数理解析研究所講究録
第 1969 巻 2015 年 101-122

L^p-INDEPENDENCE OF GROWTH BOUNDS OF FEYNMAN-KAC SEMIGROUP AND ITS APPLICATIONS

MASAYOSHI TAKEDA

1. INTRODUCTION

A. Beurling and J. Deny [2], [3] initiated the theory of Dirichlet forms. Using potential theory of Dirichlet forms, M. Fukushima [17] succeeded in the construction of symmetric Hunt processes associated with Dirichlet forms. Since then, the theory of Dirichlet forms has been developed by many persons as a useful tool for analyzing symmetric Markov processes. The theory of Dirichlet forms is an L^2-theory, and which is a reason why the theory is suitable for treating singular Markov processes. On the other hand, the theory of Markov processes is, in a sense, an L^1-theory. To bridge this gap, we have studied the L^p-independence of growth bounds of Markov semigroups, more generally, of generalized Feynman-Kac (Schrödinger) semigroups ([10],[13],[33],[35],[38]). The L^p-independence enables us to control L^∞-properties of the symmetric Markov process; in fact, we can state, in terms of the bottom of L^2-spectrum, a necessary and sufficient conditions for the integrability of Feynman-Kac functionals ([32]) and for the stability of Gaussian both side estimates of Schrödinger heat kernels ([34]).

For the proof of the L^p-independence, we apply arguments in the Donsker-Varadhan large deviation theory. The large deviation principle for a symmetric Markov process is governed by its Dirichlet form, namely, the rate function is identified with its Dirichlet form. Hence we can expect that the L^p-independence is fulfilled for symmetric Markov processes satisfying the large deviation principle. This is our key idea. Z.-Q. Chen [10] recently derives the L^p-independence by a different method (by employing, so called, the gauge theorem) and extends our results.

Let X be a locally compact separable metric space and m a positive Radon measure on X with full support. Let $\mathbb{M} = (X_t, \mathbb{P}_x, \zeta)$ be an irreducible m-symmetric Markov process on X with strong Feller property. Here ζ is the lifetime of $\mathbb{M}$. We further assume that $\mathbb{M}$ is in Class (I) or Class (II) (Definition 2.1, Definition 2.2 in Section 2). Let μ be a signed smooth Radon measure on X in Class $\mathcal{K}_\infty$ (Definition 3.1). Denote by $A_t(\mu)$ the continuous additive functional with Revuz correspondence to μ (see (2.3) below).

We define the generalized Feynman-Kac semigroup $\{p_t^\mu\}_{t>0}$ by

$$p_t^\mu f(x) = \mathbb{E}_x\left[\exp(A_t(\mu))f(X_t)\right],$$

and the Schrödinger type operator formally by

$$\mathcal{H}^\mu f = \mathcal{L}f + \mu f,$$

where $\mathcal{L}$ is the generator of the Markov process $\mathbb{M}$. We then see that the semigroup $\{p_t^\mu\}_{t>0}$ is the one generated by $\mathcal{H}^\mu$, $p_t^\mu = \exp(t\mathcal{H}^\mu)$.

We define the L^p-growth bound of $\{p_t^\mu\}_{t>0}$ by

$$\lambda_p(\mu) = -\lim_{t\to\infty}\frac{1}{t}\log\|p_t^\mu\|_{p,p} \quad 1 \le p \le \infty,$$

where $\|\cdot\|_{p,p}$ is the operator norm from $L^p(X;m)$ to $L^p(X;m)$. The L^p-independence of the growth bounds of $\{p_t^\mu\}_{t>0}$ means that

$$\lambda_p(\mu) = \lambda_2(\mu), \quad 1 \le \forall p \le \infty.$$

We now have the next theorem.

Theorem 1.1. ([35], [43]) *Let μ be a measure in the class $\mathcal{K}_\infty$.*

(i) *Assume that $\mathbb{M}$ is in Class (I). Then $\lambda_p(\mu)$ is independent of p.*

(ii) *Assume that $\mathbb{M}$ is in Class (II). Then $\lambda_p(\mu)$ is independent of p if and only if $\lambda_2(\mu) \le 0$.*

Theorem 1.1 (ii) says that the L^p-independence for a symmetric Markov process in Class (II) is completely determined by the L^2-growth bound. Z.-Q. Chen and D. Kim and K. Kuwae [13] recently extend Theorem 1.1 to Feynman-Kac semigroups generated by more general additive functionals.

As mentioned above, the idea for the proof of Theorem 1.1 lies in the Donsker-Varadhan theory, the large deviation theory for occupation distributions. We denote by $(\mathcal{E},\mathcal{F})$ the Dirichlet form generated by the symmetric Markov process $\mathbb{M}$. We then see that the semigroup $\{p_t^\mu\}_{t>0}$ generates the bilinear form $\mathcal{E}^\mu$:

$$\mathcal{E}^\mu(u,u) = \mathcal{E}(u,u) - \int_X u^2 d\mu. \quad u \in \mathcal{F},$$

Let $\mathcal{P}(X)$ be the set of probability measures on X equipped with the weak topology. We define the function $I_{\mathcal{E}^\mu}$ on $\mathcal{P}(X)$ by

$$I_{\mathcal{E}^\mu}(\nu) = \begin{cases} \mathcal{E}^\mu(\sqrt{f},\sqrt{f}) & \text{if } \nu = f\cdot m,\ \sqrt{f}\in\mathcal{F} \\ \infty & \text{otherwise.}\end{cases} \tag{1.1}$$

For $\omega \in \Omega$ with $0 < t < \zeta(\omega)$, we define the occupation distribution $L_t(\omega) \in \mathcal{P}(X)$ by

$$L_t(\omega)(A) = \frac{1}{t}\int_0^t 1_A(X_s(\omega))ds,$$

where 1_A is the indicator function of the Borel set $A \subset X$. We then have the next theorem:

Theorem 1.2. *Assume that $\mathbb{M}$ is in Class (I). Let μ be a measure in $\mathcal{K}_\infty$.*

(i) *For each open set $G \subset \mathcal{P}(X)$,*

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x \left[e^{A_t(\mu)}; L_t \in G, t < \zeta\right] \geq - \inf_{\nu\in G} I_{\mathcal{E}^\mu}(\nu).$$

(ii) *For each closed set $K \subset \mathcal{P}(X)$,*

$$\limsup_{t\to\infty} \frac{1}{t} \log \sup_{x\in X} \mathbb{E}_x \left[e^{A_t(\mu)}; L_t \in K, t < \zeta\right] \leq - \inf_{\nu\in K} I_{\mathcal{E}^\mu}(\nu).$$

