数理解析研究所講究録
第 1969 巻 2015 年 123-145

L^{∞}-decay property for parabolic-elliptic Keller-Segel systems with porous-medium diffusion

Sachiko Ishida*
Department of Mathematics
Tokyo University of Science

Abstract. This paper deals with the Keller-Segel system $(\mathrm{KS})_0$ of parabolic-elliptic type with porous-medium diffusion. In this type Sugiyama-Kunii [16] established the L^r-decay property $(1 \le r < \infty)$ of solutions to $(\mathrm{KS})_0$ with small initial data when $q \ge m + \frac{2}{N}$ (m denotes the intensity of diffusion and q denotes the nonlinearity). However, the L^{∞}-decay property was not obtained yet. Therefore this paper gives the L^{∞}-decay property of solutions to $(\mathrm{KS})_0$ with small initial data when $q > m + \frac{2}{N}$.

1. Introduction and results

In this paper we consider the following quasilinear degenerate Keller-Segel system of parabolic-elliptic type:

$$(\mathrm{KS})_0 \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \nabla \cdot (\nabla u^m - u^{q-1} \nabla v) & \text{in } \mathbb{R}^N \times (0,\infty), \\ 0 = \Delta v - v + u & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases}$$

where $N \in \mathbb{N}$, $m \ge 1$, $q \ge 2$. The initial data satisfies

$$u_0 \ge 0, \quad u_0 \in L^1(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N). \tag{1.1}$$

The minimal Keller-Segel system of parabolic-parabolic type, i.e., $(\mathrm{KS})_0$ with $m = 1$, $q = 2$ and the second equation replaced with

$$\frac{\partial v}{\partial t} = \Delta v - v + u,$$

was proposed by Keller-Segel [**6**], and power type was studied by Sugiyama-Kunii [**16**] (see also Sugiyama [**13**] and Ishida-Yokota [**2**], [**3**]). On the other hand, the system $(\mathrm{KS})_0$ of parabolic-elliptic type was considered by [**16**]. In particular, $(\mathrm{KS})_0$ with $m = 1$ and $q = 2$ is called the Nagai model, and investigated until now (see e.g., Nagai-Senba-Yoshida [**11**], Nagai [**10**], Sugiyama [**12**], [**14**], [**15**] and Kozono-Sugiyama [**7**]; see also T. Suzuki [**18**]). These models describe a part of cellular slime molds with the chemotaxis at the life cycle. Usually $u(x,t)$ shows the density of cellular slime molds and $v(x,t)$ shows the density of the semiochemical at place x and time t.

*The author is supported by Grant-in-Aid for JSPS Fellows, No. 24·5405.

The purpose of this paper is to give the L^∞-decay property of solutions to $(\mathrm{KS})_0$ with small initial data when $q \geq m + \frac{2}{N}$. Substituting the second equation $\Delta v = v - u$ into the first equation in $(\mathrm{KS})_0$ implies

$$\text{(E1)} \qquad \begin{aligned} \frac{\partial u}{\partial t} &= \Delta u^m - \nabla u^{q-1} \cdot \nabla v - u^{q-1}\Delta v \\ &= \Delta u^m - \nabla u^{q-1} \cdot \nabla v + u^q - u^{q-1}v. \end{aligned}$$

This is analogous to the following nonlinear degenerate heat equation:

$$\text{(NLD)} \qquad \frac{\partial z}{\partial t} = \Delta z^m + z^q \quad \text{in } \mathbb{R}^N \times (0, \infty).$$

The studies for (NLD) and $(\mathrm{KS})_0$ in Table 1.1 are currently known.

		(NLD) ($q > m + \frac{2}{N}$)	$(\mathrm{KS})_0$ ($q \geq m + \frac{2}{N}$)
Decay property of $z(t)$ or $u(t)$	L^r	–	$(t+1)^{-\frac{N}{N(m-1)+2}\cdot\frac{r-1}{r}}$ (Sugiyama-Kunii [16])
	L^∞	$t^{-\frac{1}{q-1}}$ (Kawanago [5])	**Unsolved (A)**
Behavior of $z(t)$ or $u(t)$ as $t \to \infty$	L^r	–	Barenblatt sol. ($m > 1$) (Luckhaus-Sugiyama [8]) Heat kernel ($m = 1$) (Luckhaus-Sugiyama [9])
	L^∞	Barenblatt sol. ($m > 1$) (Kawanago [5]) Heat kernel ($m = 1$) (Kawanago [5])	**Unsolved (B)**

Table 1.1. The known results for (NLD) and $(\mathrm{KS})_0$ with small initial data.

Therefore our aim is to give an answer to the unsolved part (A) in Table 1.1.
Before stating our result we define global weak solutions to $(\mathrm{KS})_0$.

Definition 1.1. Let $T > 0$. A pair (u, v) of non-negative functions defined on $\mathbb{R}^N \times (0, T)$ is called a *weak solution* to $(\mathrm{KS})_0$ *on* $[0, T)$ if

(a) $u \in L^\infty(0, T; L^p(\mathbb{R}^N))$ $(\forall p \in [1, \infty])$, $u^m \in L^2(0, T; H^1(\mathbb{R}^N))$,

(b) $v \in L^\infty(0, T; H^1(\mathbb{R}^N))$,

(c) (u, v) satisfies $(\mathrm{KS})_0$ in the distributional sense, i.e., for every $\varphi \in C_0^\infty(\mathbb{R}^N \times [0, T))$,

$$\int_0^T \int_{\mathbb{R}^N} (\nabla u^m \cdot \nabla\varphi - u^{q-1}\nabla v \cdot \nabla\varphi - u\varphi_t)\, dxdt = \int_{\mathbb{R}^N} u_0(x)\varphi(x, 0)\, dx,$$

$$\int_0^T \int_{\mathbb{R}^N} (\nabla v \cdot \nabla\varphi + v\varphi - u\varphi)\, dxdt = 0.$$

In particular, if $T > 0$ can be taken arbitrarily, then (u, v) is called a *global weak solution* to $(\mathrm{KS})_0$.

We now state our main result in this paper.

Theorem 1.1. *Let* $N \in \mathbb{N}$, $m \geq 1$, $q \geq 2$. *Let* m *and* q *satisfy*

$$q > m + \frac{2}{N}.$$

Assume further that u_0 *satisfy* (1.1) *and*

$$(1.2) \quad \begin{cases} \|u_0\|_{L^{\frac{N}{2}(q-m)}} \leq \min\{\delta_{u,\frac{N}{2}(q-m)}, \delta_{u,r_3}, \delta_{u,r_0}\} & \text{when } q \geq m+1 (N \geq 3),\ N = 1,2 \\ \|u_0\|_{L^{\frac{N}{2}}} \leq \min\{\delta_{u,\frac{N}{2}}, \delta_{u,r_3}, \delta_{u,r_0}\} & \text{when } q < m+1 (N \geq 3), \end{cases}$$

where

$$\delta_{u,r} = \min\Big\{1, \frac{4m}{2^{q-2} r C'}, \Big(\frac{4m(r+q-2)}{2^{q-2}(r+m-1)^2 C''}\Big)^{\frac{1}{q-m}}\Big\},$$

$C' = C'(r,m,q,N)$, $C'' = C''(r,m,q,N)$ $r_3 = r_3(m,q,N)$ *(defined in subsection 3.2) and* $r_0 = \max\{N-m+1, m-3, N(q-m)-m+1\}$ *are positive constants. Then* $(\mathrm{KS})_0$ *has a non-negative weak solution* (u,v) *on* $[0,\infty)$ *which satisfies the following decay property:*

$$(1.3) \quad \|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq K t^{-\frac{1}{q-1}} = K t^{-\frac{N}{N(m-1)+2q_*}}, \quad \text{a.a. } t \in (0,\infty),$$

$$(1.4) \quad \|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq K_\rho (t+\rho)^{-\frac{N}{N(m-1)+2}}, \quad \text{a.a. } t \in [5\rho,\infty),$$

where $q_* := \frac{N}{2}(q-m)$, $K = K(\|u_0\|_{L^{q_*}}, C_{r_3}, r_3, m, q, N) > 0$ *is a constant,* $\rho \in (0,1]$ *is arbitrary and* $K_\rho = K_\rho(\rho, C_{r_3}, r_3, \|u_0\|_{L^1}, \|u_0\|_{L^{q_*}}, \|u_0\|_{L^{r_3}}, m, q, N)$ $(\to \infty$ *as* $\rho \to 0)$ *is a positive constant, where* C_r *is the constant given in Proposition 2.1.*

The decay rate in Theorem 1.1 may be best possible, because of the following two reasons.

<u>*First Reason*</u>: As stated above, $(\mathrm{KS})_0$ can be rewritten as the equation (E1) like (NLD). From comparing the diffusion term Δu^m with the aggregation term u^q in (E1), $(\mathrm{KS})_0$ has the global solvability and the solution has L^r-decay property when $q \geq m + \frac{2}{N}$ and the initial data is sufficiently small ([**16**]). Kawanago [**5**] showed the L^∞-decay property for (NLD) when $q > m + \frac{2}{N}$, that is, if the initial data is sufficiently small, then (NLD) has a global solution which satisfies

$$\|z(t)\|_{L^\infty(\mathbb{R}^N)} \leq M_0 t^{-\frac{1}{q-1}} = M_0 t^{-\frac{N}{N(m-1)+2q_*}},$$

where $q_* = \frac{N}{2}(q-m)$ and $M_0 > 0$ is some constant. Hence we expect that the solution to $(\mathrm{KS})_0$ has

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq M_1 t^{-\frac{1}{q-1}} = M_1 t^{-\frac{N}{N(m-1)+2q_*}},$$

where M_1 is some constant.

<u>*Second Reason*</u>: Sugiyama-Kunii [**16**] showed the L^r-decay property of solutions to $(\mathrm{KS})_0$:

$$(1.5) \quad \|u(t)\|_{L^r} \leq C_r (1+t)^{-\alpha}, \quad r \in [1,\infty),$$

where

$$\alpha = \frac{N}{N(m-1)+2} \cdot \frac{r-1}{r}.$$

Giving an eye to the decay rate α, we have

$$\frac{N}{N(m-1)+2}\cdot\frac{r-1}{r}\to\frac{N}{N(m-1)+2}\quad(r\to\infty).$$

Hence we expect that the solution to $(\mathrm{KS})_0$ has

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)}\le M_2t^{-\frac{N}{N(m-1)+2}},$$

where M_2 is some constant.

