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ILL-POSEDNESS FOR THE NONLINEAR SCHRÖDINGIER EQUATIONS IN ONE SPACE DIMENSION

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1. Introduction

We consider the Cauchy problems for the quadratic nonlinear Schrödinger equations:

$$\begin{cases} i\partial_t u + \Delta u = u^2, & t \in \mathbb{R},\ x \in \mathbb{R}^n, \\ u(0,x) = \phi(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $n = 1, 2$. We consider the ill-posedness by showing that the continuous dependence on initial data does not hold in general. There are many results known for the Cauchy problems to the nonlinear Schrödinger equations. In the case $n = 1, 2, 3$, the local well-posedness in $L^2(\mathbb{R}^n) = H^0(\mathbb{R}^n)$ was shown by Y. Tsutsumi [16] (see also Ginibre-Velo [7] and Kato [9]).

Proposition 1.1 (Ginibre-Velo, Tsutsumi, Kato). *Let $n = 1, 2, 3$. Then for $u_0 \in L^2(\mathbb{R}^N)$ the initial value problem for (1.1) is time locally wellposed in the class $C([0,T); L^2(\mathbb{R}^N))$. In particular, it is unconditional when $n = 1$.*

The scaling invariance for the equation is an important factor to find the threshold for the well-posedness and the ill-posedness. On the study in the Sobolev spaces $H^s(\mathbb{R}^n)$ with nonnegative regularity $s \geq 0$, it is possible to obtain the well-posedness in the scaling invariant function spaces. For (1.1), the scaling invariant space is $H^{s_c}(\mathbb{R}^n)$ with $s_c := \frac{n}{2} - 2$. In fact, for the solution u to (1.1), let u_λ be defined by $u_\lambda(t,x) := \lambda^2 u(\lambda t, \lambda x)$ $(\lambda > 0)$. Then, u_λ is also a solution to (1.1) and we have the following invariance:

$$\|u_\lambda(0)\|_{\dot{H}^{s_c}} = \|u(0)\|_{\dot{H}^{s_c}} \text{ for any } \lambda > 0 \text{ with } s_c = \frac{n}{2} - 2.$$

For the Cauchy problem (1.1) in the case $n \geq 4$, the scaling critical regularity $s_c = \frac{n}{2} - 2$ is nonnegative and the local well-posedness in $H^s(\mathbb{R}^n)$ with $s \geq s_c$ was proved by Cazenave-Weissler [4].

For the negative regularity with scaling invariance, it is generally not clear whether the Cauchy problems for the nonlinear partial differential equations is well-posed in general or not. Indeed, some other nonlinear partial differential equations, different situation can be observed in some literatures (see for instance Ponce-Sideris [14].

In the case $n = 2$ with the negative regularity $s \leq 0$, Colliander-Delort-Kenig-Staffilani [5] showed the local well-posedness in $H^s(\mathbb{R}^2)$ with $s > -3/4$. Bejenaru- da Siva [2] improved the result to show the well-posedness in $H^s(\mathbb{R}^n)$ with $s > -1$.

Proposition 1.2 (Bejenaru-da Siva). *Let $n = 2$ and $s > -1$. For any $u_0 \in H^s(\mathbb{R}^2)$ the initial value problem for* (1.1) *is time locally wellposed in the class* $C([0,T); H^s(\mathbb{R}^N))$.

We consider the case $s \leq -1$ for two space dimensions to obtain the following theorem.