Theorem 1.2 was proven in [35] and [43]. Applying Theorem 1.2 to $G = K = \mathcal{P}(X)$, we see that

$$\lim_{t\to\infty} \frac{1}{t} \log \sup_{x\in X} \mathbb{E}_x \left[e^{A_t(\mu)}; t < \zeta\right] = - \inf_{\nu\in\mathcal{P}(X)} I_{\mathcal{E}^\mu}(\nu)$$

$$= - \inf \left\{ \mathcal{E}^\mu(u,u) : u \in \mathcal{F},\ \int_X u^2 dm = 1 \right\}. \tag{1.2}$$

The equation (1.2) leads us to Theorem 1.1 (i). Indeed, noting that

$$\sup_{x\in X} \mathbb{E}_x \left[e^{A_t(\mu)}; t < \zeta\right] = \sup_{x\in X} p_t^\mu 1(x) = \|p_t^\mu\|_{\infty,\infty}$$

and by the spectral theorem

$$\lambda_2(\mu) = \inf \left\{ \mathcal{E}^\mu(u,u) : u \in \mathcal{F},\ \int_X u^2 dm = 1 \right\}, \tag{1.3}$$

we have $\lambda_\infty(\mu) = \lambda_2(\mu)$ by (1.2), which implies that $\lambda_p(\mu)$ is independent of p by the Riesz-Thorin interpolation theorem ([12, 1.1.5]).

The method for the proof of Theorem 1.1 (ii) is different from that of Theorem 1.1 (i): we first note that if the state space X is compact, only the Feller property is necessary for the proof of the upper bound. We thus extend the Markov process $\mathbb{M}$ to the one-point compactification X_∞ by making the infinity ∞ a trap, and derive the upper bound for this extended Markov process. Then the rate function becomes a function on the set of probability measures on X_∞ not on X; in this way, the adjoined point ∞ makes a contribution to the rate function. We show that the infimum of the rate function on the set of probability measures on X_∞ is equal to the infimum of the original rate function on the set of probability measures on X, if and only if the L^2-spectral bound is non-positive. Consequently we obtain a necessary and sufficient condition for the L^p-independence. The idea of considering the contribution to the rate function from ∞ is due to A. Budhiraja and P. Dupuis [6], where a large deviation principle of occupation distributions was proved for Markov processes without stability property.

We applied Theorem 1.1 (i) to random time-changed processes of symmetric Markov processes, and considered the gaugeability, the stability of heat kernels as stated above ([18, Chapter 6]). We applied Theorem 1.1 (ii) to symmetric α-stable processes, the Lévy process on $\mathbb{R}^d$ generated by the fractional Laplacian $(-\Delta)^{\alpha/2}$, $0 < \alpha < 2$, and showed the large deviation principle for their additive functionals ([41]). In this note we give another application of Theorem 1.1 (ii); we deal with the criticality for Schrödinger operators based on recurrent symmetric α-stable processes. More precisely, let $\mathbf{M}^\alpha$ be a symmetric α-stable process. It is known that $\mathbf{M}^\alpha$ is transient for $d > \alpha$ and recurrent for $d(=1) \leq \alpha < 2$. Let $(\mathcal{E}^{(\alpha)}, \mathcal{D}(\mathcal{E}^{(\alpha)}))$ be the Dirichlet form on $L^2(\mathbb{R}^1)$ generated by $\mathbf{M}^\alpha$ (see (6.1), (6.2)). Let $\mu = \mu^+ - \mu^-$ be a signed Radon measure in the Kato class, where μ^+ (resp. μ^-) is the positive (resp. negative) part of μ. We define

$$(1.4) \qquad \lambda(\mu) = \inf\left\{\mathcal{E}^{\mu^+}(u,u) : u \in \mathcal{D}_e(\mathcal{E}^{\mu^+}), \int_{\mathbb{R}^1} u^2 d\mu^- = 1\right\},$$

where $\mathcal{E}^{\mu^+}(u,u) = \mathcal{E}^{(\alpha)}(u,u) + \int_{\mathbb{R}^1} u^2 d\mu^+$ and $\mathcal{D}_e(\mathcal{E}^{\mu^+})$ is the extended Dirichlet space of the Dirichlet form $(\mathcal{E}^{\mu^+}, \mathcal{D}(\mathcal{E}^{\mu^+}))$. Let $G^{\mu^+}(x,y)$ be the Green function of the subprocess of $\mathbf{M}^\alpha$ by $\exp(-A_t^{\mu^+})$, where $A_t^{\mu^+}$ is the positive continuous additive functional associated with μ^+. We assume that the negative part μ^- is *Green-tight* with respect to $G^{\mu^+}(x,y)$ (for definition, see (6.4)).

For the measure μ, let $\mathcal{H}^\mu$ be a Schrödinger type operator defined by $(-d^2/dx^2)^{\alpha/2} + \mu$. We say $\mathcal{H}^\mu$ *critical* (resp. *subcritical*) if $\lambda(\mu) = 1$ (resp. $\lambda(\mu) > 1$). In B. Simon [25], $\mathcal{H}^\mu$ is said to be critical if $\lambda_\infty(\mu) = 0$ but $\lambda_\infty((1+\epsilon)\mu) < 0$ for all $\epsilon > 0$, and subcritical if $\lambda_\infty((1+\epsilon)\mu) = 0$ for some $\epsilon > 0$. We see from the L^p-independence that if μ is, in addition, Green-tight with respect to the 1-resolvent density of $\mathbf{M}^\alpha$, in particular μ has a compact support, our definition is equivalent with Simon's (Lemma 6.1).

We consider properties of $\mathcal{H}^\mu$-harmonic functions when $\mathcal{H}^\mu$ is critical or subcritical. More precisely, we prove that there exists no positive bounded $\mathcal{H}^\mu$-harmonic function if $\mathcal{H}^\mu$ is subcritical (Proposition 6.8). Moreover, we show that if the measure μ has compact support and $\mathcal{H}^\mu$ is critical, then there exists a bounded $\mathcal{H}^\mu$-harmonic function uniformly lower-bounded by a positive constant (Proposition 6.5). Employing this fact, we can derive that if $\lambda_\infty(\mu) = 0$, then

$$\beta_\infty(\mu) = \sup_{t>0} \|e^{-t\mathcal{H}^\mu}\|_{\infty,\infty}$$

is finite (Lemma 6.7). When $\mathbf{M}$ is the 2-dimensional Brownian motion, Simon [25] conjecture that for a potential with compact support $\lambda_\infty(\mu) = 0$ implies $\beta_\infty(\mu) < \infty$. Murata [23] solved his conjecture completely by characterizing the criticality or subcriticality by the existence of positive $\mathcal{H}^\mu$-harmonic functions with some growth orders. Lemma 6.7 is an extension to recurrent symmetric α-stable processes.