One of the difficulties in showing the L^∞-decay estimates is that the coefficient $C_r\to\infty$ as $r\to\infty$ in (1.5) (see the definition of C_r in Proposition 2.1 below), and hence the L^∞-decay property is not obtained by the limiting process in (1.5). To evade this problem and obtain the L^∞-decay property we establish the following two kinds of L^∞-L^r estimates without assuming that the initial data is small (see Section 3):

$$\text{(I)}\qquad \|u(t)\|_{L^\infty(\mathbb{R}^N)}^{r-(q_*+q-1)}\le C(r)\Big(\tfrac{t}{2}\|u(\tfrac{t}{2})\|_{L^r(\mathbb{R}^N)}^r+\big(\tfrac{t}{2}\big)^{1-\frac{r-q_*}{q-1}}\Big),$$

$$\text{(II)}\quad \|u(t)\|_{L^\infty(\mathbb{R}^N)}^r\le\widetilde{C}(r)(t+\rho)^{-\frac{N}{N(m-1)+2}}\Big(\|u(\tfrac{t}{2}-\tfrac{\rho}{2})\|_{L^r(\mathbb{R}^N)}^r+\|u_0\|_{L^1}(t+\rho)^{-\frac{N(r-1)}{N(m-1)+2}}\Big),$$

where $q_*=\frac{N}{2}(q-m)$, $C(r)$, and $\widetilde{C}(r)$ are positive constants. We can obtain the L^∞-decay properties (1.3) and (1.4) by combining the L^r-decay estimate with (I) and (II), respectively. The condition $q>m+\frac{2}{N}$ is necessary to show that the coefficient $\widetilde{C}(r)$ is bounded as $r\to\infty$. The proofs of (I) and (II) are based on R. Suzuki [**17**] in which he studied the following equation:

$$\text{(E2)}\qquad \frac{\partial z}{\partial t}=\Delta z^m+a\cdot\nabla z^p+z^q\quad\text{in }\mathbb{R}^{\mathrm{N}}\times(0,\infty),$$

where $m\ge1$, $p,q>1$, $a\in\mathbb{R}^N$, $a\ne0$. He proved that the solution to (E2) has the following decay property when $q>m+\frac{2}{N}$: if the initial data is sufficiently small, then

$$\|z(t)\|_{L^\infty(\mathbb{R}^N)}\le M_3\min\{t^{-\frac{N}{N(m-1)+2q_*}},t^{-\frac{N}{N(m-1)+2}}\},\quad\text{a.a. }t>0,$$

where $q_*=\frac{N}{2}(q-m)$, $M_3>0$ is some constant. Also from this, we can expect that the solution to $(\mathrm{KS})_0$ has the L^∞-decay properties (1.3) and (1.4). Moreover, he showed in [**17**] that the solution to (E2) behaves like the Barenblatt solution ($m>1$) or the Heat kernel ($m=1$) when $q>m+\frac{2}{N}$ and $p>m+\frac{1}{N}$.

Finally, we glance at the unsolved part (B) in Table 1.1. From the known results for the behavior of solutions ([**5**], [**8**], [**9**] and [**17**]), we conjecture that the solution to $(\mathrm{KS})_0$ has a similar behavior in the case where $q>m+\frac{2}{N}$ and the initial data is small. This conjecture will be discussed in our forthcoming paper.

This paper is organized as follows. In Section 2 we recall the L^r-decay of solutions to $(\mathrm{KS})_0$. First we deal with the case where $N\ge2$ in Section 3, because the approximation is different between more than one dimension and $1D$. Section 3 consists of two subsections. Section 3.1 gives the L^∞-bound of solutions to $(\mathrm{KS})_0$. Section 3.2 is the main part of this paper, where the L^∞-decay of solutions to $(\mathrm{KS})_0$ is obtained. Finally we consider the case where $N=1$ in Section 4.

2. L^r-decay property

First we state the result on the global existence and L^r-decay property of solutions to $(\mathrm{KS})_0$. This proposition is stated in [**16**, Theorem 3].

Proposition 2.1 (global existence of weak solutions to $(\mathrm{KS})_0$). *Let $N \in \mathbb{N}$, $m \geq 1$, $q \geq 2$. Suppose that m and q satisfy the super-critical condition, i.e.,*

$$q \geq m + \frac{2}{N}$$

Let the initial data satisfy (1.1) *and the smallness condition* (1.2) *in Theorem 1.1. Then* $(\mathrm{KS})_0$ *has a non-negative global weak solution (u, v) which has the mass conservation law:*

$$\|u(t)\|_{L^1(\mathbb{R}^N)} = \|u_0\|_{L^1(\mathbb{R}^N)}, \quad t \geq 0. \tag{2.1}$$

Moreover, $t \mapsto \|u(t)\|_{L^r(\mathbb{R}^N)}$ $(1 \leq r < \infty)$ is a non-increasing function with the following decay property:

$$\|u(t)\|_{L^r(\mathbb{R}^N)} \leq C_r(1+t)^{-\alpha}, \quad r \in [1, \infty),\ t \geq 0, \tag{2.2}$$

where

$$\alpha = \frac{N}{N(m-1)+2} \cdot \frac{r-1}{r}, \tag{2.3}$$

$$C_r = \max\Big\{\frac{(r+m-1)^2}{r} \cdot \frac{1}{2m(m-1+\frac{2}{N})}\big(c(N)\|u_0\|_{L^1}\big)^{\frac{N}{N(m-1)+2}\cdot\frac{r-1}{r}}, \|u_0\|_{L^r}\Big\}. \tag{2.4}$$

Remark 2.1. The non-negativity of the solutions is obtained from the standard argument and the comparison principle (see [**16**]).

Remark 2.2. In [**16**], they assume the smallness only $\|u_0\|_{L^{\frac{N}{2}(q-m)}}$ $(N \geq 1)$. However from the approximation to the nonlinear term in the first equation in $(\mathrm{KS})_0$, when $m+\frac{2}{N} \leq q < m+1$ $(N \geq 3)$, we should assume the smallness of $\|u_0\|_{L^{\frac{N}{2}}}$ (see [**4**]).

Remark 2.3. In [**16**], it seems difficult to prove the L^∞-bound of the approximate solution without assuming that $u_0 = 0$. Indeed, they assume the smallness $\|u_0\|_{L^{\frac{N(q-m)}{2}}} \leq \delta_{u,r} = C_0 r^{-\frac{1}{q-m}}$ to obtain the L^r-estimate. If $r \to \infty$ in this assumption, then it should be $\|u_0\|_{L^{\frac{N(q-m)}{2}}} = 0$. To overcome the difficulty we give a proof by using Moser's iteration technique (cf. R. Suzuki [**17**, Section 3.1]).

3. The case where $N \geq 2$

In this section we establish two kinds of "L^∞-L^r estimates" of solutions to $(\mathrm{KS})_0$. The first one is for the L^∞-bound (Proposition 3.1) and the second is for L^∞-decay property (Proposition 3.5). In the end of this section we prove Theorem 1.1 $(N \geq 2)$. Now we introduce the approximate problem:

$$(\mathrm{KS})_\varepsilon \quad \begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} = \nabla \cdot (\nabla(u_\varepsilon+\varepsilon)^m - (u_\varepsilon + \varepsilon^{\frac{m}{q-2}})^{q-2} u_\varepsilon \nabla v_\varepsilon) & \text{in } \mathbb{R}^N \times (0,T), \quad \cdots (1)_\varepsilon \\ 0 = \Delta v_\varepsilon - v_\varepsilon + u_\varepsilon & \text{in } \mathbb{R}^N \times (0,T), \quad \cdots (2)_\varepsilon \\ u_\varepsilon(x,0) = u_{0\varepsilon}(x),\ v_\varepsilon(x,0) = v_{0\varepsilon}(x), & x \in \mathbb{R}^N, \end{cases}$$

where $N \geq 2$, $m \geq 1$, $q \geq 2$ and $\varepsilon \in (0,1)$. The initial data $u_{0\varepsilon} \in C_0^\infty(\mathbb{R}^N)$ is given as $u_{0\varepsilon} := (\rho_\varepsilon * u_0)\,\zeta_\varepsilon$, where ρ_ε is a mollifier such that

$$0 \leq \rho_\varepsilon \in C_0^\infty(\mathbb{R}^N), \quad \operatorname{supp}\rho_\varepsilon \subset \overline{B(0,\varepsilon)}, \quad \int_{\mathbb{R}^N} \rho_\varepsilon(x)\,dx = 1,$$

and ζ_ε is a cut-off function, i.e., $\zeta_\varepsilon(x) := \zeta(\varepsilon x)$, where ζ is a fixed function in $C_0^\infty(\mathbb{R}^N)$ such that

$$0 \leq \zeta \leq 1, \quad \zeta(x) = \begin{cases} 1 & (|x| \leq 1), \\ 0 & (|x| \geq 2). \end{cases}$$

Remark 3.1. Let $T > 0$. Let u_ε be a solution to $(\mathrm{KS})_\varepsilon$ on $[0,T)$. Then the following continuity holds:

$$\|u_\varepsilon(t)\|_{L^r(\mathbb{R}^N)} \in C([0,T]) \quad (\forall r \in [1,\infty)). \tag{3.1}$$

Indeed, reading the standard argument to construct the local (approximate) solution again (see [**16**, Proposition 8, Lemmas 11 and 12], Amann [**1**, Theorem IV.1.5.1]), we see that $u_\varepsilon \in C([0,T]; L^\alpha(\mathbb{R}^N))$ for every $\alpha \in (N,\infty)$. This fact together with the mass conservation law (2.1) implies the continuity (3.1). This continuity will be used in Lemma 3.3.

Remark 3.2. If u_0 satisfies the smallness condition as in Theorem 1.1. then the approximate solution u_ε has the same L^r-decay as (2.2) and $t \mapsto \|u_\varepsilon(t)\|_{L^r(\mathbb{R}^N)}$ is a non-increase function.

3.1. L^∞-bounds

The next proposition shows the L^∞-bound of the solution u to $(\mathrm{KS})_0$. Indeed, (3.3) (in Proposition 3.1) implies that $\|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq K_0$ a.a. $t \in (\rho, T)$ for every $\rho > 0$.

Proposition 3.1 (L^∞-estimate of solutions to $(\mathrm{KS})_0$). *Let $N \geq 2$, $m \geq 1$, $q \geq 2$, $\varepsilon \in (0,1)$ and $T > 0$. Let (u,v) be a weak solution to $(\mathrm{KS})_0$ on $[0,T)$. Assume that m and q satisfy*

$$q \geq m + \frac{2}{N} \tag{3.2}$$

and u_0 satisfies (1.1) and the smallness condition (1.2) in Theorem 1.1. Then the following estimate holds:

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq K_1 t^{-\frac{N}{N(m-1)+2q_*}}, \qquad \text{a.a. } t \in (0,T), \tag{3.3}$$

where $q_* = \frac{N}{2}(q-m)$, $K_1 = K_1(\|u_0\|_{L^{q_*}}, C_{r_1}, m, q, N) > 0$ and C_{r_1} is the same constant as in Proposition 2.1.

The proof of this proposition employs the similar method to R. Suzuki [**17**, Section 4]. For this purpose we prepare three lemmas.

Lemma 3.2. *Let $N \geq 2$, $m \geq 1$, $q \geq 2$, $\varepsilon \in (0,1)$, $T > 0$ and $0 \leq t_1 < t_2 \leq T$. Let $(u_\varepsilon, v_\varepsilon)$ be a unique solution to $(\mathrm{KS})_\varepsilon$ on $[0,T)$. Let $\psi(t) \in C^1([t_1,t_2])$ with $0 \leq \psi \leq 1$, $\psi(t_1) = 0$, $\psi(t_2) = 1$. Assume that m and q satisfy (3.2). Then for $r > q$,*

$$\begin{aligned}(3.4)\qquad & \|u_\varepsilon(t_2)\|^{r-q+1}_{L^{r-q+1}(\mathbb{R}^N)} + \frac{4m(r-q+1)(r-q)}{(r-q+m)^2}\int_{t_1}^{t_2}\psi(t)\|\nabla u_\varepsilon^{\frac{r-q+m}{2}}(t)\|^2_{L^2(\mathbb{R}^N)}\,dt \\ &\quad + \varepsilon^{m-1}\frac{4m(r-q)}{r-q+1}\int_{t_1}^{t_2}\psi(t)\|\nabla u_\varepsilon^{\frac{r-q+1}{2}}(t)\|^2_{L^2(\mathbb{R}^N)}\,dt \\ &\leq \int_{t_1}^{t_2}\psi'(t)\|u_\varepsilon(t)\|^{r-q+1}_{L^{r-q+1}(\mathbb{R}^N)}\,dt \\ &\quad + 2^{q-2}(r-q)\Big(\int_{t_1}^{t_2}\|u_\varepsilon(t)\|^r_{L^r(\mathbb{R}^N)}\,dt + \varepsilon^m\int_{t_1}^{t_2}\|u_\varepsilon(t)\|^{r-q+2}_{L^{r-q+2}(\mathbb{R}^N)}\,dt\Big).\end{aligned}$$

