of finite groups on finite T_0-spaces, i.e. topological spaces having finitely many points with the T_0-separation axioms, that is, for each pair of distinct two points, there exists an open set containing one but not the other. Many well-known properties about finite (T_0-)spaces may be found in [2], [4], [7] and [11]. Throughout this note, assume that any finite topological space (for short, finite space) has the T_0-separation axiom. Moreover we consider the finite space with a finite group G-action, called a *finite G-space*. Let X, Y be finite G-spaces. Let X denote a finite space. Let x be an element of X. Then we define a subset C_x of X by $C_x = U_x \cup F_x$, where a set U_x be the minimal open set of X which contains x, and a set F_x be the closure of one point set $\{x\}$. For a G-map $f : X \to Y$, we consider a condition: for any $x \in X$,

$$(*) \qquad f(C_x) \subset C_{f(x)}.$$

Let $\mathcal{F}top_{ex}^G$ be the category consisting of the following data: objects are finite G-spaces and morphisms are G-maps satisfying $(*)$. On the other hand, let $\mathcal{F}_{SC}^G$ be the category which consists of finite G-simplicial complexes and simplicial G-maps. Remark that a finite G-space correspondences to a finite G-partially ordered set(for short, a G-poset). Therefore a finite G-space X determines a finite G-simplicial complex $\mathcal{K}(X)$. Then

Theorem A. *Let X, Y be finite G-spaces. Then X is G-homotopy equivalent to Y in $\mathcal{F}top_{ex}^G$ if and only if $\mathcal{K}(X)$ is strong G-homotopy equivalent to $\mathcal{K}(Y)$.*

We shall explain some notations and terminologies. In $\mathcal{F}top_{ex}^G$, we define the homotopy. Let f, g be morphisms from a finite G-space X to another finite G-space Y satisfying $(*)$. Let $\mathcal{I} = \{0, 1\}$ be a finite space whose topology is $\{\emptyset, \{0\}, \{0, 1\}\}$ with the trivial G-action. Then f is *G-homotopic* to g if there is a sequence $f = f_0, f_1, \cdots, f_n = g$ such that for each i $(1 \leq i \leq n)$ there exist two maps $F_i, G_i : X \times \mathcal{I} \to Y$ satisfying $(*)$ with

$$F_i(x, 0) = G_i(x, 1) = f_{i-1}(x) \quad \text{and} \quad F_i(x, 1) = G_i(x, 0) = f_i(x),$$

denoted by $f \simeq_{ex}^G g$. Moreover X is *G-homotopy equivalent* to Y, denoted by $X \simeq_{ex}^G Y$, if there are G-maps $f : X \to Y$ and $g : Y \to X$ satisfying $(*)$ such that $g \circ f \simeq_{ex}^G 1_X$ and $f \circ g \simeq_{ex}^G 1_Y$.

Let K and L be finite simplicial G-complexes, and φ and ψ simplicial G-maps from K to L. Let σ be any simplex of K. If $\varphi(\sigma) \cup \psi(\sigma)$ is also a simplex of L, two simplicial G-maps φ and ψ are said to be *adjacent*. A *fence* in L^K (the set of all simplicial G-maps from K to L) is a sequence $\varphi_0, \varphi_1, \cdots, \varphi_n$ of simplicial G-maps from K to L such that any two consecutive are adjacent. A simplicial G-map φ is *strong G-homotopic* to a simplicial G-map ψ if there exists a fence starting in φ and ending in ψ, and it is denoted by $\varphi \sim_G \psi$. Note that the geometric realization is G-homotopic, i.e., $|\varphi| \sim_G |\psi| : |K| \to |L|$.

When there are simplicial G-maps $\varphi : K \to L$ and $\psi : L \to K$ such that $\psi \circ \varphi \sim_G 1_K$ and $\varphi \circ \psi \sim_G 1_L$, K is said to be *strong G-homotopy equivalent* to L (or two simplicial G-complexes K and L have the *same strong G-homotopy type*), denoted by $K \sim_G L$.