We would like to emphasis that when $\mathcal{H}^\mu$ is critical, $\lambda(\mu) = 1$, the function h attaining the infimum in (1.4) is just an $\mathcal{H}^\mu$-harmonic function. Indeed, we show in Section 4 that the function h is continuous and possesses a probabilistic property of $\mathcal{H}^\mu$-harmonicity: for any relatively compact domain $D \subset \mathbb{R}^1$,

$$h(x) = E_x \left[\exp(-A^\mu_{\tau_D})h(X_{\tau_D})\right], \quad x \in D, \tag{1.5}$$

where τ_D is the first exit time from D.

Throughout this paper, m is the Lebesgue measure and $B(x,r)$ is an open ball with radius r centered at x. We write $B(r)$ when x is the origin. We use $c, C, ..., etc$ as positive constants which may be different at different occurrences.

2. Dirichlet Forms and Symmetric Markov Processes

In this section we briefly review the theory of Dirichlet forms, symmetric Markov processes and Feynman-Kac semigroups. Let X be a locally compact separable metric space and X_∞ the one-point compactification of X with adjoined point ∞. Let m be a positive Radon measure on X with full support. Let $\mathbb{M} = (\Omega, \mathcal{M}, \mathcal{M}_t, \theta_t, X_t, \mathbb{P}_x, \zeta)$ be an m-symmetric Markov process on X. Here, $\{\mathcal{M}_t\}$ is the minimal (augmented) admissible filtration, $\{\theta_t\}_{t\geq 0}$ is the shift operator satisfying $X_s(\theta_t) = X_{s+t}$ identically for $s, t \geq 0$, and ζ is the lifetime of $\mathbb{M}$, $\zeta = \inf\{t > 0 : X_t = \infty\}$. Let $\{p_t\}_{t>0}$ and $\{G_\beta\}_{\beta>0}$ be the semigroup and the resolvent of $\mathbb{M}$: for a bounded Borel function f on X

$$p_t f(x) = \mathbb{E}_x[f(X_t); t < \zeta], \quad G_\beta f(x) = \int_0^\infty e^{-\beta t} p_t f(x) dt.$$

Throughout this paper, we make two assumptions on $\mathbb{M}$.

Assumption I. (**Irreducibility**) If a Borel set A is p_t-invariant, i.e.,

$$p_t(1_A f)(x) = 1_A p_t f(x), \quad m\text{-a.e. for } \forall t > 0, \ \forall f \in L^2(X; m) \cap \mathcal{B}_b(X),$$

then A satisfies either $m(A) = 0$ or $m(X \setminus A) = 0$. Here $\mathcal{B}_b(X)$ is the space of bounded Borel functions on X.

Assumption II. (**Strong Feller Property**) For each $t > 0$, $p_t(\mathcal{B}_b(X)) \subset C_b(X)$, where $C_b(X)$ is the space of bounded continuous functions on X.

We introduce two classes of symmetric Markov processes.

Definition 2.1. *A symmetric Markov process* $\mathbb{M}$ *is said to be in* **Class (I)**, *if for any* $\epsilon > 0$, *there exists a compact set* $K \subset X$ *such that*

$$\sup_{x\in X} G_1 1_{K^c}(x) \leq \epsilon, \tag{2.1}$$

Here 1_{K^c} *is the indicator function of the complement of* K.

Definition 2.2. *A symmetric Markov process* $\mathbb{M}$ *is said to be in* **Class (II)**, *if its semigroup* $\{p_t\}_{t\geq 0}$ *is conservative,* $p_t 1 = 1$, *and satisfies* $p_t(C_\infty(X)) \subset C_\infty(X)$. *Here* $C_\infty(X)$ *is the space of continuous functions on* X *vanishing at the infinity.*

Let $\{G_\beta(x,y)\}_{\beta\geq 0}$ be the resolvent kernel defined by

$$G_\beta(x,y) = \int_0^\infty e^{-\beta t} p(t,x,y)dt, \quad \beta \geq 0.$$

If the Markov process $\mathbb{M}$ is transient, then $G_0(x,y) < \infty$ $x \neq y$. In this case, we simply write $G(x,y)$ for $G_0(x,y)$ and call it the *Green function*. By [18, Lemma 4.2.4] the density $G_\beta(x,y)$ is assumed to be a non-negative Borel function such that $G_\beta(x,y)$ is symmetric and β-excessive in x and in y.

By the right continuity of sample paths of $\mathbb{M}$, $\{p_t\}_{t>0}$ can be extended to an $L^2(X;m)$-strongly continuous contraction semigroup, $\{T_t\}_{t>0}$ ([18, Lemma 1.4.3]). The *Dirichlet form* $(\mathcal{E},\mathcal{F})$ generated by $\mathbb{M}$ is defined by

$$\tag{2.2} \begin{cases} \mathcal{F} = \left\{ u \in L^2(X;m) : \lim_{t\to 0} \dfrac{1}{t}(u - T_t u, u)_m < \infty \right\}, \\ \mathcal{E}(u,v) = \lim_{t\to 0} \dfrac{1}{t}(u - T_t u, v)_m, \quad u, v \in \mathcal{F}, \end{cases}$$

where $(u,v)_m$ is the inner product on $L^2(X;m)$.

If an AF $\{A_t\}_{t\geq 0}$ is positive and continuous with respect to t for each $\omega \in \Lambda$, the AF is called a *positive continuous additive functional* (PCAF in abbreviation). Under the absolute continuity condition, "quasi everywhere" statements are strengthened to "everywhere" ones. Moreover, we can defined notions without exceptional set, for example, *smooth measures in the strict sense* or *positive continuous additive functional in the strict sense* (cf. [18, Section 5.1]). Here we only treat the notions in the strict sense and omit the phrase "in the strict sense".