Theorem 1.3. *Let $n = 2$. For any fixed $s \leq -1$, there exists a sequence of time $\{T_N\}_N$ with $T_N \to 0\,(N \to \infty)$ and an initial data $\{\phi_N\}_N \subset L^2(\mathbb{R}^2)$ $(N = 1, 2, \cdots)$ such that the corresponding sequence of the solution $\{u_N\}_N \subset C([0,T); L^2(\mathbb{R}^2))$ to* (1.1) *with $u_N(0) = \phi_N$ satisfies*

$$\lim_{N\to\infty} \|\phi_N\|_{H^s} \to 0, \qquad \lim_{N\to\infty} \|u_N(T_N)\|_{H^s} = \infty.$$

In the case $n = 1$ the local well-posedness in $H^s(\mathbb{R})$ $(s > -3/4)$ was shown by Kenig-Ponce-Vega [10]. Bejenaru-Tao [3] showed the local well-posedness in $H^s(\mathbb{R})$ with $s \geq -1$ and the ill-posedness in $H^s(\mathbb{R})$ with $s < -1$. The scaling critical regularity is $s_c = -3/2$ and there is a difference between the critical regularities for the scaling, and the well-posedness and the ill-posedness.

Proposition 1.4 (Bejenaru-Tao). *Let $n = 1$. For any fixed $s \leq -1$, there exists a sequence of initial data $\{\phi_N\}_N \subset L^2(\mathbb{R}^2)$ $(N = 1, 2, \cdots)$ such that the corresponding sequence of the solution $\{u_N\}_N \subset C([0,1]; L^2(\mathbb{R}^2))$ to* (1.1) *with $u_N(0) = \phi_N$ satisfies*

$$\lim_{N\to\infty} \|\phi_N\|_{H^s} \to 0, \qquad \lim_{N\to\infty} \sup_{t\in(0,1)} \|u_N(t)\|_{H^s} > C.$$

We introduce Besov spaces to specify the ill-posedness class by not only the regularity index nor regularity scale in the Sobolev space but a finer index involving the interpolation space for $n = 1$.

Let $\{\psi_j\}_j$ be the Littlewood-Paley dyadic decomposition of the unity, namely it satisfies

$$\widehat{\psi}(\xi) \in C_0^\infty(\mathbb{R}^n), \quad \widehat{\psi_j}(\xi) = \psi\Big(\frac{\xi}{2^j}\Big) \text{ for all } j \in \mathbb{Z}, \quad \sum_{j\in\mathbb{Z}} \widehat{\psi_j}(\xi) \equiv 1 \text{ if } \xi \neq 0.$$

Let $\widetilde{\psi} \in \mathcal{S}(\mathbb{R}^n)$ satisfy

$$\widehat{\widetilde{\psi}}(\xi) = 1 - \sum_{j\geq 0} \widehat{\psi_j}(\xi) \qquad \text{for } \xi \in \mathbb{R}^n.$$

Definition 1.5. *(the inhomogeneous Besov spaces) For any $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$, the inhomogeneous Besov space $B^s_{p,q}(\mathbb{R}^n)$ is given by*

$$B^s_{p,q}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}^*; \|f\|_{B^s_{p,q}} \equiv \left(\|\widetilde{\psi} * f\|_p^q + \sum_{j\geq 0} 2^{sjq} \|\psi_j * f\|_p^q \right)^{1/q} < \infty \right\}.$$

Theorem 1.6. *Let $n = 1$. For any fixed $q > 2$, there exists a sequence of time $\{T_N\}_N$ with $T_N \to 0\,(N \to \infty)$ and a sequence of initial data $\{\phi_N\}_N \subset L^2(\mathbb{R})$ $(N = 1, 2, \cdots)$ such that the corresponding sequence of the solution $\{u_N\}_N \subset C([0,T); L^2(\mathbb{R}))$ to* (1.1) *with $u_N(0) = \phi_N$ satisfies*

$$\lim_{N\to\infty} \|\phi_N\|_{B^{-1}_{2,q}} \to 0, \qquad \lim_{N\to\infty} \|u_N(T_N)\|_{B^{-1}_{2,q}} = \infty.$$

Remark. As for comparison of our theorem and Proposition 1.4 by Bejenaru-Tao [3], it is known that

$$H^{-1}(\mathbb{R}) \subset B^{-1}_{2,q}(\mathbb{R}) \subset H^s(\mathbb{R}) \quad \text{if } 2 < q \leq \infty \text{ and } s < -1.$$

It is impossible to take the Besov space larger for the well-posedness than the Sobolev space $H^{-1}(\mathbb{R})$ in which the well-posedness was obtained by Bejenaru-Tao [3]. Furthermore, we treat the solutions arbitrary large with small initial data although each norm of solutions are bounded below by an abosolute constant in the result [3].