Proof. Let $r > 2$. Multiplying the first approximate equation $(1)_\varepsilon$ by u_ε^{r-1} and integrating it over $\mathbb{R}^N$, we obtain

$$\begin{aligned}(3.5)\qquad & \frac{1}{r}\frac{d}{dt}\|u_\varepsilon(t)\|^r_{L^r(\mathbb{R}^N)} \\ &\leq -\frac{4m(r-1)}{(r+m-1)^2}\|\nabla u_\varepsilon^{\frac{r+m-1}{2}}(t)\|^2_{L^2(\mathbb{R}^N)} - \frac{4m(r-1)\varepsilon^{m-1}}{r^2}\|\nabla u_\varepsilon^{\frac{r}{2}}(t)\|^2_{L^2(\mathbb{R}^N)} \\ &\quad + \int_{\mathbb{R}^N}(u_\varepsilon + \varepsilon^{\frac{m}{q-2}})^{q-2}u_\varepsilon\nabla v_\varepsilon\cdot\nabla u_\varepsilon^{r-1}\,dx.\end{aligned}$$

Multiplying (3.5) by $\psi(t)$ and integrating it by parts over (t_1,t_2), we see that

$$\begin{aligned}(3.6)\qquad & \|u_\varepsilon(t_2)\|^r_{L^r(\mathbb{R}^N)} - \int_{t_1}^{t_2}\psi'(t)\|u_\varepsilon(t)\|^r_{L^r(\mathbb{R}^N)}\,dt \\ &\leq -\frac{4mr(r-1)}{(r+m-1)^2}\int_{t_1}^{t_2}\psi(t)\|\nabla u_\varepsilon^{\frac{r+m-1}{2}}(t)\|^2_{L^2(\mathbb{R}^N)}\,dt \\ &\quad - \frac{4m(r-1)\varepsilon^{m-1}}{r}\int_{t_1}^{t_2}\psi(t)\|\nabla u_\varepsilon^{\frac{r}{2}}(t)\|^2_{L^2(\mathbb{R}^N)}\,dt + r\int_{t_1}^{t_2}\psi(t)\boldsymbol{I}_2\,dt.\end{aligned}$$

We denoted by $\boldsymbol{I}_2$ the third term on the right-hand side of (3.5). We make an estimation of $\boldsymbol{I}_2$. Letting

$$F(s) := \int_0^s(\tau + \varepsilon^{\frac{m}{q-2}})^{q-2}\tau^{r-1}\,d\tau, \quad \tau \geq 0,\ s \geq 0,\ \varepsilon \in (0,1)$$

and noting that

$$0 \leq F(s) \leq 2^{q-2}\Big[\frac{s^{r+q-2}}{r+q-2} + \frac{\varepsilon^m s^r}{r}\Big],$$

we find by $(2)_\varepsilon$ that

$$
\begin{aligned}
(3.7)\qquad \boldsymbol{I}_2 &= -(r-1)\int_{\mathbb{R}^N} F(u_\varepsilon)\Delta v_\varepsilon\,dx \\
&= -(r-1)\int_{\mathbb{R}^N} (v_\varepsilon - u_\varepsilon)F(u_\varepsilon)\,dx \\
&\le (r-1)\int_{\mathbb{R}^N} u_\varepsilon F(u_\varepsilon)\,dx \\
&\le \frac{2^{q-2}(r-1)}{r+q-2}\int_{\mathbb{R}^N} u_\varepsilon^{r+q-1}\,dx + \frac{2^{q-2}\varepsilon^m(r-1)}{r}\int_{\mathbb{R}^N} u_\varepsilon^{r+1}\,dx.
\end{aligned}
$$

Hence it follows from (3.6), (3.7) and $0 \le \psi \le 1$ that

$$
\begin{aligned}
(3.8)\qquad &\|u_\varepsilon(t_2)\|^r_{L^r(\mathbb{R}^N)} + \frac{4mr(r-1)}{(r+m-1)^2}\int_{t_1}^{t_2} \psi(t)\|\nabla u_\varepsilon^{\frac{r+m-1}{2}}(t)\|^2_{L^2(\mathbb{R}^N)}\,dt \\
&\qquad + \frac{\varepsilon^{m-1}4m(r-1)}{r}\int_{t_1}^{t_2} \psi(t)\|\nabla u_\varepsilon^{\frac{r}{2}}(t)\|^2_{L^2(\mathbb{R}^N)}\,dt \\
&\le \int_{t_1}^{t_2} \psi'(t)\|u_\varepsilon(t)\|^r_{L^r(\mathbb{R}^N)}\,dt \\
&\quad + 2^{q-2}(r-1)\int_{t_1}^{t_2}\Big(\frac{r}{r+q-2}\|u_\varepsilon(t)\|^{r+q-1}_{L^{r+q-1}(\mathbb{R}^N)} + \varepsilon^m\|u_\varepsilon(t)\|^{r+1}_{L^{r+1}(\mathbb{R}^N)}\Big)\,dt.
\end{aligned}
$$

Replacing r with $r-q+1$ in (3.8), we obtain (3.4) for $r > q$. □

Lemma 3.3. *Let $N \ge 2$, $m \ge 1$, $q \ge 2$, $\varepsilon \in (0,1)$ and $T > 0$. Let $(u_\varepsilon, v_\varepsilon)$ be a unique solution to $(\mathrm{KS})_\varepsilon$ on $[0,T)$. Put $I = [\tau, \tau+s]$ and $I' = [\tau-\sigma, \tau+s]$ with $0 < \sigma < \tau < \tau+s < T$. Put $q_* := \frac{N}{2}(q-m)$, $h := \sup_{t\in[0,T]} \|u_\varepsilon(t)\|^{q_*}_{q_*}$ and $r_* > q_* + q - 1$ is some constant. Assume further that m and q satisfy (3.2) and $\delta > 0$ satisfies*

$$
(3.9)\qquad \sigma\delta^{q-1} \le 1.
$$

Then for $r \ge r_*$,

$$
\begin{aligned}
(3.10)\qquad &\mu_0\big(Y_{I,k(r-q+1)+m-1} + Z_{I,k(r-q+1)}\big)^{\frac{1}{k}} \\
&\qquad \le \Big(\frac{4}{\sigma}\delta^{-q+1} + 2^{q-1}(r-q)\Big)Y_{I',r} + 2^{q-1}\varepsilon^m(r-q)Z_{I',r-q+2},
\end{aligned}
$$

where $k := 1 + \frac{2}{N}$, $\mu_0 = \mu_0(h, m, q, N)$ and

$$
Y_{I,r} := \int_I\int_{\mathbb{R}^N} u_\varepsilon^r\,dxdt + \frac{(s+\sigma)h}{\delta^{q_*}}\delta^r, \qquad Z_{I,r} := \int_I\int_{\mathbb{R}^N} u_\varepsilon^r\,dxdt.
$$

Proof. Let $r > q$. From (3.1) we can take $\tilde{t} \in I$ such that

$$
\max_{t\in I}\int_{\mathbb{R}^N} u_\varepsilon^{r-q+1}(t)\,dx = \int_{\mathbb{R}^N} u_\varepsilon^{r-q+1}(\tilde{t})\,dx.
$$

Let

$$\tilde{\psi}(t) := \frac{t-\tau+\sigma}{\tilde{t}-\tau+\sigma}, \quad \tilde{t_1} := \tau - \sigma, \quad \tilde{t_2} := \tilde{t}$$

and note that $0 \le \tilde{\psi} \le 1$, $\tilde{\psi}(\tilde{t_1}) = 0$, $\tilde{\psi}(\tilde{t_2}) = 1$, $0 \le \tilde{\psi}'(t) = \frac{1}{\tilde{t}-\tau+\sigma} \le \frac{1}{\sigma}$ and $[\tilde{t_1}, \tilde{t_2}] \subset I'$ Then we can substitute $\tilde{\psi}$, $\tilde{t_1}$ and $\tilde{t_2}$ into ψ, t_1 and t_2 in (3.4) and thus, we have

(3.11)

$$\max_{t\in I} \int_{\mathbb{R}^N} u_\varepsilon^{r-q+1}(t)\,dx$$
$$\le \frac{1}{\sigma}\int_{I'}\int_{\mathbb{R}^N} u_\varepsilon^{r-q+1}\,dxdt + 2^{q-2}(r-q)\int_{I'}\Big(\|u_\varepsilon(t)\|^r_{L^r(\mathbb{R}^N)} + \varepsilon^m\|u_\varepsilon(t)\|^{r-q+2}_{L^{r-q+2}(\mathbb{R}^N)}\Big)\,dt.$$

Next letting

$$\hat{\psi}(t) := \begin{cases} 1, & t \in [\tau, \tau+s], \\ -\sigma^{-2}(t-\tau)^2 + 1, & t \in [\tau-\sigma, \tau], \end{cases} \quad \hat{t_1} := \tau - \sigma, \quad \hat{t_2} := \tau + s$$

and noting that $0 \le \hat{\psi} \le 1$, $\hat{\psi}(\hat{t_1}) = 0$, $\hat{\psi}(\hat{t_2}) = 1$, $0 \le \hat{\psi}'(t) \le \frac{2}{\sigma}$ and $I \subset [\hat{t_1}, \hat{t_2}] \subset I'$, we can substitute $\hat{\psi}$, $\hat{t_1}$ and $\hat{t_2}$ into ψ, t_1 and t_2 in (3.4). Hence we see that

(3.12)

$$\nu_0\int_I \|\nabla u_\varepsilon^{\frac{r+m-q}{2}}(t)\|^2_{L^2(\mathbb{R}^N)}\,dt + \varepsilon^{m-1}\nu_1\int_I \|\nabla u_\varepsilon^{\frac{r-q+1}{2}}(t)\|^2_{L^2(\mathbb{R}^N)}\,dt$$
$$\le \frac{2}{\sigma}\int_{I'}\int_{\mathbb{R}^N} u_\varepsilon^{r-q+1}\,dxdt + 2^{q-2}(r-q)\int_{I'}\Big(\|u_\varepsilon(t)\|^r_{L^r(\mathbb{R}^N)} + \varepsilon^m\|u_\varepsilon(t)\|^{r-q+2}_{L^{r-q+2}(\mathbb{R}^N)}\Big)\,dt,$$

where $\nu_0 := \min\{1, \inf_{r\ge r_*} \frac{4m(r-q+1)(r-q)}{(r+m-q)^2}\}$, $\nu_1 := \min\{1, \inf_{r\ge r_*} \frac{4m(r-q)}{r-q+1}\}$ and $r_* > \frac{N}{2}(q-m)+q-1$ is some constant. Combining (3.11) with (3.12), we have

$$\text{(3.13)} \quad \max_{t\in I}\int_{\mathbb{R}^N} u_\varepsilon^{r-q+1}(t)\,dx$$
$$+ \nu_0\int_I \|\nabla u_\varepsilon^{\frac{r+m-q}{2}}(t)\|^2_{L^2(\mathbb{R}^N)}\,dt + \varepsilon^{m-1}\nu_1\int_I \|\nabla u_\varepsilon^{\frac{r-q+1}{2}}(t)\|^2_{L^2(\mathbb{R}^N)}\,dt$$
$$\le \frac{3}{\sigma}\int_{I'}\int_{\mathbb{R}^N} u_\varepsilon^{r-q+1}\,dxdt + 2^{q-1}(r-q)\Big(\int_{I'}\|u_\varepsilon(t)\|^r_{L^r}\,dt + \varepsilon^m\int_{I'}\|u_\varepsilon(t)\|^{r-q+2}_{L^{r-q+2}}\,dt\Big).$$

We estimate the first term on the right-hand side of (3.13). Set

$$\boldsymbol{E}_\delta(t) := \{x \in \mathbb{R}^N; u_\varepsilon(x,t) \ge \delta\}, \; q_* := \frac{N}{2}(q-m), \; h := \sup_{t\in[0,T]}\|u_\varepsilon(t)\|^{q_*}_{L^{q_*}(\mathbb{R}^N)}.$$