We denote S_{00} the set of positive Borel measures μ such that $\mu(X) < \infty$ and $G_1\mu(x)(= \int_X G_1(x,y)\mu(dy))$ is uniformly bounded in $x \in X$. A positive Borel measure μ on X is said to be *smooth* if there exists a sequence $\{E_n\}_{n=1}^\infty$ of Borel sets increasing to X such that $1_{E_n}\cdot\mu \in S_{00}$ for each n and

$$\mathbb{P}_x\left(\lim_{n\to\infty} \sigma_{X\setminus E_n} \geq \zeta\right) = 1, \quad \forall x \in X, \tag{5.1.28}$$

where $\sigma_{X\setminus E_n}$ is the first hitting time of $X \setminus E_n$. We denote by S_1 the totality of smooth measures. By [18, Theorem 5.1.4], there exists a one-to-one correspondence (**Revuz correspondence**) between smooth measures and PCAFs as follows: for each smooth measure μ, there exists a unique PCAF $\{A_t\}_{t\geq 0}$ such that for any $f \in \mathcal{B}_+(X)$ and

γ-excessive function h ($\gamma \geq 0$), $e^{-\gamma t}p_t h \leq h$,

$$\lim_{t\to 0}\frac{1}{t}\mathbb{E}_{h\cdot m}\left[\int_0^t f(X_s)dA_s\right] = \int_X f(x)h(x)\mu(dx). \tag{2.3}$$

Here, $\mathbb{E}_{h\cdot m}[\,\cdot\,] = \int_X \mathbb{E}_x[\,\cdot\,]h(x)m(dx)$. We denote by $A_t(\mu)$ the PCAF of the smooth measure μ. For a signed smooth measure $\mu = \mu^+ - \mu^-$, we define $A_t(\mu) = A_t(\mu^+) - A_t(\mu^-)$.

3. Generalized Feynman-Kac Semigroups

In this section we introduce classes of local and non-local potentials. For a signed Borel measure μ, we write its total variation by $|\mu|$. Following Chen [8], [9], we define classes of potentials.

Definition 3.1 (**Kato measure, Green tight measure**).

(I) A signed Borel measure μ is said to be the Kato measure *(in notation, $\mu \in \mathcal{K}$) if $|\mu| \in S_1$ and*

$$\lim_{t\to 0}\sup_{x\in X}\mathbb{E}_x[A_t(|\mu|)] = 0.$$

*(II) A measure $\mu \in \mathcal{K}$ is said to be the β-*Green tight measure *(in notation, $\mu \in \mathcal{K}_{\infty,\beta}$) if for any $\epsilon > 0$ there exist a compact subset K and a positive constant $\delta > 0$ such that*

$$\sup_{x\in X}\int_{K^c} G_\beta(x,y)|\mu|(dy) \leq \epsilon,$$

and for any Borel set $B \subset K$ with $|\mu|(B) < \delta$,

$$\sup_{x\in X}\int_B G_\beta(x,y)|\mu|(dy) < \epsilon.$$

For a positive measure μ on X, denote

$$G_\beta\mu(x) = \int_X G_\beta(x,y)\mu(dy).$$

We note that for any $\beta > 0$, $\mathcal{K}_{\infty,\beta} = \mathcal{K}_{\infty,1}$. Indeed, for a positive measure μ on X, let $\mu_{K^c}(\cdot) = \mu(K^c\cap\cdot)$. Since by the resolvent equation

$$G_\beta\mu_{K^c} = G_\gamma\mu_{K^c} + (\gamma-\beta)G_\beta G_\gamma\mu_{K^c}, \quad 0 < \beta < \gamma,$$

we have

$$\|G_\beta\mu_{K^c}\|_\infty \leq \|G_\gamma\mu_{K^c}\|_\infty + \frac{\gamma-\beta}{\beta}\|G_\gamma\mu_{K^c}\|_\infty = \frac{\gamma}{\beta}\|G_\gamma\mu_{K^c}\|_\infty.$$

We simply write $\mathcal{K}_\infty$ for $\mathcal{K}_{\infty,1}$ and call a measure in $\mathcal{K}_\infty$ a 1-Green tight measure. *Moreover, if the Markov process is transient, a measure $\mu \in \mathcal{K}_{\infty,0}$ is called a* Green tight measure. *We remark that $\mathcal{K}_{\infty,0} \subset \mathcal{K}_\infty \subset \mathcal{K}$ ([8]).*

We now provide an inequality proved in P. Stollmann and J. Voigt [26].

Theorem 3.1. *Let $\mu \in \mathcal{K}$. Then for each $\beta \geq 0$,*

$$\int_X u^2(x)\mu(dx) \leq \|G_\beta\mu\|_\infty \cdot \mathcal{E}_\beta(u,u), \quad u \in \mathcal{F}, \tag{3.1}$$

where $\mathcal{E}_\beta(u,u) = \mathcal{E}(u,u) + \beta(u,u)_m$.

Let $\{p_t^\mu\}_{t>0}$ be the L^2-semigroup generated by $\mathcal{H}^\mu$: $p_t^\mu = \exp(t\mathcal{H}^\mu)$. The semigroup $\{p_t^\mu\}_{t>0}$ is expressed by

$$p_t^\mu f(x) = \mathbb{E}_x[\exp(A_t(\mu))f(X_t)].$$

Next two theorems on the generalized Feynman-Kac semigroups $\{p_t^\mu\}_{t>0}$ follows from Albeverio, Blanchard and Ma [1, Theorem 4.1] and Chung [11, Theorem 2] respectively.

Theorem 3.2. *Let $\mu \in \mathcal{K}_\infty$. There exist constants c and $\kappa(\mu)$ such that*

$$\|p_t^\mu\|_{p,p} \leq ce^{\kappa(\mu)t}, \quad 1 \leq \forall p \leq \infty, \ t > 0.$$

Here, $\|\cdot\|_{p,p}$ means the operator norm from $L^p(X;m)$ to $L^p(X;m)$.

Theorem 3.3. *Suppose that a symmetric Markov process $\mathbb{M}$ is in Class (II). Then for $\mu \in \mathcal{K}_\infty$, $p_t^\mu(C_\infty(X)) \subset C_\infty(X)$ and $p_t^\mu(\mathcal{B}_b(X)) \subset C_b(X)$.*

4. Donsker-Varadhan Type Large Deviation Principle

For a symmetric Markov process, its Dirichlet form governs the Donsker-Varadhan large deviation principle, that is, the rate function is identified with the Dirichlet form. Therefore, we can expect that if the symmetric Markov process obeys the large deviation principle, then the L^2-theory is more dominant. In this section, we extend Donsker-Varadhan type large deviations to symmetric Markov processes with Feynman-Kac functional. In this case the rate function is identified with not a Dirichlet form but a Schrödinger form.

Let $\mu \in \mathcal{K}_\infty$. We define the function $I_{\mathcal{E}^\mu}$ on $\mathcal{P}(X)$ by

$$I_{\mathcal{E}^\mu}(\nu) = \begin{cases} \mathcal{E}^\mu(\sqrt{f},\sqrt{f}) & \text{if } \nu = f\cdot m, \ \sqrt{f} \in \mathcal{F}, \\ \infty & \text{otherwise.} \end{cases}$$

Let $L_t \in \mathcal{P}(X)$ be the *normalized occupation distribution*, that is, for $0 < t < \zeta$

$$L_t(A) = \frac{1}{t}\int_0^t 1_A(X_s)ds, \quad A \in \mathcal{B}(X). \tag{4.1}$$

We then have the lower bound estimate.