Next we presented the second topic. Let G be a finite group. Let X be a finite G-CW-complex, $S(G)$ be the set of all subgroups of G. For each $H \in S(G)$, let X^H be the H-fixed point set, and $\pi_0(X^H)$ be the connected components of X^H. Then we put

$$\Pi(X) := \coprod_{H \in S(G)} \pi_0(X^H) \quad \text{(disjoint union)},$$

called a *G-poset associated to X*. On the ordering of $\Pi(X)$, we define

$$\alpha \leq \beta \text{ if and only if } \rho(\alpha) \supset \rho(\beta) \text{ and } |\alpha| \subset |\beta| \quad (\alpha, \beta \in \Pi(X)),$$

where $\rho : \Pi(X) \to S(G)\,;\, \alpha \mapsto H$ s.t. $\alpha \in \pi_0(X^H)$, and $|\alpha|$ is the underlying space of α. A finite G-CW-complex Z with a basepoint q is called a *$\Pi(X)$-complex* if it is equipped with a specified set $\{Z_\alpha \,|\, \alpha \in \Pi\}$ of subcomplexes Z_α of Z, satisfying the following four conditions:
(i) $q \in Z_\alpha$,
(ii) $gZ_\alpha = Z_{g\alpha}$ for $g \in G$, $\alpha \in \Pi$,
(iii) $Z_\alpha \subseteqq Z_\beta$ if $\alpha \leqq \beta$ in Π, and
(iv) for any $H \in S(G)$,

$$Z^H := \bigvee_{\alpha \in \Pi \text{ with } \rho(\alpha) = H} Z_\alpha \, .$$

where $\chi(Z_\alpha)$ is the Euler characteristic of Z_α. Here we define a equivalence relation:

$$Z \sim W \overset{\text{def}}{\Longleftrightarrow} \chi(Z_\alpha) = \chi(W_\alpha) \quad \text{for all } \alpha \in \Pi(X).$$

We put $\Omega(G, \Pi(X)) := \{\Pi(X)\text{-complexes}\}/\sim$. Then $\Omega((G, \Pi(X))$ is an abelian group via $[Z] + [W] := [Z \vee W]$.

Let X and Y be pointed finite G-spaces. Let $|\mathcal{K}(X)|$(resp. $|\mathcal{K}(Y)|$) be the geometric realizations of $\mathcal{K}(X)$(resp. $\mathcal{K}(Y)$). Now, we simply write $\Pi(X)$ for $\Pi(|\mathcal{K}(X)|)$. Similarly $\Pi(Y)$ for $\Pi(|\mathcal{K}(Y)|)$. Note that $\Omega(G, \Pi(X))$ and $\Omega(G, \Pi(Y))$ are finitely generated free abelian groups. Then we have a group homomorphism

$$\Omega(G, \Pi(f)) : \Omega(G, \Pi(X)) \to \Omega(G, \Pi(Y)) \quad ; \; [Z]_{\Omega(G, \Pi(X))} \mapsto [Z]_{\Omega(G, \Pi(Y))}$$

Let $\mathcal{F}top_*^G$ be the category of pointed finite G-spaces. Let Ab be the category of abelian groups. Hence we have

Theorem B. *There exist a functor $F : \mathcal{F}top_*^G \to Ab$ such that*

$$F(X) = \Omega(G, \Pi(X)) \text{ and } F(f) = \Omega(G, \Pi(f)).$$

2 Outline of proofs

Proof of Theorem A.

We need some preliminaries to prove Theorem 1. First we prepare the following lemma.

Lemma 1. Let $f, g : X \to Y$ be two G-homotopic maps between finite G-spaces satisfying $(*)$ in $\mathcal{F}top^G_{ex}$. Then there exists a sequence $f = f_0, f_1, \cdots, f_n = g$ such that for every $0 \le i < n$ and there is a point $x_i \in X$ with the following properties:
1. f_i and f_{i+1} coincide in $X \backslash Gx_i$, where $Gx_i = \{hx_i \mid h \in G\}$ and
2. $f_i(x_i) \lneq f_{i+1}(x_i)$ or $f_{i+1}(x_i) \lneq f_i(x_i)$.