Noting that $|I'| = s + \sigma$, we see that for $r \ge \max\{q, r_*\} = r_*(> q_* + q - 1)$,

$$\text{(3.14)} \quad \int_{I'}\int_{\mathbb{R}^N} u_\varepsilon^{r-q+1}\,dxdt = \Big(\int_{I'}\int_{\boldsymbol{E}_\delta(t)} + \int_{I'}\int_{\mathbb{R}^N\setminus\boldsymbol{E}_\delta(t)}\Big)u_\varepsilon^{r-q+1}\,dxdt$$
$$\le \delta^{-q+1}\int_{I'}\int_{\mathbb{R}^N} u_\varepsilon^r\,dxdt + \delta^{r-q_*-q+1}h(s+\sigma).$$

To estimate the left-hand side of (3.13), we use the Sobolev type inequality in [**17**, Lemma 2.9]:

$$(3.15)\qquad \Big[\int_I\int_{\mathbb{R}^N} |f|^{\tilde\alpha}\,dxdt\Big]^{\frac{1}{k}} \le C_0^{\frac{1}{k}}\Big[\max_{t\in I}\int_{\mathbb{R}^N}|f|^{\alpha}\,dx + \int_I\int_{\mathbb{R}^N}|\nabla f|^2\,dxdt\Big],$$

where $\alpha \ge 0$, $\tilde\alpha = 2(\frac{\alpha}{N}+1)$, $k = 1+\frac{2}{N}$, $f \in C(I;L^{\alpha}(\mathbb{R}^N)) \cap L^2(I;H^1(\mathbb{R}^N))$ and C_0 is a positive constant depending only on N. Applying (3.15) with $f = u_\varepsilon^{\frac{r+m-q}{2}}$ and $\alpha = \frac{2(r-q+1)}{r-q+m}$ or $f = u_\varepsilon^{\frac{r-q+1}{2}}$ and $\alpha = 2$, we find that for $r \ge q-1$,

(3.16)

$$\Big\{\frac{1}{C_0}\int_I\int_{\mathbb{R}^N} u_\varepsilon^{k(r-q+1)+m-1}\,dxdt\Big\}^{\frac{1}{k}} \le \max_{t\in I}\int_{\mathbb{R}^N} u_\varepsilon^{r-q+1}(t)\,dx + \int_I \|\nabla u_\varepsilon^{\frac{r+m-q}{2}}(t)\|^2_{L^2(\mathbb{R}^N)}\,dt,$$

$$(3.17)\quad \Big\{\frac{1}{C_0}\int_I\int_{\mathbb{R}^N} u_\varepsilon^{k(r-q+1)}\,dxdt\Big\}^{\frac{1}{k}} \le \max_{t\in I}\int_{\mathbb{R}^N} u_\varepsilon^{r-q+1}(t)\,dx + \int_I \|\nabla u_\varepsilon^{\frac{r-q+1}{2}}(t)\|^2_{L^2(\mathbb{R}^N)}\,dt.$$

Let $r \ge \max\{r_*, q-1\} = r_*$. Plugging (3.16)–(3.17) into (3.14) to left- and right-hand sides of (3.13), respectively, we have

$$(3.18)\quad \begin{aligned} &\frac{\nu_0}{2C_0^{1/k}}\Big\{\int_I\int_{\mathbb{R}^N} u_\varepsilon^{k(r-q+1)+m-1}\,dxdt\Big\}^{\frac{1}{k}} + \varepsilon^{m-1}\frac{\nu_1}{2C_0^{1/k}}\Big\{\int_I\int_{\mathbb{R}^N} u_\varepsilon^{k(r-q+1)}\,dxdt\Big\}^{\frac{1}{k}} \\ &\le \Big[\frac{3}{\sigma}\delta^{-q+1} + 2^{q-1}(r-q)\Big]\int_{I'}\int_{\mathbb{R}^N} u_\varepsilon^r\,dxdt + \frac{3(s+\sigma)}{\sigma}\delta^{r-q_*-q+1}h \\ &\qquad + 2^{q-1}\varepsilon^m(r-q)\int_{I'}\int_{\mathbb{R}^N} u_\varepsilon^{r-q+2}\,dxdt. \end{aligned}$$

Adding $\frac{s+\sigma}{\sigma}\delta^{r-q_*-q+1}h$ to the both sides of (3.18), we obtain

$$(3.19)\quad \begin{aligned} &\frac{\nu_0}{2C_0^{1/k}}\Big\{\int_I\int_{\mathbb{R}^N} u_\varepsilon^{k(r-q+1)+m-1}\,dxdt\Big\}^{\frac{1}{k}} + \frac{s+\sigma}{\sigma}\delta^{r-q_*-q+1}h \\ &\qquad + \varepsilon^{m-1}\frac{\nu_1}{2C_0^{1/k}}\Big\{\int_I\int_{\mathbb{R}^N} u_\varepsilon^{k(r-q+1)}\,dxdt\Big\}^{\frac{1}{k}} \\ &\le \Big[\frac{3}{\sigma}\delta^{-q+1} + 2^{q-1}(r-q)\Big]\int_{I'}\int_{\mathbb{R}^N} u_\varepsilon^r\,dxdt + \frac{4}{\sigma}\delta^{-q+1}(s+\sigma)h\delta^{r-q_*} \\ &\qquad + 2^{q-1}\varepsilon^m(r-q)\int_{I'}\int_{\mathbb{R}^N} u_\varepsilon^{r-q+2}\,dxdt \\ &\le \Big[\frac{4}{\sigma}\delta^{-q+1} + 2^{q-1}(r-q)\Big]\Big\{\int_{I'}\int_{\mathbb{R}^N} u_\varepsilon^r\,dxdt + \frac{(s+\sigma)h}{\delta^{q_*}}\delta^r\Big\} \\ &\qquad + 2^{q-1}\varepsilon^m(r-q)\int_{I'}\int_{\mathbb{R}^N} u_\varepsilon^{r-q+2}\,dxdt. \end{aligned}$$

Since $\sigma\delta^{q-1} \le 1$ and

$$k(r-q+1)+m-1 = k(r-q_*-q+1)+q_*+q-1,$$

it follows that

$$\frac{(s+\sigma)h}{\sigma}\delta^{r-q_*-q+1} = \Big\{\frac{(s+\sigma)h}{\delta^{q_*}}\delta^{k(r-q+1)+m-1}\Big\}^{\frac{1}{k}}\Big(\frac{1}{\sigma\delta^{q-1}}\Big)^{\frac{1}{k}}\Big(\frac{s+\sigma}{\sigma}h\Big)^{1-\frac{1}{k}} \geq h^{1-\frac{1}{k}}\Big\{\frac{(s+\sigma)h}{\delta^{q_*}}\delta^{k(r-q+1)+m-1}\Big\}^{\frac{1}{k}}. \tag{3.20}$$

Taking (3.20) in the left-hand side of (3.19) and using the inequality $(A+B)^{\frac{1}{k}} \leq A^{\frac{1}{k}} + B^{\frac{1}{k}}$ $(A, B > 0)$, we have

$$\begin{aligned}
&\mu_0\Big\{\int_I\int_{\mathbb{R}^N} u_\varepsilon^{k(r-q+1)+m-1}\,dxdt + \frac{(s+\sigma)h}{\delta^{q_*}}\delta^{k(r-q+1)+m-1} \\
&\qquad\qquad + \varepsilon^{m-1}\int_I\int_{\mathbb{R}^N} u_\varepsilon^{k(r-q+1)}\,dxdt\Big\}^{\frac{1}{k}} \\
&\leq \Big[\frac{4}{\sigma}\delta^{-q+1} + 2^{q-2}(r-q)\Big]\cdot\Big\{\int_{I'}\int_{\mathbb{R}^N} u_\varepsilon^r\,dxdt + \frac{(s+\sigma)h}{\delta^{q_*}}\delta^r\Big\} \\
&\qquad\qquad + 2^{q-1}\varepsilon^m(r-q)\int_{I'}\int_{\mathbb{R}^N} u_\varepsilon^{r-q+2}\,dxdt,
\end{aligned}$$

where $\mu_0 := \min\{\frac{\nu_0}{2C_0^{1/k}}, \frac{\nu_1}{2C_0^{1/k}}, h^{1-\frac{1}{k}}\}$. Thus we obtain (3.10). □

Lemma 3.4. *Let $N \geq 2$, $m \geq 1$, $q \geq 2$, $\varepsilon \in (0,1)$, $T > 0$ and $0 < \chi < \tau < \tau + s < T$. Let $(u_\varepsilon, v_\varepsilon)$ be a unique solution to* $(\mathrm{KS})_\varepsilon$ *on $[0,T)$. Assume that m and q satisfy* (3.2) *and δ satisfies*

$$\chi\delta^{q-1} \leq 1.$$

Then the following estimate holds:

$$\begin{aligned}
&\|u_\varepsilon\|^{r_1-(q_*+q-1)}_{L^\infty(\tau,\tau+s;L^\infty(\mathbb{R}^N))} \\
&\quad\leq \big[2B(2k)^{\frac{1}{k-1}}\big]^{\frac{k}{k-1}}\Big\{\big(1+\varepsilon^m\big)\int_{\tau-\chi}^{\tau+s}\int_{\mathbb{R}^N} u_\varepsilon^{r_1}\,dxdt + \Big(s+\frac{\chi}{2}\Big)h\delta^{r_1-q_*}\Big\},
\end{aligned} \tag{3.21}$$

where $k = 1 + \frac{2}{N}$, $q_* = \frac{N}{2}(q-m)$, $r_1 = r_1(m,q,N) \geq 1$, $h := \sup_{t\in[0,T]}\|u_\varepsilon(t)\|_{q_*}^{q_*}$ and $B = B(h, r_1, \chi, \delta, m, q, N) > 0$ are constants.

Proof. Let $q_* := \frac{N}{2}(q-m)$, $k := 1+\frac{2}{N}$, $\lambda_0 := q_* + q - 1$, $\Lambda_0 := \frac{N}{2} + q - 1$ and let $r_* > \lambda_0$ be some constant. First let the sequence $\{\lambda_n\}_n \subset \mathbb{R}$ be defined by

$$\begin{cases} \lambda_n = (\lambda_{n-1} - q + 1)k + m - 1, \\ \lambda_1 = r_1 := \max\{r_*, \lambda_0, \Lambda_0\}. \end{cases}$$

Thus

$$\lambda_n = \lambda_0 + (r_1 - \lambda_0)k^{n-1}. \tag{3.22}$$

Since $k = 1 + \frac{2}{N} > 1$, it follows that

$$\lambda_{n+1} > \lambda_n, \quad r_1 \leq \lambda_n \leq r_1 k^{n-1} \quad \text{and} \quad \lim_{n\to\infty} \lambda_n = \infty.$$

Next define the sequence $\{\Lambda_n\}_n \subset \mathbb{R}$ as

$$\begin{cases} \Lambda_n - q + 2 = (\Lambda_{n-1} - q + 1)k, \\ \Lambda_1 - q + 2 = r_1, \end{cases}$$

and then,

$$\Lambda_n = \Lambda_0 + (r_1 - \Lambda_0)k^{n-1}.$$

Since $k = 1 + \frac{2}{N} > 1$, it follows that

$$\Lambda_{n+1} > \Lambda_n, \quad r_1 \leq \Lambda_n \leq r_1 k^{n-1} \quad \text{and} \quad \lim_{n\to\infty} \Lambda_n = \infty.$$