Theorem 4.1 ([20, Theorem 4.1]). *For each open set $G \subset \mathcal{P}(X)$,*

$$\liminf_{t\to\infty} \frac{1}{t}\log \mathbb{E}_x\left[\exp(A_t(\mu)); L_t \in G, t < \zeta\right] \geq -\inf_{\nu\in G} I_{\mathcal{E}^\mu}(\nu). \tag{4.2}$$

We have the next theorem by the same argument as in [36].

Theorem 4.2. *Assume that a symmetric Markov process* $\mathbb{M}$ *is in Class (I). Then for each closed set* $K \subset \mathcal{P}(X)$,

$$\limsup_{t\to\infty} \frac{1}{t} \log \sup_{x\in X} \mathbb{E}_x \left[\exp(A_t(\mu)); L_t \in K, t < \zeta\right] \le - \inf_{\nu\in K} I_{\mathcal{E}^\mu}(\nu).$$

We will show in section 6 that the infimum of $I_{\mathcal{E}^\mu}(\nu)$ is attained at the normalized ground state of the generalized Schrödinger operator $\mathcal{H}^\mu$. In this sense, Theorem 4.1 and Theorem 4.2 is regarded as a large deviation principle form not the invariant measure but the ground state. The essential idea of the proof of Theorem 4.1 and Theorem 4.2 lies in Donsker-Varadhan [14]; however, since $A_t(\mu)$ is not a function of L_t, we need to extend Donsker-Varadhan's argument to Markov processes with Feynman-Kac functional.

A key to the proof of Theorem 4.1 is the fact that any irreducible symmetric Markov process can be transformed to a symmetric ergodic process by a certain supermartingale multiplicative functional. A one-dimensional absorbing Brownian motion can be transformed to a symmetric ergodic diffusion by a drift transform. Using this fact, they proved in Donsker-Varadhan [14] the lower estimate for the one-dimensional Brownian motion. To prove the ergodicity, they used the Feller test, while we apply an ergodic theorem in the Dirichlet form theory.

A key to the proof of Theorem 4.2 is the definition of a suitable *I-function.* More precisely, define $\kappa(\mu)$ by

$$\kappa(\mu) = \lim_{t\to\infty} \frac{1}{t} \log \|p_t^\mu\|_{\infty,\infty}.$$

We see from Theorem 3.2 that $\kappa(\mu)$ is finite. For $\alpha > \kappa(\mu)$, the resolvent G_α^μ is defined by

$$G_\alpha^\mu f(x) = \mathbb{E}_x \left[\int_0^\infty e^{-\alpha t + A_t(\mu)} f(X_t)dt\right], \quad f \in \mathcal{B}_b(X).$$

We set

$$\mathcal{D}_+(\mathcal{H}^\mu) = \left\{G_\alpha^\mu f : \alpha > \kappa(\mu), f \in L^2(X;m) \cap C_b(X), f \ge 0 \text{ and } f \not\equiv 0\right\}.$$

Each function $\phi = G_\alpha^\mu f \in \mathcal{D}_+(\mathcal{H}^\mu)$ is strictly positive because $\mathbb{P}_x(\sigma_O < \zeta) > 0$ for any $x \in X$ by Assumption I. Here O is a non-empty open set $\{x \in X : f(x) > 0\}$. We define the generator $\mathcal{H}^\mu$ by

$$\mathcal{H}^\mu u = \alpha u - f, \quad u = G_\alpha^\mu f \in \mathcal{D}_+(\mathcal{H}^\mu).$$

Suppose that $\mu \in \mathcal{K}_\infty$ is *gaugeable*, that is,

$$\sup_{x\in X} \mathbb{E}_x \left[e^{A_\zeta(\mu)}\right] < \infty$$

and let $h(x) = \mathbb{E}_x\left[\exp\left(A_\zeta(\mu)\right)\right]$. The function $h(x)$ is strictly positive, $h(x) \ge c > 0$. Indeed, it follows from Proposition 2.2 in [8] and the definition of $\mathcal{J}_\infty$ that for $\mu \in \mathcal{K}_\infty$ and $F \in \mathcal{J}_\infty$, $\sup_{x\in E}\mathbb{E}_x(A^\mu_\zeta) < \infty$. Hence, by Jensen's inequality,

$$\inf_{x\in X}\mathbb{E}_x(\exp(A^\mu_\zeta)) > 0.$$

After consideration of the Feynman-Kac functional, we define the modified I-function by

$$I_\mu(\nu) = -\inf_{\substack{\phi\in\mathcal{D}_+(\mathcal{H}^\mu)\\ \epsilon>0}} \int_X \frac{\mathcal{H}^\mu\phi}{\phi+\epsilon h}d\nu, \quad \nu\in\mathcal{P}. \tag{4.3}$$

We need to add strictly positive functions ϵh, because the function $\mathcal{H}^\mu\phi/\phi$ is not always in $C_b(X)$. Since $\mathcal{P}(X)$ is equiped with the weak topology, it is crucial for the proof of Theorem 4.2 that the function $\frac{\mathcal{H}^\mu\phi}{\phi+\epsilon h}$ belongs to $C_b(X)$; in fact, we show the upper bound with this modified I-function I_μ. The function h is said to be a *gauge function* and a necessary and sufficient condition for the measure μ being gaugeable is known (cf. [9]). An important remark on the proof of Theorem 4.1 and Theorem 4.2 is that we have only to prove these theorems for the β-subprocess of $\mathbb{M}$, the killed process by $\exp(-\beta t)$, $\beta > 0$. Owing to this, we may assume that $\mathbb{M}$ is transient. In addition, we may assume that μ is gaugeable because every Green-tight measure becomes gaugeable with respect to the β-subprocess of $\mathbb{M}$ for a large enough β([9, Theorem 3.4]). The β-subprocess is a useful tool of studying Markov processes. It is worth to point out that this tool becomes available by extending the large deviation to symmetric Markov processes with finite lifetime.

The next proposition says that this modified I-function can be identified with the Schrödinger form.

Proposition 4.3. It holds that for $\nu\in\mathcal{P}(X)$,

$$I_\mu(\nu) = I_{\mathcal{E}^\mu}(\nu).$$

In [28] we proved Theorem 4.1 for symmetric Markov processes without Feynman-Kac functional. We there used the identity function 1 for the gauge function h in order to define the I-function. Note that the identity function is excessive for the Markov semigroup generated by $\mathcal{L}$ and the gauge function h is excessive for the Schrödinger semigroup generated by $\mathcal{H}^\mu$. Hence we can regard the function I_μ as an extension of the I-function in [28]. In [29] we proved the upper bound (ii) for each compact set of $\mathcal{P}$ without assuming (2.1). We did not need to add ϵh in (4.3) because the Markov process was supposed to be conservative and the I-function was defined by taking the infimum over uniformly positive functions in a domain of $\mathcal{H}^\mu$. We would like to emphasize that the function I_μ is independent of h if the function h is uniformly positive and bounded, that is, I_μ is identical to the Schrödinger form (1.1).