Proposition 2. *Let $f, g : X \to Y$ be G-homotopic maps satisfying $(*)$ between finite G-spaces. Then $\mathcal{K}(f) \sim_G \mathcal{K}(g)$.*

Let $\mathcal{X}(K)$ be a face poset for a simplicial complex K. Giving a simplicial map $\varphi : K \to L$ between simplicial complexes, we can induce a map $\mathcal{X}(\varphi) : \mathcal{X}(K) \to \mathcal{X}(L)$.

Proposition 3. *Let $\varphi, \psi : K \to L$ be simplicial G-maps which is strong G-homotopic between finite G-simplicial complexes. Then $\mathcal{X}(\varphi) \simeq^G_{ex} \mathcal{X}(\psi)$.*

Under these preliminaries, we show the following.

Theorem A. *Let X, Y be finite G-spaces. X is G-homotopy equivalent to Y in $\mathcal{F}top^G_{ex}$ if and only if $\mathcal{K}(X)$ is strong G-homotopy equivalent to $\mathcal{K}(Y)$.*

Proof. Suppose $f : X \to Y$ is a G-homotopy equivalence between finite G-spaces with G-homotopy inverse $g : Y \to X$. By Propositon 2, $\mathcal{K}(f)\mathcal{K}(g) \sim_G 1_{\mathcal{K}(Y)}$ and $\mathcal{K}(g)\mathcal{K}(f) \sim_G 1_{\mathcal{K}(X)}$. If $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ are G-simplicial complexes with the same strong G-homotopy type, there exist $\varphi : \mathcal{K}(X) \to \mathcal{K}(Y)$ and $\psi : \mathcal{K}(Y) \to \mathcal{K}(X)$ such that $\varphi \circ \psi \sim_G 1_{\mathcal{K}(Y)}$ and $\psi \circ \varphi \sim_G 1_{\mathcal{K}(X)}$. By Proposition 3, $\mathcal{X}(\varphi) : \mathcal{X}\mathcal{K}(X) \to \mathcal{X}\mathcal{K}(Y)$ is a G-homotopy equivalence with a G-homotopy inverse $\mathcal{X}(\psi)$. Hence, it suffices that $\mathcal{X}\mathcal{K}(X) \simeq^G_{ex} X$. Note that $X \subset \mathcal{X}\mathcal{K}(X)$. Let x_0 be the maximal element of a simplex σ of $\mathcal{K}(X)$. We define a G-map f from $\mathcal{X}\mathcal{K}(X)$ to X by $f(\sigma) = x_0$. Then $f \circ \iota \simeq^G_{ex} id_X$ and $\iota \circ f \simeq^G_{ex} id_{\mathcal{X}\mathcal{K}(X)}$, where ι is an inclusion map from X to $\mathcal{X}\mathcal{K}(X)$. In fact, $f \circ \iota = id_X$ and $(\iota \circ f)(\sigma) \subset \sigma$ for every $\sigma \in \mathcal{K}(X)$. □

As a corollary, we have the following.

Corollary 4. *A functor $\mathcal{K} : \mathcal{F}top^G_{ex} \to \mathcal{F}^G_{SC}$* induces a fully faithful functor between homotopy categories:

$$\mathcal{H}\mathcal{K} : \mathcal{H}\mathcal{F}top^G_{ex} \to \mathcal{H}\mathcal{F}^G_{SC}.$$

Proof of Theorem B.

The following is the key lemma to prove Theorem B.

Lemma 5. Given a G-map $f : X \to Y$ and $\Pi(X)$-complex Z, there exist a G-map $p : Z \to |\mathcal{K}(X)|$ such that the following diagram

$$\begin{array}{ccc} Z & \xrightarrow{p} & |\mathcal{K}(X)| \\ & {\scriptstyle |\mathcal{K}(f)|\circ p} \searrow & \downarrow {\scriptstyle |\mathcal{K}(f)|} \\ & & |\mathcal{K}(Y)| \end{array}$$

commutes. Moreover Z has also a $\Pi(Y)$-complex structure.

Let $\mathcal{F}top_*^G$ be the category of pointed finite G-spaces and Ab be the category of abelian groups. Then we show the following.

Theorem B. *There exist a functor* $F : \mathcal{F}top_*^G \to Ab$ *such that*

$$F(X) = \Omega(G, \Pi(X)) \text{ and } F(f) = \Omega(G, \Pi(f)).$$

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