Let $I_n := [\tau - 2^{-n+1}\chi, \tau + s]$ and $\delta > 0$ such that $\chi\delta^{q-1} \leq 1$. Then (3.9) holds for δ:

$$\{(\tau - 2^{-n}\chi) - (\tau - 2^{-n+1}\chi)\}\delta^{q-1} = (2^{-n}\chi)\delta^{q-1} \leq 1 \quad (n \geq 1)$$

and thus, we can put $I = I_{n+1}$ and $I' = I_n$ in (3.10). Setting

$$\boldsymbol{J}_n := \int_{I_n}\int_{\mathbb{R}^N} u_\varepsilon^{\lambda_n}\,dxdt + \frac{(s + 2^{-n}\chi)h}{\delta^{q_*}}\delta^{\lambda_n} + \varepsilon^m \int_{I_n}\int_{\mathbb{R}^N} u_\varepsilon^{\Lambda_n - q + 2}\,dxdt,$$

we see from (3.10) that

$$\mu_0 \boldsymbol{J}_{n+1}{}^{\frac{1}{k}} \leq \left\{\frac{4}{2^{-n}\chi\delta^{q-1}} + 2^{q-1}(\lambda_n - q) + 2^{q-1}(\Lambda_n - q)\right\}\boldsymbol{J}_n. \tag{3.23}$$

Now we evaluate the coefficients in (3.23). Noting that $2^{-n}\chi\delta^{q-1} \leq 1$, $\lambda_n \leq r_1 k^{n-1}$ and $\Lambda_n \leq r_1 k^{n-1}$, we find that

$$\begin{aligned} &\frac{4}{2^{-n}\chi\delta^{q-1}} + 2^{q-1}(\lambda_n - q) + 2^{q-1}(\Lambda_n - q) \\ &\leq \frac{1}{2^{-n}\chi\delta^{q-1}}\left\{4 + 2^{q-1}(\lambda_n - q) + 2^{q-1}(\Lambda_n - q)\right\} \\ &\leq \frac{2^{q-1}}{2^{-n}\chi\delta^{q-1}}(\lambda_n + \Lambda_n) \\ &\leq \frac{2^q r_1}{\chi\delta^{q-1}}2^n k^{n-1}. \end{aligned} \tag{3.24}$$

From (3.23) and (3.24) it follows that

$$\boldsymbol{J}_{n+1}{}^{\frac{1}{k}} \leq \frac{2^q r_1}{\mu_0\chi\delta^{q-1}}2^n k^{n-1}\boldsymbol{J}_n =: B \cdot 2^n k^{n-1}\boldsymbol{J}_n. \tag{3.25}$$

Therefore we obtain

$$
\begin{aligned}
(3.26)\qquad & \left(\boldsymbol{J}_{n+1}\right)^{\frac{1}{k^n}} \\
& \leq \left(B\cdot 2^n k^{n-1}\right)^{\frac{1}{k^{n-1}}}\left(B\cdot 2^{n-1}k^{n-2}\right)^{\frac{1}{k^{n-2}}}\times\cdots\times\left(B\cdot 2\right)\boldsymbol{J}_1 \\
& = (2B)^{\frac{1}{k^{n-1}}+\frac{1}{k^{n-2}}+\cdots+1}(2k)^{\frac{n-1}{k^{n-1}}+\frac{n-2}{k^{n-2}}+\cdots+\frac{1}{k}}\boldsymbol{J}_1.
\end{aligned}
$$

From the definition of $\boldsymbol{J}_n$ and (3.22) we see that

$$
\begin{aligned}
\liminf_{n\to\infty}\left(\boldsymbol{J}_{n+1}\right)^{\frac{1}{k^n}} &\geq \liminf_{n\to\infty}\left(\int_{\tau-2^{-n}\chi}^{\tau+s}\int_{\mathbb{R}^N} u_\varepsilon^{\lambda_{n+1}}\,dxdt\right)^{\frac{r_1-\lambda_0}{\lambda_{n+1}-\lambda_0}} \\
&\geq \liminf_{n\to\infty}\|u_\varepsilon\|_{L^{\lambda_{n+1}}(\tau,\tau+s;L^{\lambda_{n+1}}(\mathbb{R}^N))}^{\frac{\lambda_{n+1}}{\lambda_{n+1}-\lambda_0}\cdot r_1-\lambda_0} \\
&= \|u_\varepsilon\|_{L^\infty(\tau,\tau+s;L^\infty(\mathbb{R}^N))}^{r_1-\lambda_0}.
\end{aligned}
$$

Hence it follows from (3.26) that

$$
\begin{aligned}
&\|u_\varepsilon\|_{L^\infty(\tau,\tau+s;L^\infty(\mathbb{R}^N))}^{r_1-\lambda_0} \\
&\leq \liminf_{n\to\infty}\left(\boldsymbol{J}_{n+1}\right)^{\frac{1}{k^n}} \\
&\leq \limsup_{n\to\infty}(2B)^{\frac{1}{k^{n-1}}+\frac{1}{k^{n-2}}+\cdots+1}(2k)^{\frac{n-1}{k^{n-1}}+\frac{n-2}{k^{n-2}}+\cdots+\frac{1}{k}}\boldsymbol{J}_1 \\
&= (2B)^{\frac{k}{k-1}}(2k)^{\frac{k}{(k-1)^2}}\left((1+\varepsilon^m)\int_{\tau-\chi}^{\tau+s}\int_{\mathbb{R}^N} u_\varepsilon^{r_1}\,dxdt+\left(s+\frac{\chi}{2}\right)h\delta^{r_1-q_*}\right).
\end{aligned}
$$

Therefore we obtain (3.21). □

Now we prove Proposition 3.1. From Lemmas 3.2–3.4 we can obtain L^∞-L^r estimate without assuming that the initial data is small. In the proof of Proposition 3.1 we assume the smallness condition of the initial data to apply the L^r-decay property of u_ε.

Proof of Proposition 3.1. Put $q_* := \frac{N}{2}(q-m)$, $k := 1+\frac{2}{N}$ and let $r_* > q_*+q-1$ be some constant. Let $r_1 := \max\{m+q-2, r_*, q_*+q-1, \frac{N}{2}+q-1\}$ and $0<\chi<t<T$. From Lemma 3.4, u_ε satisfies (3.21). Moreover, (3.21) implies that for a.a. $0<t<T$,

$$
\begin{aligned}
(3.27)\qquad & \|u_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)}^{r_1-(q_*+q-1)} \\
& \leq (2B)^{\frac{k}{k-1}}(2k)^{\frac{k}{(k-1)^2}}\left((1+\varepsilon^m)\int_{t-\chi}^{t}\int_{\mathbb{R}^N} u_\varepsilon^{r_1}\,dxds+\frac{\chi}{2}h\delta^{r_1-q_*}\right),
\end{aligned}
$$

where $B = \frac{2^q r_1}{\mu_0\chi\delta^{q-1}} > 0$, $h := \sup_{t\in[0,T]}\|u_\varepsilon(t)\|_{L^{q_*}(\mathbb{R}^N)}^{q_*} = \|u_{0\varepsilon}\|_{L^{q_*}}^{q_*}$ and $\mu_0 = \mu_0(m,q,N,h)$ is the same constant as in the proof of Lemma 3.3. Let $0<t<T$. Taking χ and δ such that $\chi = \delta^{-(q-1)} = \frac{t}{2}$ in (3.27), and noting that $t\mapsto\|u_\varepsilon(t)\|_{L^{r_1}(\mathbb{R}^N)}$ is a non-increasing function

on $[0,T)$ and using the L^r-decay property (see Proposition 2.1 and Remark 3.2), we see that

$$\begin{aligned}\|u_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)}^{r_1-(q_*+q-1)} &\le C_1\Big\{(1+\varepsilon^m)\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^N} u_\varepsilon^{r_1}\,dxds + \frac{h}{2}\Big(\frac{t}{2}\Big)^{1-\frac{r_1-q_*}{q-1}}\Big\}\\
&\le C_1\Big\{(1+\varepsilon^m)\frac{t}{2}\int_{\mathbb{R}^N} u_\varepsilon^{r_1}\Big(\frac{t}{2}\Big)\,dx + \frac{h}{2}\Big(\frac{t}{2}\Big)^{1-\frac{r_1-q_*}{q-1}}\Big\}\\
&\le C_1\Big\{(1+\varepsilon^m)C_{r_1}\frac{t}{2}\Big(\frac{t}{2}+1\Big)^{-\frac{N(r_1-1)}{N(m-1)+2}} + \frac{h}{2}\Big(\frac{t}{2}\Big)^{1-\frac{r_1-q_*}{q-1}}\Big\}\\
&\le C_1 2^{\frac{r_1-q_*-q+1}{q-1}}\Big\{(1+\varepsilon^m)C_{r_1}+\frac{h}{2}\Big\}t^{-\frac{r_1-q_*-q+1}{q-1}},\end{aligned}$$

where

$$C_1 = \Big(\frac{2^{q+1}r_1}{\mu_0}\Big)^{\frac{k}{k-1}}(2k)^{\frac{k}{(k-1)^2}}$$

and C_{r_1} is the same constant as in (2.2). Thus we obtain

$$\begin{aligned}\|u_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)} &\le K_0(\varepsilon)t^{-\frac{1}{q-1}}\\
&= K_0(\varepsilon)t^{-\frac{N}{N(m-1)+2q_*}}, \quad \text{a.a. } t\in(0,T),\end{aligned} \tag{3.28}$$

where

$$K_0(\varepsilon) = 2^{\frac{1}{q-1}}\Big\{C_1(1+\varepsilon^m)C_{r_1}+\frac{h}{2}\Big\}^{\frac{1}{r_1-(q_*+q-1)}}.$$

It follows from (3.28) that for a.a. $t\in(0,T)$,

$$\begin{aligned}\|u_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)} &\le K_0(\varepsilon)t^{-\frac{N}{N(m-1)+2q_*}}\\
&\le K_0(1)t^{-\frac{N}{N(m-1)+2q_*}}.\end{aligned} \tag{3.29}$$

This inequality and $\|u_{0\varepsilon}\|_{L^r}\le\|u_0\|_{L^r}$ $(1\le r\le\infty)$ show that the right-hand side of this inequality is independent of ε. Hence we see that

$$\begin{aligned}\|u(t)\|_{L^\infty(\mathbb{R}^N)} &\le \liminf_{\varepsilon\to0}\|u_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)}\\
&\le \liminf_{\varepsilon\to0}K_0(\varepsilon)t^{-\frac{N}{N(m-1)+2q_*}}\\
&= K_1t^{-\frac{N}{N(m-1)+2q_*}},\end{aligned}$$

where $K_1 := K_0(1) > 0$ is a constant which depends on $\|u_0\|_{L^{q_*}}$, C_{r_1}, r_1, m, q and N. Therefore we obtain the desired inequality (3.3). □

Remark 3.3. The estimate (3.3) holds for some $r\ge r_1$. In fact, by recalling the definitions of λ_n and Λ_n, we see that if $\lambda_1=\Lambda_1=r$, then (3.3) holds with $r_1=r$.

3.2. L^∞-decay property

In this subsection we prove the L^∞-decay property of solutions to $(\mathrm{KS})_0$.

Proposition 3.5. (L^∞-decay property) *Let $N \geq 2$, $m \geq 1$, $q \geq 2$ and $\rho \in (0,1]$. Let (u,v) be a global weak solution to $(\mathrm{KS})_0$ on $[0,\infty)$. Assume further that m and q satisfy*

$$q > m + \frac{2}{N} \tag{3.30}$$

and u_0 satisfies (1.1) and the smallness condition as in Theorem 1.1. Then the solution u has the following decay property:

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq K_\rho (t+\rho)^{-\frac{N}{N(m-1)+2}}, \quad \text{a.a. } t \in [5\rho, \infty), \tag{3.31}$$

where $K_\rho = K_\rho(\rho, r, C_r, \|u_0\|_{L^1}, \|u_0\|_{L^{q_*}}, \|u_0\|_{L^r}, m, q, N)$ with $q_* = \frac{N}{2}(q-m)$ and $r \geq r_3 = r_3(m,q,N)$ are positive constants and C_r is the same constant as in Proposition 2.1.