When the Markov process $\mathbb{M}$ be in Class (II), Theorem 4.2 does not hold generally. We thus first extend the Markov process $\mathbb{M}$ and the

I-function; we define the transition density $\bar{p}_t(x, dy)$ on $(X_\infty, \mathcal{B}(X_\infty))$: for $E \in \mathcal{B}(X_\infty)$,

$$\bar{p}_t(x, E) = \begin{cases} p_t(x, E \setminus \{\infty\}), & x \in X, \\ \delta_\infty(E), & x = \infty. \end{cases}$$

Let $\bar{\mathbb{M}}$ be the Markov process on X_∞ with transition probability $\bar{p}_t(x, dy)$, that is, an extension of $\mathbb{M}$ with ∞ being a trap. Furthermore, for $\mu \in \mathcal{K}_\infty$, let the semigroup $\{\bar{p}_t^\mu\}_{t>0}$ and the resolvent $\{\bar{G}_\alpha^\mu\}_{\alpha > \kappa(\mu)}$ of $\bar{\mathbb{M}}$:

$$\bar{p}_t^\mu f(x) = \bar{\mathbb{E}}_x[\exp(A_t(\mu)) f(X_t)],$$
$$\bar{G}_\alpha^\mu f(x) = \int_0^\infty e^{-\alpha t} \bar{p}_t^\mu f(x) dt, \quad f \in \mathcal{B}_b(X_\infty).$$

Here, $\kappa(\mu)$ is the constant in Theorem 3.2. Then $\bar{G}_\alpha^\mu f(x) = G_\alpha^\mu f(x)$ for $x \in X$ and $\bar{G}_\alpha^\mu f(\infty) = f(\infty)/\alpha$. Set

$$\mathcal{D}_{++}(\bar{\mathcal{H}}^\mu) = \{\phi = \bar{G}_\alpha^\mu g : \alpha > \kappa(\mu), g \in C(X_\infty) \text{ with } g > 0\}.$$

We see that for $\phi = \bar{G}_\alpha^\mu g \in \mathcal{D}_{++}(\bar{\mathcal{H}}^\mu)$, $\lim_{x \to \infty} \phi(x) = g(\infty)/\alpha$. Let us define the function $\bar{I}_\mu$ on $\mathcal{P}(X_\infty)$, the set of probability measures on X_∞, by

$$\bar{I}_\mu(\nu) = - \inf_{\phi \in \mathcal{D}_{++}(\bar{\mathcal{H}}^\mu)} \int_{X_\infty} \frac{\bar{\mathcal{H}}^\mu \phi}{\phi} d\nu,$$

where $\bar{\mathcal{H}}^\mu \phi = \alpha \bar{G}_\alpha^\mu g - g$ for $\phi = \bar{G}_\alpha^\mu g \in \mathcal{D}_{++}(\bar{\mathcal{H}}^\mu)$.

Note that $\bar{\mathbb{M}}$ has the Feller property, while it has no longer the strong Feller property. In the proof of the large deviation upper bound for a Markov process with compact state space, we need only the Feller property. Hence we have

Theorem 4.4 (Kim [20, Remark 4.1]). *For each closed set $K \subset \mathcal{P}(X_\infty)$,*

$$\limsup_{t \to \infty} \frac{1}{t} \log \sup_{x \in X} \mathbb{E}_x [\exp(A_t(\mu)); L_t \in K] \leq - \inf_{\nu \in K} \bar{I}_\mu(\nu). \tag{4.4}$$

5. L^p-Independence of Growth Bounds

When the symmetric Markov process $\mathbb{M}$ is in Class (I), we have the next theorem by applying Theorem 4.1 and Theorem 4.3 to $G = K = \mathcal{P}(X)$.

Theorem 5.1. *If $\mathbb{M}$ is in Class (I), then $\lambda_p(\mu)$ is independent of p.*

In the remainder of this section, we assume that $\mathbb{M}$ is in Class (II). We note that the rate function $\bar{I}_\mu$ in Theorem 4.4 is defined on the space of probability measures on X_∞ not on X. In this sense the adjoined point ∞ makes a contribution to the rate function. We see that

$$\bar{I}_\mu(\delta_\infty) = 0, \tag{5.1}$$

because $\bar{\mathcal{H}}^\mu\phi(\infty) = \alpha\phi(\infty) - g(\infty) = g(\infty) - g(\infty) = 0$ for any $\phi \in \mathcal{D}_{++}(\bar{\mathcal{H}}^\mu)$. $\mathcal{P}(X_\infty) \setminus \{\delta_\infty\}$ and $(0,1] \times \mathcal{P}(X)$ are in one-to-one correspondence through the map:

(5.2) $\nu \in \mathcal{P}(X_\infty)\setminus\{\delta_\infty\} \mapsto (\nu(X), \hat{\nu}(\cdot) = \nu(\cdot)/\nu(X)) \in (0,1]\times\mathcal{P}(X)$.

Lemma 5.2. *For* $\nu \in \mathcal{P}(X_\infty) \setminus \{\delta_\infty\}$,

$$\bar{I}_\mu(\nu) = I_\mu(\nu) = \nu(X) \cdot I_{\mathcal{E}^\mu}(\hat{\nu}).$$

Proof. For $\phi = \bar{G}^\mu_\alpha g \in \mathcal{D}_{++}(\bar{\mathcal{H}}^\mu)$, $\bar{\mathcal{H}}^\mu\phi(\infty) = 0$ and $\bar{\mathcal{H}}^\mu\phi(x) = \mathcal{H}^\mu\phi(x)$ for $x \in X$. Hence for $\nu \in \mathcal{P}(X_\infty)$,

$$\begin{aligned}\bar{I}_\mu(\nu) &= -\inf_{\phi\in\mathcal{D}_{++}(\bar{\mathcal{H}}^\mu)} \int_{X_\infty} \frac{\bar{\mathcal{H}}^\mu\phi}{\phi} d\nu \\ &= -\inf_{\phi\in\mathcal{D}_{++}(\mathcal{H}^\mu)} \int_{X} \frac{\mathcal{H}^\mu\phi}{\phi} d\nu \\ &= -\inf_{\phi\in\mathcal{D}_{++}(\mathcal{H}^\mu)} \nu(X) \int_{X} \frac{\mathcal{H}^\mu\phi}{\phi} d\hat{\nu} \\ &= \nu(X) \cdot I_{\mathcal{E}^\mu}(\hat{\nu}).\end{aligned}$$

□

We have the next equality through the one-to-one map (5.2).

$$\inf_{\nu\in\mathcal{P}(X_\infty)\setminus\{\delta_\infty\}} \bar{I}_\mu(\nu) = \inf_{0<\theta\leq 1, \nu\in\mathcal{P}(X)} (\theta \cdot I_{\mathcal{E}^\mu}(\nu))$$

In addition, noting that $\bar{I}_\mu(\delta_\infty) = 0$, we have the next corollary.