Remark. We should note that in Theorem 1.6 and Theorem 1.3 the solution u_N is considered in $C([0,\widetilde{T}_N); L^2(\mathbb{R}^n))$, where $\widetilde{T}_N$ is the maximal existence time of the solution u_N, and $T_N < \widetilde{T}_N$ for all N. Note that the sequence in our theorem is choosen to be satified as $\|u_N(T_N)\|_{L^2} \to \infty$ as $N \to \infty$ by the embedding $L^2(\mathbb{R}^n) \subset H^s(\mathbb{R}^n), B^{-1}_{2,q}(\mathbb{R}^n)$ $(s \leq -1, 2 < q \leq \infty)$. For the nonlinearity $\bar{u}^2$ instead of u^2, it is possible to obtain the same results as Theorem 1.3 and Theorem 1.6. For other nonlinearities such as $|u|^2$, $|u|\bar{u}$ or $|u|u$, the threshold of the regularity may be different from the above theorems. Indeed, $H^0(\mathbb{R}) = L^2(\mathbb{R})$ is critical for the gauge invariant nonlinearity $|u|u$ (see Kenig-Ponce-Vega [11]). We refer to [12] for the other results of the local well-posedness and the ill-posedness.

Remark. We note that the blow-up solutions are not treated in our theorems. In fact, the norm $\|u(T_N)\|_X$ ($X = H^s(\mathbb{R}^2)$ with $s \leq -1$ or $B^{-1}_{2,q}(\mathbb{R})$ with $q > 2$) is large but is finite for each $N \in \mathbb{N}$. Let $u[\phi]$ be solution for initial data ϕ, we see the following by our theorems:

$$\sup_{\phi \in L^2,\ \|\phi\|_X \leq 1,\ t \in (0,1)} \|u[\phi](t)\|_X = \infty.$$

Remark. The power of nonlinear term in Theorem 1.3 for two space dimensions is critical for long or short range scattering. In fact for the nonlinearity $|u|^{p-1}u$ with gauge invariance, short range scattering was shown for $1 + \frac{2}{n} < p < 1 + \frac{4}{n}$ by Tsutsumi-Yajima [17], and nonexistence of short range scattering was shown for $1 < p \leq 1 + \frac{2}{n}$ by Barab [1]. Also, short range scattering for u^2 and $\bar{u}^2$ was shown by Moriyama-Tonegawa-Tsutsumi [13]. We treat the nonlinearities u^2 and $\bar{u}^2$ only but expect that it can be possible to classify the nonlinear terms with the critical regularity of the well-posedness and the ill-posedness, and relate to the classification of the long range scattering and the short range scattering.

Remark. For the nonlinear heat equations:

$$\begin{cases} \partial_t u - \Delta u = u^2, & t \in \mathbb{R},\ x \in \mathbb{R}^n, \\ u(0,x) = \phi(x), & x \in \mathbb{R}^n, \end{cases}$$

it is also possible to obtain the same results as our theorems. Indeed, by replacing $e^{-it|\xi|^2}$ for Schrödinger equations by $e^{-t|\xi|^2}$ for heat equations, we can apply the same proof as the proof for Schrödinger equations to obtain the ill-posedness in $B^{-1}_{2,q}(\mathbb{R})$ $(2 < q \leq \infty)$ and $H^s(\mathbb{R}^2)$ $(s \leq -1)$.

In this paper, we consider the case $n = 1$ to prove the ill-posedness in $H^s(\mathbb{R})$ with $s < -1$ only to show the method of the proof simply. For the detailed proof of Theorem 1.3 and Theorem 1.6, see [8]. In next section, we introduce an asymptotic expansion of the solution u to (1.1) and a proposition for the justification of the expansion by the use of modulation spaces. The followings are definition of modulation spaces and its properties.