The proof is based on [**17**, Sections 5–7]. To this end we need three lemmas.

Lemma 3.6. *Let $N \geq 2$, $m \geq 1$, $q \geq 2$, $\varepsilon \in (0,1)$, $\rho \in (0,1]$ and r_1 is the same constant as in Section 3.1. Let $(u_\varepsilon, v_\varepsilon)$ be a unique solution to $(\mathrm{KS})_\varepsilon$ on $[0,\infty)$. Assume that m and q satisfy (3.2) and u_0 satisfies the smallness condition as in Theorem 1.1. Then for $r \geq r_1$ and a.a. $t \in [2\rho, \infty)$,*

$$\|u_\varepsilon(t)\|^{r-(q_*+q-1)}_{L^\infty(\mathbb{R}^N))} \leq C'_\rho \|u_\varepsilon(t-\rho)\|^{r\{1-\frac{q-1}{r-q_*}(1+\frac{N}{2})\}}_{L^r(\mathbb{R}^N)}, \tag{3.32}$$

where $q_* = \frac{N}{2}(q-m)$ and $C'_\rho = C'_\rho(\rho, \varepsilon, r, \|u_{0\varepsilon}\|_{L^r}, \|u_{0\varepsilon}\|_{L^{q_*}}, m, q, N) > 0$ is a constant.

Proof. Let $\rho \in (0,1]$, $r \geq r_1$ (see Section 3.1), $t \geq 2\rho$, $q_* = \frac{N}{2}(q-m)$ and $k = 1 + \frac{2}{N}$. Since $t \mapsto \|u_\varepsilon(t)\|_{L^r(\mathbb{R}^N)}$ is a non-increasing function, we can take χ and δ such that

$$\chi = \rho, \quad \delta = \Big(\|u_\varepsilon(t-\rho)\|^{\frac{r}{r-q_*}}_{L^r(\mathbb{R}^N)}\Big)\Big(\|u_{0\varepsilon}\|^{\frac{r}{r-q_*}}_{L^r(\mathbb{R}^N)}\Big)^{-1} (\leq 1)$$

in (3.27) with $r_1 = r$ (see Remark 3.3). Hence it follows that for a.a. $t \geq 2\rho$,

$$\begin{aligned}
&\|u_\varepsilon(t)\|^{r-(q_*+q-1)}_{L^\infty(\mathbb{R}^N)} \\
&\leq C_\rho\Big(\|u_\varepsilon(t-\rho)\|^{\frac{r}{r-q_*}}_{L^r(\mathbb{R}^N)}\Big)^{-\frac{(q-1)k}{k-1}} \\
&\quad \times \left\{(1+\varepsilon^m)\int_{t-\rho}^t \|u_\varepsilon(s)\|^r_{L^r(\mathbb{R}^N)}\,ds + \frac{\rho h}{2}\Big(\frac{\|u_\varepsilon(t-\rho)\|_{L^r(\mathbb{R}^N)}}{\|u_{0\varepsilon}\|_{L^r(\mathbb{R}^N)}}\Big)^r\right\} \\
&\leq C_\rho\Big(\|u_\varepsilon(t-\rho)\|^{\frac{r}{r-q_*}}_{L^r(\mathbb{R}^N)}\Big)^{-\frac{(q-1)k}{k-1}}\Big(\rho(1+\varepsilon^m) + \frac{\rho h}{2}\|u_{0\varepsilon}\|^{-r}_{L^r(\mathbb{R}^N)}\Big)\|u_\varepsilon(t-\rho)\|^r_{L^r(\mathbb{R}^N)} \\
&= C_\rho\Big(\rho(1+\varepsilon^m) + \frac{\rho h}{2}\|u_{0\varepsilon}\|^{-r}_{L^r(\mathbb{R}^N)}\Big)\Big(\int_{\mathbb{R}^N} u_\varepsilon(t-\rho)^r\,dx\Big)^{1-\frac{q-1}{r-q_*}(1+\frac{N}{2})},
\end{aligned}$$

where

$$C_\rho = \Big(\frac{2^{q+1}r}{\rho\mu_0}\|u_{0\varepsilon}\|_{L^r(\mathbb{R}^N)}^{\frac{r(q-1)}{r-q_*}}\Big)^{\frac{k}{k-1}}(2k)^{\frac{k}{(k-1)^2}}$$

and μ_0 is the same constant as in the proof of Lemma 3.3. Therefore we obtain (3.32), where $C'_\rho = C_\rho(\rho(1+\varepsilon^{m-1}) + \frac{\rho h}{2}\|u_{0\varepsilon}\|_{L^r}^{-r})$. □

Lemma 3.7. *Let $N \geq 2$, $m \geq 1$, $q \geq 2$, $\varepsilon \in (0,1)$, $\rho \in (0,1]$ and $t \geq 2\rho$. Let $(u_\varepsilon, v_\varepsilon)$ be a unique solution to* $(\mathrm{KS})_\varepsilon$ *on* $[0,\infty)$. *Assume that m and q satisfy* (3.2). *Put*

$$G(s) := (r-1)\int_0^s (\tau + \varepsilon^{\frac{m}{q-2}})^{q-2}\,d\tau, \tag{3.33}$$

$$w_\varepsilon(x,t) := u_\varepsilon e^{-\int_{2\rho}^t G(\|u_\varepsilon(s)\|_{L^\infty(\mathbb{R}^N)})\,ds}. \tag{3.34}$$

Then w_ε satisfies the following:

$$\|w_\varepsilon(t)\|_{L^1(\mathbb{R}^N)} \leq \|u_{0\varepsilon}\|_{L^1}, \quad t \geq 2\rho, \tag{3.35}$$

$$\frac{d}{dt}\int_{\mathbb{R}^N} w_\varepsilon^{r-m+1}(t)\,dx + \mu_1\int_{\mathbb{R}^N}|\nabla w_\varepsilon^{\frac{r}{2}}(t)|^2\,dx \leq 0, \quad r > m,\ t \geq 2\rho, \tag{3.36}$$

$$t \mapsto \|w_\varepsilon(t)\|_{L^r(\mathbb{R}^N)}\ (1 \leq r < \infty) \text{ is a non-increasing function on } [2\rho,\infty), \tag{3.37}$$

where $\mu_1 = \mu_1(m)$ is a positive constant.

Proof. First we prove (3.35). From the definition of w_ε and the mass conservation law, we see that for $t \geq 2\rho$,

$$\|w_\varepsilon(t)\|_{L^1(\mathbb{R}^N)} \leq \|u_\varepsilon(t)\|_{L^1(\mathbb{R}^N)} = \|u_{0\varepsilon}\|_{L^1}.$$

Thus we obtain (3.35). Next we prove (3.36). Let $r > 1$ and $t \geq 2\rho$. Differentiating w_ε about t, we see by the first approximate equation $(1)_\varepsilon$ (see $(\mathrm{KS})_\varepsilon$ in the top of Section 3) that

$$\begin{aligned}\frac{dw_\varepsilon}{dt} &= e^{-\int_{2\rho}^t G(\|u_\varepsilon(s)\|_{L^\infty})\,ds} \\ &\quad \times \big(\nabla\cdot(\nabla(u_\varepsilon+\varepsilon)^m - (u_\varepsilon + \varepsilon^{\frac{m}{q-2}})^{q-2}u_\varepsilon\nabla v_\varepsilon) - u_\varepsilon G(\|u_\varepsilon(t)\|_{L^\infty})\big).\end{aligned} \tag{3.38}$$

Multiplying (3.38) by w_ε^{r-1} and integrating it over $\mathbb{R}^N$, we have

$$\begin{aligned}&\frac{1}{r}\frac{d}{dt}\|w_\varepsilon(t)\|_{L^r(\mathbb{R}^N)}^r \\ &= \Big(e^{-r\int_{2\rho}^t G(\|u_\varepsilon(s)\|_{L^\infty})\,ds}\Big) \times \Big(\int_{\mathbb{R}^N}\nabla\cdot(\nabla(u_\varepsilon+\varepsilon)^m - (u_\varepsilon+\varepsilon^{\frac{m}{q-2}})^{q-2}u_\varepsilon\nabla v_\varepsilon)u_\varepsilon^{r-1}\,dx \\ &\qquad\qquad - \int_{\mathbb{R}^N} u_\varepsilon^r G(\|u_\varepsilon(t)\|_{L^\infty})\,dx\Big) \\ &=: \Big(e^{-r\int_{2\rho}^t G(\|u_\varepsilon(s)\|_{L^\infty})\,ds}\Big) \times (\boldsymbol{I}_4 - \boldsymbol{I}_5).\end{aligned} \tag{3.39}$$

By a similar argument from (3.5) to (3.7) in Lemma 3.2, it follows that

$$(3.40)\quad \boldsymbol{I}_4 \le -\frac{4m(r-1)}{(r+m-1)^2}\|\nabla u_\varepsilon^{\frac{r+m-1}{2}}(t)\|^2_{L^2(\mathbb{R}^N)} - \frac{4m(r-1)\varepsilon^{m-1}}{r^2}\|\nabla u_\varepsilon^{\frac{r}{2}}(t)\|^2_{L^2(\mathbb{R}^N)} + (r-1)\int_{\mathbb{R}^N} u_\varepsilon F(u_\varepsilon)\,dx,$$

where

$$F(s) := \int_0^s (\tau + \varepsilon^{\frac{m}{q-2}})^{q-2}\tau^{r-1}\,d\tau.$$

Recalling the definition of the function G, we see that

$$\begin{aligned}(3.41)\quad &(r-1)\int_{\mathbb{R}^N} u_\varepsilon F(u_\varepsilon)\,dx - \boldsymbol{I}_5\\ &= (r-1)\int_{\mathbb{R}^N}\Big\{u_\varepsilon\int_0^{u_\varepsilon}(\tau+\varepsilon^{\frac{m}{q-2}})^{q-2}\tau^{r-1}\,d\tau - u_\varepsilon^r\int_0^{\|u_\varepsilon(t)\|_{L^\infty}}(\tau+\varepsilon^{\frac{m}{q-2}})^{q-2}\,d\tau\Big\}\,dx\\ &\le (r-1)\int_{\mathbb{R}^N}\Big\{u_\varepsilon^r\Big(\int_0^{u_\varepsilon} - \int_0^{\|u_\varepsilon(t)\|_{L^\infty}}\Big)(\tau+\varepsilon^{\frac{m}{q-2}})^{q-2}\,d\tau\Big\}\,dx\\ &\le 0.\end{aligned}$$

Hence it follows from (3.39)–(3.41) that

$$(3.42)\quad \frac{d}{dt}\|w_\varepsilon(t)\|^r_{L^r(\mathbb{R}^N)} \le \Big(-e^{-r\int_{2\rho}^t G(\|u_\varepsilon(s)\|_{L^\infty})\,ds}\Big)\cdot 4mr(r-1) \times\Big(\frac{1}{(r+m-1)^2}\|\nabla u_\varepsilon^{\frac{r+m-1}{2}}(t)\|^2_{L^2(\mathbb{R}^N)} + \frac{\varepsilon^{m-1}}{r^2}\|\nabla u_\varepsilon^{\frac{r}{2}}(t)\|^2_{L^2(\mathbb{R}^N)}\Big).$$

Since

$$\|\nabla u_\varepsilon^{\frac{r+m-1}{2}}\|^2_{L^2(\mathbb{R}^N)} = \Big(e^{(r+m-1)\int_{2\rho}^t G(\|u_\varepsilon\|_{L^\infty})\,ds}\Big)\cdot\|\nabla w_\varepsilon^{\frac{r+m-1}{2}}\|^2_{L^2(\mathbb{R}^N)}$$

by the definition of w_ε, we see from (3.42) that

$$(3.43)\quad \frac{d}{dt}\|w_\varepsilon(t)\|^r_{L^r(\mathbb{R}^N)} \le -e^{(m-1)\int_{2\rho}^t G(\|u_\varepsilon(s)\|_{L^\infty})\,ds}\frac{4mr(r-1)}{(r+m-1)^2}\|\nabla w_\varepsilon^{\frac{r+m-1}{2}}(t)\|^2_{L^2(\mathbb{R}^N)}.$$

Replacing r by $r-m+1$ in (3.43) and setting $\mu_1 := \inf_{r\ge m}\frac{4m(r-m+1)(r-m)}{r^2}$, we obtain (3.36) for $r > m$. Finally we prove (3.37). From (3.43) we see that for $r \ge 1$,

$$\frac{d}{dt}\|w_\varepsilon(t)\|^r_{L^r(\mathbb{R}^N)} \le 0, \quad t \ge 2\rho,$$

so $t \mapsto \|w_\varepsilon(t)\|_{L^r(\mathbb{R}^N)}$ $(1 \le r < \infty)$ is a non-increasing function on $[2\rho, \infty)$. □

The next lemma gives the L^∞-estimate of w_ε. The lemma similar to Lemma 3.8 is proved in [**17**, Section 6], where they considered the following function $\widetilde{w}_\varepsilon$ instead of w_ε:

$$\widetilde{w}_\varepsilon(x,t) := u_\varepsilon \exp\Big(-\int_{2\rho}^t \|u_\varepsilon(s)\|^{q-1}_{L^\infty(\mathbb{R}^N)}\,ds\Big).$$

The proof starts with (3.36) and uses (3.37) with $r = \frac{2N}{N-1}$, $r = 2$ and (3.35). Thus the next lemma is proved by using not the definition of w_ε but the property of w_ε.