Corollary 5.1.

$$\inf_{\nu\in\mathcal{P}(X_\infty)} \bar{I}_\mu(\nu) = \inf_{0\leq\theta\leq 1} (\theta \inf_{\nu\in\mathcal{P}(X)} I_{\mathcal{E}^\mu}(\nu)). \tag{5.3}$$

Let us denote by $\|p^\mu_t\|_{p,p}$ the operator norm of p^μ_t from $L^p(X;m)$ to $L^p(X;m)$ and define

$$\lambda_p(\mu) = -\lim_{t\to\infty} \frac{1}{t} \log \|p^\mu_t\|_{p,p}, \quad 1 \leq p \leq \infty.$$

We then have:

Corollary 5.2. For $\mu \in \mathcal{K}_\infty$,

$$\lambda_\infty(\mu) \geq \inf_{0\leq\theta\leq 1} \left(\theta \inf_{\nu\in\mathcal{P}(X)} I_{\mathcal{E}^\mu}(\nu)\right) = \inf_{0\leq\theta\leq 1} (\theta\lambda_2(\mu)). \tag{5.4}$$

Noting that if $\lambda_2(\mu) \leq 0$, then $\inf_{0\leq\theta\leq 1}(\theta\lambda_2(\mu)) = \lambda_2(\mu)$, we have:

Corollary 5.3. If $\lambda_2(\mu) \leq 0$, then

$$\lambda_\infty(\mu) \geq \lambda_2(\mu).$$

The inequality, $\lambda_2(\mu) \geq \lambda_\infty(\mu)$, generally holds. Indeed,

$$\begin{aligned} p_t^\mu f(x) &= \mathbb{E}_x[\exp(A_t(\mu))f(X_t)] \\ &\leq (\mathbb{E}_x[\exp(A_t(\mu))f^2(X_t)])^{1/2} \cdot (\mathbb{E}_x[\exp(A_t(\mu))])^{1/2} \end{aligned}$$

and

$$\|p_t^\mu f\|_2^2 \leq \|p_t^\mu(f^2)\|_1 \sup_{x\in X} \mathbb{E}_x[\exp(A_t(\mu))].$$

By the symmetry and the positivity of p_t^μ,

$$\|p_t^\mu(f^2)\|_1 = \int_X f(x)^2(p_t^\mu 1(x))m(dx) \leq \|f\|_2^2 \cdot \|p_t^\mu\|_{\infty,\infty}.$$

Hence we have $\|p_t^\mu\|_{2,2} \leq \|p_t^\mu\|_{\infty,\infty}$, and thus $\lambda_2(\mu) \geq \lambda_\infty(\mu)$. Moreover, by the Riesz-Thorin interpolation theorem,

$$\|p_t^\mu\|_{2,2} \leq \|p_t^\mu\|_{p,p} \leq \|p_t^\mu\|_{\infty,\infty}, \quad 1 \leq \forall p \leq \infty.$$

Therefore, we can conclude that

$$\lambda_2(\mu) \leq 0 \Longrightarrow \lambda_p(\mu) = \lambda_2(\mu), \quad 1 \leq \forall p \leq \infty. \tag{5.5}$$

We see that if $\lambda_2(\mu) > 0$, then $\lambda_\infty(\mu) = 0$. Indeed, if $\lambda_2(\mu) > 0$, then by Corollary 5.2

$$\lambda_\infty(\mu) \geq \inf_{0\leq\theta\leq 1} \theta \inf_{\nu\in\mathcal{P}(X)} I_{\mathcal{E}^\mu}(\nu) = \inf_{0\leq\theta\leq 1} \theta(\lambda_2(\mu)) = 0.$$

On the other hand, since $\lim_{x\to\infty} p_t^\mu 1(x) = 1$, $\|p_t^\mu\|_{\infty,\infty} \geq 1$, and thus $\lambda_\infty(\mu) \leq 0$.

Theorem 5.3. *Assume that $\mathbb{M}$ is in Class (II). Let $\mu \in \mathcal{K}_\infty$. Then $\lambda_2(\mu) = \lambda_p(\mu)$ for all $1 \leq p \leq \infty$ if and only if $\lambda_2(\mu) \leq 0$. In particular, if $\lambda_2(\mu) > 0$, then $\lambda_\infty(\mu) = 0$.*

Example 5.1. *(Brownian motion on $\mathbb{H}^d$) We consider the Brownian motion on the hyperbolic space $\mathbb{H}^d$ $(d \geq 2)$, the diffusion process generated by the Laplace-Beltrami operator $(1/2)\Delta$. The corresponding Dirichlet form $(\mathcal{E}, \mathcal{F})$ is as follows:*

$$\begin{cases} \mathcal{E}(u,u) = \dfrac{1}{2}\displaystyle\int_{\mathbb{H}^d} (\nabla u, \nabla v)\, dm, \quad u,\ v \in \mathcal{F} \\ \mathcal{F} = \text{the closure of } C_0^\infty(\mathbb{H}^d) \text{with respect to } \mathcal{E} + (\ ,\ )_m, \end{cases}$$

where m is the Riemannian volume.

The Brownian motion is in Class (II). Hence $\lambda_\infty = 0$, while

$$\lambda_2 = \inf\{\mathcal{E}(u,u) \mid u \in \mathcal{F}, \|u\|_2 = 1\} = \frac{1}{2}\left(\frac{d-1}{2}\right)^2.$$

Hence the L^p-independence does not hold; However, by adding a Kato measure $\mu \in \mathcal{K}_\infty$ with $\lambda_2(\mu) \leq 0$, the L^p-independence is recovered. In fact, we consider $\mathcal{H}^\mu = 1/2\Delta + \delta_r$, where δ_r is the surface measure of the sphere centerd the origin with radius r.