Definition 1.7. *(the modulation spaces) Let $\{\chi_k\}_{k\in\mathbb{Z}}$ be a sequence of the Fourier window function that satisfies*

$$\operatorname{supp}\widehat{\chi_k} \subset \{\, \xi \in \mathbb{R} \mid k-1 \leq \xi \leq k+1\}, \quad \sum_{k\in\mathbb{Z}} \widehat{\chi_k}(\xi) \equiv 1.$$

Then, for any $1 \le p, q \le \infty$, *the modulation space* $M_{p,q}(\mathbb{R})$ *is defined by*

$$M_{p,q}(\mathbb{R}) := \left\{ f \in \mathcal{S}'(\mathbb{R}) \,\middle|\, \|f\|_{M_{p,q}} := \left\| \left\{ \|\chi_k * f\|_{L^p(\mathbb{R})} \right\}_{k\in\mathbb{Z}} \right\|_{\ell^q(\mathbb{Z})} < \infty \right\}.$$

Lemma 1.8. [6, 15, 18]
(i) *If* $1 \le p \le r \le \infty$. $M_{p,1}(\mathbb{R}) \subset L^r(\mathbb{R}^n)$.
(ii) *Let* $1 \le q \le \infty$.

$$\left\|e^{it\Delta} f\right\|_{M_{2,q}} = \|f\|_{M_{2,q}}. \tag{1.2}$$

(iii) *Let* $1 \le p \le \infty$. *Then, there exists* $C_M > 0$ *such that*

$$\|fg\|_{M_{p,1}} \le C_M \|f\|_{M_{p,1}} \|g\|_{M_{p,1}}. \tag{1.3}$$

After the propostion for justification in modulation spaces, we give sequences of initial data $\{\phi_N\}$, time $\{T_N\}$ and solutions $\{u_N\}$ which satisfy

$$\lim_{N\to\infty} \|\phi_N\|_{H^s} = 0, \quad \lim_{N\to\infty} \|u_N(T_N)\|_{H^s} = \infty$$

by the use of the asymptotic expansion.

2. Outlined Proof for Theorem

In the case of $n = 1$, the threshold of the regularity is -1 for the well-posedness and the ill-posedness but the scaling critical regularity is $-3/2$. Therefore, there is a gap for these two exponents. On the other hand, in the case of $n = 2$, both of the threshold for the well-posedness and the ill-posedness and the scaling critical regularity are -1. We explain roughly the reason why there is a difference between $n = 1$ and $n = 2$. For initial data, let a test function ϕ be taken as $\phi := 2^{(-s-\frac{n}{2})N}\psi_N$, where ψ_N is the one component of the Littlewood-Paley decomposition and $\|\phi\|_{H^s}$ is independent of N. By the Duhamel formula, we consider the solution u satisfying the following integral equation:

$$u(t) = e^{it\Delta}\phi - i\int_0^t e^{i(t-\tau)\Delta} u^2 d\tau.$$

Since the linear part is bounded, we have to consider the nonlinear part for the divergence of the solution. Then, we approximate the solution by the linear part $u \simeq e^{it\Delta}\phi_N$ to see

$$\left\| \int_0^t e^{i(t-\tau)\Delta}(e^{i\tau\Delta}\phi)^2 d\tau \right\|_{H^s} \simeq \max\{2^{-2(s+1)N}, 2^{(-(s-\frac{n}{2}+2)N}\}.$$

by the direct calculation. Therefore, the nonlinear part of the solution diverges as $N \to \infty$ if $s < -1$ for $n = 1$ and $s < \frac{n}{2} - 2$ for $n \ge 2$. This divergence occurs from the scaling difference between the lower and higher frequency weight $(1 + |\xi|^2)^{s/2}$ in the definition of inhomogeneous Sobolev spaces $H^s(\mathbb{R}^n)$. One may think that this difficulty divergence can be avoided if we apply the homogeneous Sobolev space $\dot{H}^s(\mathbb{R})$, however the well-posedness in the homogeneous space is not be suitable for our problem since $\||\xi|^s\|_{L^2(|\xi|\le 2^j)} = \infty$ if $n = 1$ and $s < -\frac{1}{2}$. Therefore, we can not expect to obtain the well-posedness with scaling critical regularity $s_c = -\frac{3}{2}$ if the space dimension is one.