Lemma 3.8. *Let $N \geq 2$, $m \geq 1$, $q \geq 2$, $\varepsilon \in (0,1)$ and $\rho \in (0,1]$. Let $(u_\varepsilon, v_\varepsilon)$ be a unique solution to* $(\mathrm{KS})_\varepsilon$ *on $[0,\infty)$. Assume that m and q satisfy* (3.2) *and u_0 satisfies* (1.1) *and the smallness condition as in Theorem 1.1. Put G and w_ε as in* (3.33) *and* (3.34). *Assume further that $\delta' > 0$ satisfies*

$$t^{\frac{1}{2}}\delta'^{\frac{1}{N}+\frac{m-1}{2}} \leq 1.$$

Then

$$\|w_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)} \leq C_3\big((t+\rho)\delta'^{m-1}\big)^{-\frac{N}{2}}\Big(\int_{\mathbb{R}^N} w_\varepsilon^r\Big(\frac{t}{2}-\frac{\rho}{2}\Big)\,dx + \|u_{0\varepsilon}\|_{L^1}\delta'^{r-1}\Big), \quad t \geq 5\rho, \tag{3.44}$$

where $C_3 = C_3(\|u_{0\varepsilon}\|_{L^1}, m, q, N)$ is a positive constant.

Proof of Proposition 3.5. Let $\rho \in (0,1]$, $r \geq r_1$ (see Section 3.1) and $t \geq 5\rho$. We use the same notation as (3.33) and (3.34). Recalling the definition of w_ε, we see that

$$\int_{\mathbb{R}^N} w_\varepsilon^r\Big(\frac{t}{2}-\frac{\rho}{2}\Big)\,dx \leq \int_{\mathbb{R}^N} u_\varepsilon^r\Big(\frac{t}{2}-\frac{\rho}{2}\Big)\,dx, \tag{3.45}$$

$$\|u_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)} \leq \exp\Big(\int_{2\rho}^t G(\|u_\varepsilon(s)\|_{L^\infty})\,ds\Big)\|w_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)}. \tag{3.46}$$

It follows from (3.46), (3.44) in Lemma 3.8 and (3.45) that

$$\begin{aligned} &\|u_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)}^r \\ &\leq C_3 e^{r\int_{2\rho}^t G(\|u_\varepsilon(s)\|_{L^\infty})\,ds}\big((t+\rho)\delta'^{m-1}\big)^{-\frac{N}{2}}\Big(\int_{\mathbb{R}^N} u_\varepsilon^r\Big(\frac{t}{2}-\frac{\rho}{2}\Big)\,dx + \|u_{0\varepsilon}\|_{L^1}\delta'^{r-1}\Big). \end{aligned} \tag{3.47}$$

Take $\delta' = (t+\rho)^{-\frac{N}{N(m-1)+2}}$ in (3.47). It follows from the L^r-decay property of u_ε (see (2.2) in Proposition 2.1 and Remark 3.2) that

$$\begin{aligned} \|u_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)}^r &\leq C_3 e^{r\int_{2\rho}^t G(\|u_\varepsilon(s)\|_{L^\infty})\,ds} \\ &\quad\times (t+\rho)^{-\frac{N}{N(m-1)+2}}\Big(\int_{\mathbb{R}^N} u_\varepsilon^r\Big(\frac{t}{2}-\frac{\rho}{2}\Big)\,dx + \|u_{0\varepsilon}\|_{L^1}(t+\rho)^{-\frac{N(r-1)}{N(m-1)+2}}\Big) \\ &\leq C_3 C_r^r e^{r\int_{2\rho}^t G(\|u_\varepsilon(s)\|_{L^\infty})\,ds} \\ &\quad\times (t+\rho)^{-\frac{N}{N(m-1)+2}}\Big(\Big(\frac{t}{2}-\frac{\rho}{2}+1\Big)^{-\frac{N(r-1)}{N(m-1)+2}} + \|u_{0\varepsilon}\|_{L^1}(t+\rho)^{-\frac{N(r-1)}{N(m-1)+2}}\Big) \\ &= C_4 e^{r\int_{2\rho}^t G(\|u_\varepsilon(s)\|_{L^\infty})\,ds}(t+\rho)^{-\frac{Nr}{N(m-1)+2}}, \end{aligned}$$

where

$$C_4 = C_3 C_r^r\big(2^{\frac{N(r-1)}{N(m-1)+2}} + \|u_{0\varepsilon}\|_{L^1}\big),$$

C_3 and C_r are the same constants as in Lemma 3.8 and Proposition 2.1, respectively. Hence we have

$$\|u_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)} \leq C_4^{\frac{1}{r}}\exp\Big(\int_{2\rho}^t G(\|u_\varepsilon(s)\|_{L^\infty(\mathbb{R}^N)})\,ds\Big)(t+\rho)^{-\frac{N}{N(m-1)+2}}, \quad t \geq 5\rho. \tag{3.48}$$

Here we estimate the function G. From (3.32) in Lemma 3.6 and the L^r-decay property (2.2), it follows that a.a. $t \geq 2\rho$,

$$
\begin{aligned}
(3.49)\quad & \int_{2\rho}^{t} G(\|u_\varepsilon(s)\|_{L^\infty(\mathbb{R}^N)})\,ds \\
&= (r-1)\int_{2\rho}^{t}\int_{0}^{\|u_\varepsilon(s)\|_{L^\infty}} (\tau+\varepsilon^{\frac{m}{q-2}})^{q-2}\,d\tau ds \\
&= \frac{r-1}{q-1}\int_{2\rho}^{t}\Big\{(\|u_\varepsilon(s)\|_{L^\infty(\mathbb{R}^N)}+\varepsilon^{\frac{m}{q-2}})^{q-1}-\varepsilon^{\frac{m(q-1)}{q-2}}\Big\}\,ds \\
&\leq \frac{r-1}{q-1}\int_{2\rho}^{t}\Big\{\Big(\big(C'_\rho\|u_\varepsilon(s-\rho)\|_{L^r(\mathbb{R}^N)}^{r\{1-\frac{q-1}{r-q_*}(1+\frac{N}{2})\}}\big)^{\frac{1}{r-(q_*+q-1)}}+\varepsilon^{\frac{m}{q-2}}\Big)^{q-1}-\varepsilon^{\frac{m(q-1)}{q-2}}\Big\}\,ds \\
&\leq \frac{r-1}{q-1}\int_{2\rho}^{t}\Big\{(C_5(s-\rho+1)^{-\beta}+\varepsilon^{\frac{m}{q-2}})^{q-1}-\varepsilon^{\frac{m(q-1)}{q-2}}\Big\}\,ds,
\end{aligned}
$$

where
$$\beta=\frac{N(r-1)}{N(m-1)+2}\Big\{1-\frac{q-1}{r-q_*}\Big(1+\frac{N}{2}\Big)\Big\}\frac{1}{r-(q_*+q-1)},$$
C'_ρ is the same constant as in (3.32) and $C_5=C_5(C'_\rho,C_r,r,m,q,N)$ is a positive constant. From (3.29) (see the proof of Proposition 3.1), (3.48) and (3.49) we see that a.a. $t\geq 1$,

$$
\begin{aligned}
(3.50)\quad & \|u(t)\|_{L^\infty(\mathbb{R}^N)} \\
&\leq \liminf_{\varepsilon\to 0}\|u_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)} \\
&\leq \liminf_{\varepsilon\to 0}\Big\{C_4^{\frac{1}{r}}\exp\Big(\int_{2\rho}^{t}G(\|u_\varepsilon(s)\|_{L^\infty(\mathbb{R}^N)})\,ds\Big)(t+\rho)^{-\frac{N}{N(m-1)+2}}\Big\} \\
&\leq \liminf_{\varepsilon\to 0}\Big[C_4^{\frac{1}{r}}\exp\Big(\frac{r-1}{q-1}\int_{2\rho}^{t}\Big\{\Big(C_5(s-\rho+1)^{-\beta}+\varepsilon^{\frac{m}{q-2}}\Big)^{q-1}-\varepsilon^{\frac{m(q-1)}{q-2}}\Big\}\,ds\Big) \\
&\qquad\qquad \times (t+\rho)^{-\frac{N}{N(m-1)+2}}\Big] \\
&= C_4^{\frac{1}{r}}\exp\Big(\int_{2\rho}^{t}C_6(s-\rho+1)^{-\beta(q-1)}\,ds\Big)(t+\rho)^{-\frac{N}{N(m-1)+2}} \\
&\leq C_4^{\frac{1}{r}}\exp\Big(\int_{2\rho}^{\infty}C_6(s-\rho+1)^{-\beta(q-1)}\,ds\Big)(t+\rho)^{-\frac{N}{N(m-1)+2}},
\end{aligned}
$$

where $C_6=\frac{C_5(r-1)}{q-1}$. When $q>m+\frac{2}{N}$, we have

$$
\begin{aligned}
-\beta(q-1) &= -\frac{N(q-1)}{N(m-1)+2}\Big\{1-\frac{q-1}{r-q_*}\Big(1+\frac{N}{2}\Big)\Big\}\frac{r-1}{r-(q_*+q-1)} \\
&\to -\frac{N(q-1)}{N(m-1)+2}<-1 \quad (r\to\infty).
\end{aligned}
$$

Hence there exists r_2 such that $-\beta(q-1) < -1$ for $r \geq r_2$. It follows that for $r \geq r_2$,

$$\int_{2\rho}^{\infty} C_6(s-\rho+1)^{-\beta(q-1)}\,ds = \frac{C_6(\rho+1)^{-\beta(q-1)+1}}{\beta(q-1)-1}. \tag{3.51}$$

Therefore we see from (3.50) and (3.51) that for $r \geq r_3 := \max\{r_1, r_2\}$,

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq K_\rho(t+\rho)^{-\frac{N}{N(m-1)+2}},$$

where K_ρ $C_4^{\frac{1}{r}} \exp(\frac{C_6(\rho+1)^{-\beta(q-1)+1}}{\beta(q-1)-1})$. This is the required decay property. □

Proof of Theorem 1.1 when $N \geq 2$. From Propositions 3.1 and 3.5 with $r = r_3$ (see the proof of Proposition 3.5) we see that