We introduce the asymptotic expansion of the solution (1.1) by some small parameter $\varepsilon > 0$ as

$$u = U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \varepsilon^3 U_3 + \cdots. \tag{2.1}$$

with the initial data

$$\phi = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots.$$

If u satisfies the equation (1.1), we have the followings on the term of order ε^k with $k = 0, 1, 2, , \cdots$

$$\varepsilon^0: \quad \begin{cases} (i\partial_t + \Delta)U_0 = U_0 U_0, \\ U_0 = \phi_0 \end{cases}$$

$$\varepsilon^1: \quad \begin{cases} (i\partial_t + \Delta)U_1 = U_0 U_1 + U_1 U_0, \\ U_1(0) = \phi_1, \end{cases}$$

$$\vdots \qquad\qquad \vdots$$

respectively. For our proof, we consider the initial data with $\phi_k = 0$ for all $k \geq 0$ except for ϕ_1. Then, the term of order ε^k can be reduced as follows:

$$\varepsilon: \quad \begin{cases} (i\partial_t + \Delta)U_1 = 0, \\ U_1(0) = \phi, \end{cases}$$

$$\varepsilon^2: \quad \begin{cases} (i\partial_t + \Delta)U_2 = U_1 U_1, \\ U_2(0) = 0, \end{cases}$$

$$\varepsilon^3: \quad \begin{cases} (i\partial_t + \Delta)U_3 = \big(U_1 U_2 + U_2 U_1\big), \\ U_3(0) = 0, \end{cases}$$

$$\vdots \qquad\qquad \vdots$$

$$\varepsilon^k: \quad \begin{cases} (i\partial_t + \Delta)U_k = \displaystyle\sum_{k_1+k_2=k,\ k_1,k_2\geq 1} U_{k_1} U_{k_2}, \\ U_k(0) = 0, \end{cases} \qquad \text{for } k \geq 2.$$

Therefore, it is natural to introduce $U_k = U_k[\phi]$ for $k = 1, 2, 3, \cdots$ inductively

$$\begin{cases} U_1[\phi](t) := e^{it\Delta}\phi, \\ U_k[\phi](t) := -i \displaystyle\sum_{k_1+k_2=k,\ k_1,k_2\geq 1} \int_0^t e^{i(t-\tau)\Delta} U_{k_1}[\phi](\tau) U_{k_2}[\phi](\tau) d\tau \quad \text{for any } k = 2, 3, \cdots. \end{cases}$$

Therefore, we obtain a formal expansion

$$u(t) = \sum_{k=1}^{\infty} \varepsilon^k U_k[\phi](t) \tag{2.2}$$

as a solution (1.1) with the initial data $u(0) = \varepsilon\phi$. It is possible to show the following proposition for justification of this expansion by the use of the linear estimate of the propagator and the bilinear estimate in modulation spaces.