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq \begin{cases} Kt^{-\frac{N}{N(m-1)+2q_*}}, & \text{a.a. } t \in (0,\infty), \\ K_\rho(t+\rho)^{-\frac{N}{N(m-1)+2}}, & \text{a.a. } t \in (5\rho,\infty), \end{cases}$$

where $q_* = \frac{N}{2}(q-m)$, $\rho \in (0,1]$, $K = K(\|u_0\|_{L^{r_3}}, C_{r_3}, r_3, m, q, N) > 0$ and $K_\rho = K_\rho(\rho, \|u_0\|_{L^1}, \|u_0\|_{L^{q_*}}, \|u_0\|_{L^{r_3}}, C_{r_3}, r_3, m, q, N) > 0$ are constants, where C_r is the same constant as in Proposition 2.1. Thus we obtain (1.3) and (1.4). □

4. The case where $N = 1$

In this section we consider the case where $N = 1$. First we introduce the approximate problem when $N = 1$.

$$(\mathrm{KS})_{\varepsilon,N=1} \quad \begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} = \dfrac{\partial^2}{\partial x^2}(u_\varepsilon+\varepsilon)^m - \dfrac{\partial}{\partial x}\Big(u_\varepsilon^{q-1}\dfrac{\partial v_\varepsilon}{\partial x}\Big) & \text{in } \mathbb{R}\times(0,T), \quad \cdots(1)_{\varepsilon,N=1} \\ 0 = \dfrac{\partial^2 v_\varepsilon}{\partial x^2} - v_\varepsilon + u_\varepsilon & \text{in } \mathbb{R}\times(0,T), \quad \cdots(2)_{\varepsilon,N=1} \\ u_\varepsilon(x,0) = u_{0\varepsilon}(x), & x \in \mathbb{R}, \end{cases}$$

where $m \geq 1$, $q \geq 2$ and $\varepsilon \in (0,1)$. The initial data $u_{0\varepsilon} \in C_0^\infty(\mathbb{R})$ is given as $u_{0\varepsilon} := (\rho_\varepsilon * u_0)\,\zeta_\varepsilon$; ρ_ε is the mollifier and ζ_ε is the standard cut function.

Note that the nonlinear term in the first equation of $(\mathrm{KS})_{\varepsilon,N=1}$ is different from the approximate nonlinear term in the case where $N \geq 2$ (see $(\mathrm{KS})_\varepsilon$ in Section 3). The reason is that the condition $q \geq m + \frac{2}{N}$ gives $q \geq 3$ when $N = 1$. This condition relates with $\|\nabla u_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)}$ (see [**16**, Proposition 9]). Differentiating the nonlinear term $\nabla(u^{q-1}\nabla v)$ in $(\mathrm{KS})_0$ about x formally to obtain the estimate of $\|\nabla u_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)}$, we see that

$$\begin{aligned} \frac{\partial}{\partial x_j}\nabla(u^{q-1}\nabla v) &= (q-1)(q-2)\sum_{i=1}^{N} u^{q-3}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} \\ &\quad + (q-1)\sum_{i=1}^{N} u^{q-2}\frac{\partial}{\partial x_j}\Big(\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\Big) + \frac{\partial}{\partial x_j}(u^{q-1}\Delta v), \quad j = 1,\cdots,N. \end{aligned}$$

Therefore if $q \geq 3$, then it is not necessary to approximate the nonlinear term to non-degenerate type (see Remark 2.2).

We obtain the following two propositions by the proofs parallel to Propositions 3.1 and 3.5.

Proposition 4.1 (L^∞-estimate when $N = 1$). *Let $m \geq 1$, $q \geq 2$, and $T > 0$. Let (u, v) be a weak solution to* $(\mathrm{KS})_0$ *on $[0, T)$. Assume further that m and q satisfy*

$$q \geq m + 2$$

and u_0 satisfies (1.1) *and the smallness condition* (1.2) *in Theorem 1.1. Then the following estimate holds:*

$$\|u(t)\|_{L^\infty(\mathbb{R})} \leq K_2 t^{-\frac{1}{m-1+2q_{**}}}, \quad \text{a.a. } t \in (0, T),$$

where $q_{**} = \frac{q-m}{2}$, $K_2 = K_2(\|u_0\|_{L^{q_{**}}}, C_r, m, q, N)$, $r \geq r' = r'(m, q, N)$ are positive constants and C_r is the same constant as in Proposition 2.1.

Proposition 4.2 (L^∞-decay property when $N = 1$). *Let $m \geq 1$, $q \geq 2$ and $\rho \in (0, 1]$. Let (u, v) be a global weak solution to* $(\mathrm{KS})_0$. *Assume further that m and q satisfy*

$$q > m + 2$$

and u_0 satisfies (1.1) *and the smallness condition* (1.2) *in Theorem 1.1. Then the solution u has the following decay property:*

$$\|u(t)\|_{L^\infty(\mathbb{R})} \leq K'_\rho (t + \rho)^{-\frac{1}{m+1}}, \quad \text{a.a. } t \in [5\rho, \infty),$$

where $K'_\rho = K'_\rho(\rho, \|u_0\|_{L^1}, \|u_0\|_{L^{\frac{q-m}{2}}}, \|u_0\|_{L^r}, r, C_r, m, q, N)$, $r \geq r'' = r''(m, q, N)$ are positive constants, where C_r is the same constant as in Proposition 2.1.

To prove Proposition 4.2 we need that $t \mapsto \|w_\varepsilon(t)\|_{L^\infty(\mathbb{R})}$ is a non-increasing function on $[2\rho, \infty)$, where

$$w_\varepsilon(x, t) := u_\varepsilon \exp\Bigl(-\int_{2\rho}^{t} \|u_\varepsilon(s)\|_{L^\infty(\mathbb{R}^N)}^{q-1}\, ds\Bigr)$$

(because we use (3.37) with $r = \frac{2N}{N-1} = \infty$ for $N = 1$ as stated in the front of Lemma 3.8). This property of w_ε is proved as follows. From a similar proof to Lemma 3.7 we see that $t \mapsto \|w_\varepsilon(t)\|_{L^r(\mathbb{R})}$ $(1 \leq r < \infty)$ is a non-increasing function on $[2\rho, \infty)$. Let $t \geq s \geq 2\rho$. Then we have

$$\|w_\varepsilon(t)\|_{L^r(\mathbb{R})} \leq \|w_\varepsilon(s)\|_{L^r(\mathbb{R})} \quad (1 \leq r < \infty). \tag{4.1}$$

It follows from Proposition 4.1 that

$$\|w_\varepsilon(t)\|_{L^\infty(\mathbb{R})} \leq \|u_\varepsilon(t)\|_{L^\infty(\mathbb{R})} \leq K_2 t^{-\frac{1}{m-1+2q_{**}}}, \quad \text{a.a. } t \geq 2\rho,$$

and hence $w_\varepsilon(t) \in L^\infty(\mathbb{R})$ (a.a. $t \geq 2\rho$). Letting $r \to \infty$ in (4.1), we have

$$\|w_\varepsilon(t)\|_{L^\infty(\mathbb{R})} \leq \|w_\varepsilon(s)\|_{L^\infty(\mathbb{R})} \quad \text{a.a. } t \geq s \geq 2\rho.$$

Therefore we see that $t \mapsto \|w_\varepsilon(t)\|_{L^\infty(\mathbb{R})}$ is a non-increasing function on $[2\rho, \infty)$.

Proof of Theorem 1.1 when $N = 1$. Combining Propositions 4.1 and 4.2 with $r = \tilde{r} := \max\{r', r''\}$, we obtain

$$\|u(t)\|_{L^\infty(\mathbb{R})} \leq \begin{cases} \widetilde{K} t^{-\frac{1}{m-1+2q_{**}}}, & \text{a.a. } t \in (0,\infty), \\ \widetilde{K}_\rho (t+\rho)^{-\frac{1}{m+1}}, & \text{a.a. } t \in [5\rho,\infty), \end{cases}$$

where $q_{**} = \frac{q-m}{2}$, $\rho \in (0,1]$, $\widetilde{K} = \widetilde{K}(K_2, \tilde{r})$ and $\widetilde{K}_\rho = \widetilde{K}_\rho(K'_\rho, \tilde{r})$ are positive constants. Thus we obtain (1.3) and (1.4). □

Acknowledgments

The author would like to express her sincere gratitude to Professor Masashi Misawa and Professor Takayoshi Ogawa for giving her an opportunity to talk in RIMS conference. The author also wishes to express her thank to Professor Tomomi Yokota for his constructive comments and warm encouragement.

References

[1] H. Amann, "Linear and Quasi-linear Parabolic Problems, Volume I, Abstract Linear Theory", Birkhäuser, Basel, 1995.

[2] S. Ishida, T. Yokota, *Global existence of weak solutions to quasilinear degenerate Keller-Segel systems of parabolic-parabolic type*, J. Differential Equations **252** (2012), 1421–1440.

[3] S. Ishida, T. Yokota, *Global existence of weak solutions to quasilinear degenerate Keller-Segel systems of parabolic-parabolic type with small data*, J. Differential Equations **252** (2012), 2469–2491.

[4] S. Ishida, T. Yokota, *Remarks on the global existence of weak solutions to quasilinear degenerate Keller-Segel systems*, submitted.

[5] T. Kawanago, *Existence and behavior of solutions for $u_t = \Delta(u^m) + u^l$*, Adv. Math. Sci. Appl. **7** (1997), 367–400.

[6] E. F. Keller, L. A. Segel, *Initiation of slime mold aggregation viewed as an instability*, J. Theor. Biol. **26** (1970), 399–415.

[7] H. Kozono, Y. Sugiyama, *Strong solutions to the Keller-Segel system with the weak $L^{\frac{n}{2}}$ initial data and its application to the blow-up rate*, Math. Nachr. **283** (2010), 732–751.

[8] S. Luckhaus, Y. Sugiyama, *Large time behavior of solutions in super-critical cases to degenerate Keller-Segel systems*, M2AN Math. Model. Numer. Anal. **40** (2006), 597–621.

[9] S. Luckhaus, Y. Sugiyama, *Asymptotic profile with the optimal convergence rate for a parabolic equation of chemotaxis in super-critical cases*, Indiana Univ. Math. J. **56** (2007), 1279–1297.

[10] T. Nagai, *Blowup of nonradial solutions to parabolic-elliptic systems modeling chemotaxis in two-dimensional domains*, J. Inequal. Appl. **6** (2001), 37-55.

[11] T. Nagai, T. Senba, K. Yoshida, *Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis*, Funkcial. Ekvac. **40** (1997), 411–433.

[12] Y. Sugiyama, *Global existence in sub-critical cases and finite time blow-up in super-critical cases to degenerate Keller-Segel systems*, Differential Integral Equations **19** (2006) 841–876.

[13] Y. Sugiyama, *Time global existence and asymptotic behavior of solutions to degenerate quasi-linear parabolic systems of chemotaxis*, Differential Integral Equations **20** (2007), 133–180.

[14] Y. Sugiyama, *Asymptotic profile of blow-up solutions of Keller-Segel systems in super-critical cases*, Differential Integral Equations **23**, (2010), 601–618.

[15] Y. Sugiyama, *Blow-up criterion via scaling invariant quantities with effect on coefficient growth in Keller-Segel system*, Differential Integral Equations **23**, (2010), 619–634.

[16] Y. Sugiyama, H. Kunii, *Global existence and decay properties for a degenerate Keller-Segel model with a power factor in drift term*, J. Differential Equations **227** (2006), 333–364.

[17] R. Suzuki, *Existence and nonexistence of global solutions to quasilinear parabolic equations with convection*, Hokkaido Mathematical Journal **27** (1998), 147–196.

[18] T. Suzuki, "Free Energy and Self-Interacting Particles", Birkhäuser, Boston, 2005.

Department of Mathematics
Tokyo University of Science
1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, JAPAN
E-mail address: j1111702@ed.tus.ac.jp

東京理科大学大学院・理学研究科 石田祥子