Proposition 2.1. [8] *For any $\phi \in M_{2,1}(\mathbb{R})$, there exists a small $T > 0$ and a unique local solution $u = u(t, x)$ in $C([0, T); M_{2,1}(\mathbb{R}))$ to* (1.1). *It satisfies the following expansion in $C((0, T); M_{2,1}(\mathbb{R}))$:*

$$u(t) = \sum_{k=1}^{\infty} \varepsilon^k U_k[\phi](t), \tag{2.3}$$

where $0 < \varepsilon \le 1$ *and*

$$\begin{cases} U_1[\phi](t) := e^{it\Delta}\phi, \\ U_k[\phi](t) := -i \displaystyle\sum_{k_1+k_2=k,\ k_1,k_2\ge 1} \int_0^t e^{i(t-\tau)\Delta} U_{k_1}[\phi](\tau) U_{k_2}[\phi](\tau) d\tau \quad \textit{for any } k = 2, 3, \cdots. \end{cases} \tag{2.4}$$

The proof for the ill-posedness in $H^s(\mathbb{R})$ with $s < -1$ is based on constructing sequences of the initial data $\{\phi_N\}$, time $\{T_N\}$ and the solution $\{u_N\}$ to (1.1) with the initial data $u_N(0) = \phi_N$ satisfying

$$\begin{aligned} &\phi_N \to 0 \quad \text{in } H^s(\mathbb{R}) \quad \text{as } N \to \infty, \\ &T_N \to 0 \quad \text{as } N \to \infty, \\ &\|u_N(T_N)\|_{H^s} \to \infty \quad \text{as } N \to \infty. \end{aligned} \tag{2.5}$$

Let the initial data ϕ_N satisfy

$$\phi_N = N(\log N)\mathcal{F}^{-1}[\chi_N + \chi_{-N}],$$

where χ_N is a characteristic function which support is $[N-1, N+1]$. On the linear part of the solution, we have from $s < -1$

$$\|\phi_N\|_{H^s} \le CN^s N \log N \to 0 \quad \text{as } N \to \infty. \tag{2.6}$$

We consider the solution u_N which is given by

$$u_N = \sum_{k\ge 1} U_k[\phi_N].$$

It is easy to see that $U_1[\phi] \to 0$ in $H^s(\mathbb{R})$ as $N \to \infty$ by (1.2) and (2.6). On the estimate of $U_k[\phi_N]$ $(k \ge 2)$, we use the following proposition in the similar way as that in [8].

Proposition 2.2. (i) *Let* $t = N^{-2}$. *Then, for* ξ *with* $|\xi| \le \frac{1}{2}$, *it holds that*

$$\left|\widehat{U_2[\phi_N]}(t, \xi)\right| \ge c\{\log N\}^2. \tag{2.7}$$

(ii) *For* $k \ge 3$, *it holds that there exists* $C_0 > 0$ *such that*

$$\|U_k[\phi_N]\|_{M_{2,1}} \le C_0^k t^{k-1} N^k (\log N)^k. \tag{2.8}$$

We show (2.5) by this proposition. We have from the asymptotic expansion of the solution, the triangle's inequality and the embedding $M_{2,1}(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$

$$\begin{aligned} \|u(t)\|_{H^s} &\ge \left\|\widehat{u(t)}\right\|_{L^2(|\xi|\le 2^{-1})} \\ &\ge \left\|\widehat{U_2[\phi_N]}\right\|_{L^2(|\xi|\le 2^{-1})} - \sum_{k\ge 3} \left\|\widehat{U_k[\phi_N]}\right\|_{L^2(|\xi|\le 2^{-1})} \\ &\ge \left\|\widehat{U_2[\phi_N]}\right\|_{L^2(|\xi|\le 2^{-1})} - \sum_{k\ge 3} \left\|U_k[\phi_N]\right\|_{M_{2,1}}. \end{aligned} \tag{2.9}$$

Then, for $t = N^{-2}$, we have from (2.7), (2.8) and (2.9)

$$\begin{aligned} \|u(t)\|_{H^s} &\ge c(\log N)^2 - \sum_{k\ge 3} C^k (N^{-2})^{k-1} N^k (\log N)^k \\ &\ge c(\log N)^2 - \sum_{k\ge 3} C^k N^{-k+2} (\log N)^k. \end{aligned} \tag{2.10}$$

If we take $N \gg 1$, we have from (2.10)

$$\|u(N^{-2})\|_{H^s} \geq c(\log N)^2 \to \infty \quad \text{as } N \to \infty.$$

Therefore, the solution do not depend on initial data continuously in general in the Sobolev space $H^s(\mathbb{R}^n)$ with $s < -1$.

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References

[1] J. E. Barab, Jacqueline, *Nonexistence of asymptotically free solutions for a nonlinear Schrödinger equation*, J. Math. Phys. **25** (1984), no. 11, 3270–3273.

[2] I. Bejenaru, D. De Silva, *Low regularity solutions for a 2D quadratic nonlinear Schrödinger equation*, Trans. Amer. Math. Soc. **360** (2008), no. 11, 5805–5830.

[3] I. Bejenaru, T. Tao, *Sharp well-posedness and ill-posedness results for a quadratic non-linear Schrödinger equation*, J. Funct. Anal. **233** (2006), no. 1, 228–259.

[4] T. Cazenave, F. B. Weissler, *The Cauchy problem for the critical nonlinear Schrödinger equation in H^s*, Nonlinear Anal. **14** (1990), no. 10, 807–836.

[5] J. E. Colliander, J. -M. Delort, C. E. Kenig, G. Staffilani, *Bilinear estimates and applications to 2D NLS*, Trans. Amer. Math. Soc. **353** (2001), no. 8, 3307–3325.

[6] H. G. Feichtinger, *Modulation spaces on locally compact Abelian groups*, Technical Report, University of Vienna, 1983, in: "Proc. Internat. Conf. on Wavelets and Applications" (Radha, R.; Krishna, M.; Yhangavelu, S. eds.), New Delhi Allied Publishers, 2003, 1–56.

[7] J. Ginibre, G. Velo, *On a class of nonlinear Schrödinger equations I: The Cauchy problem, II: Scattering theory, general case*, J. Funct. Anal. **32** (1979), 1–71.

[8] T. Iwabuchi, T. Ogawa, *Ill-posedness for nonlinear Schrödingier equation with quadratic non-linearity in low dimensions*, preprint

[9] T. Kato, *On nonlinear Schrödinger equations*, Ann. Inst. H. Poincaré Phys: Théor. 46 (1987), 113–129.

[10] C. E. Kenig, G. Ponce, L. Vega, *Quadratic forms for the 1-D semilinear Schrödinger equation*, Trans. Amer. Math. Soc. **348** (1996), no. 8, 3323–3353.

[11] C. E. Kenig, G. Ponce, L. Vega, *On the ill-posedness of some canonical dispersive equations*, Duke Math. J. **106** (2001), no. 3, 617-633.

[12] N, Kishimoto, *Low-regularity bilinear estimates for a quadratic nonlinear Schrödinger equation*, J. Differential Equations **247** (2009), no. 5, 1397–1439.

[13] K. Moriyama, S. Tonegawa, Satoshi, Y. Tsutsumi, *Wave operators for the nonlinear Schrödinger equation with a nonlinearity of low degree in one or two space dimensions*, Commun. Contemp. Math. **5** (2003), no. 6, 983–996.

[14] G. Ponce, T. Sideris, *Local wellposedness result and illposedenss for nonlinear wave equations*, Comm. P. D. E., (1998), .

[15] J. Toft, *Continuity properties for modulation spaces, with applications to pseudo-differential calculus, I*, J. Funct. Anal., **207** (2004), 399–429.

[16] Y. Tsutsumi, *L^2-solutions for nonlinear Schrödinger equations and nonlinear groups*, Funkcial. Ekvac. **30** (1987), no. 1, 115–125.

[17] Y. Tsutsumi, K. Yajima, *The asymptotic behavior of nonlinear Schrödinger equations*, Bull. Amer. Math. Soc. (N.S.) **11** (1984), no. 1, 186–188.

[18] B. Wang, L. Zhao, B. Guo, *Isometric decomposition operators, function spaces $E^\lambda_{p,q}$ and applications to nonlinear evolution equations*, J. Funt. Anal., **233** (2006), 1-39.