(a) $d = 2$

(i) $0 \le r < r_0 \Longrightarrow \lambda_\infty(\delta_r) = 0, \quad \lambda_2(\delta_r) > 0.$
(ii) $r \ge r_0 > 0 \Longrightarrow \lambda_p(\delta_r) = \lambda_2(\delta_r), \quad 1 \le \forall p \le \infty.$

Here r_0 is a unique solution of

$$(e^r - e^{-r}) \log\left(\frac{e^r + 1}{e^r - 1}\right) = 1.$$

(b) $d \ge 3$

$$\lambda_\infty(\delta_r) = 0, \quad \lambda_2(\delta_r) > 0, \quad r \ge 0.$$

The uniform upper bound in Theorem 4.2 is crucial for the proof of L^p-independence, and so is the condition (2.1). We see that a one-dimensional diffusion process satisfies (2.1), if no boundaries are natural in Feller's boundary classification. As a result, the L^p-independence holds if no boundaries are natural. We see by exactly the same argument as in [?] that if one of the boundary points is natural, then the L^p-independence holds if and only if the L^2-growth bound is non-positive. For example, consider the one-dimensional diffusion process with generator $(1/2)\Delta + k \cdot d/dx$ on $(-\infty, \infty)$. Here k is a constant. Then the both boundaries are natural and $\lambda_2(0)$ equals $k^2/2$, while $\lambda_\infty(0) = 0$ because of the conservativeness. Consequently, Theorem 4.2 does not hold when K are the whole space $\mathcal{P}$. This example was given in [16]. Next consider the Ornstein-Uhlenbeck process, the diffusion process generated by $(1/2)\Delta - x \cdot d/dx$ on $(-\infty, \infty)$. Then both boundaries are natural and $\lambda_2(0)$ and $\lambda_\infty(0)$ are zero, consequently the L^p-independence follows. We would like to remark that the uniform upper bound (ii) is not known, while the locally uniform upper bound was shown in [16]. In this sense, we can say that the L^p-independence of the Ornstein-Uhlenbeck operator holds for the reason that $\lambda_2(0) = 0$.

Let $\mathbb{M} = (\mathbb{P}_x, X_t)$ be a symmetric Lévy process with Lévy exponent ψ

$$\mathbb{E}_x(\exp(i(\xi, X_t)) = \exp(-t\psi(\xi)).$$

Assume that

$$\int_{\mathbb{R}^d} e^{-t\psi(\xi)} d\xi < \infty, \quad \forall t > 0, \tag{5.6}$$

We can show that the assumption (5.6) implies the strong Feller property and $\lambda_2(0)$ equals to 0. Hence, $\lambda_2(\mu) \le 0$ for any $\mu \in \mathcal{K}_\infty$ and The L^p-independence of $\lambda_p(\mu)$ follows.

If the Lévy measure J of $\mathbb{M}$ is *exponentially localized*, that is, there exists a positive constant δ such that

$$\int_{|x|>1} e^{\delta|x|} J(dx) < \infty, \tag{5.7}$$

we can prove in the same way as in [29] that for μ in the class $\mathcal{K}$, $\lambda_p(\mu)$ is independent of p. For example, the Lévy measure of the *relativistic Schrödinger process*, the symmetric Lévy process generated by

$\sqrt{-\Delta + m^2} - m$, $m > 0$, satisfies (5.7) (Carmona, Master and Simon [7]). On the other hand, the Lévy measure of the symmetric α-stable process on $\mathbb{R}^d$ is $\left(K(d,\alpha)/|x|^{d+\alpha}\right) dx$, and is not exponentially localized, though its Lévy exponent satisfies (5.6). This is the reason why we need to restrict the class of potentials to $\mathcal{K}_\infty$.

6. Related topics

Let $\mathbb{M}^\alpha = (\Omega, \mathcal{F}, \mathcal{F}_t, \theta_t, \mathbb{P}_x, X_t)$ be a symmetric α-stable process on $\mathbb{R}^1$ with $0 < \alpha < 2$. Here $\{\mathcal{F}_t\}_{t\geq 0}$ is the minimal (augmented) admissible filtration and θ_t, $t \geq 0$, is the shift operators satisfying $X_s(\theta_t) = X_{s+t}$ identically for $s, t \geq 0$. When $\alpha \geq 1$ (resp. $\alpha < 1$), the process $\mathbb{M}^\alpha$ is recurrent (resp. transient). Moreover, if $\alpha > 1$, then $\mathbb{M}^\alpha$ is pointwise recurrent. In this paper, we consider the recurrent case.

Let $p(t,x,y)$ be the transition density function of $\mathbb{M}^\alpha$ and $G(x,y)$ the so-called *compensated Green kernel*: for $\alpha = d = 1$,

$$G(x,y) = \frac{1}{\pi} \log \frac{1}{|x-y|},$$

and for $\alpha > d = 1$,

$$G(x,y) = \frac{|x-y|^{\alpha-1}}{2\Gamma(\alpha)\cos(\pi\alpha/2)}.$$

Let $(\mathcal{E}^{(\alpha)}, \mathcal{D}(\mathcal{E}^{(\alpha)}))$ be the Dirichlet form generated by $\mathbf{M}^\alpha$. It is given by

(6.1)

$$\mathcal{E}^{(\alpha)}(u,v) = \mathcal{A}(1,\alpha) \iint_{\mathbb{R}^1\times\mathbb{R}^1\setminus\Delta} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{1+\alpha}} dxdy$$

(6.2)

$$\mathcal{D}(\mathcal{E}^{(\alpha)}) = \left\{ u \in L^2(\mathbb{R}^1) : \iint_{\mathbb{R}^1\times\mathbb{R}^1\setminus\Delta} \frac{(u(x)-u(y))^2}{|x-y|^{1+\alpha}} dxdy < \infty \right\},$$

where

$$\mathcal{A}(1,\alpha) = \frac{\alpha 2^{1-1}\Gamma(\frac{\alpha+1}{2})}{\pi^{1/2}\Gamma(1-\frac{\alpha}{2})}$$

([18, Example 1.4.1]).

It is known that $\mu \in \mathcal{K}$ is equivalent with

$$\lim_{a\to 0} \sup_{x\in\mathbb{R}^1} \int_{|x-y|\leq a} G(x,y)|\mu|(dy) = 0. \tag{6.3}$$

Let $G^\mu(x,y)$ be the Green function defined by

$$\int_0^\infty p_t^\mu f(x)dt = \int_{\mathbb{R}^1} G^\mu(x,y)f(y)dy.$$

For a positive measure $\mu \in \mathcal{K}$ denote by $\mathbb{M}^\mu = (\mathbb{P}_x^\mu, X_t, \zeta)$ the subprocess by the multiplicative functional $\exp(-A_t^\mu)$:

$$\mathbb{P}_x^\mu(d\omega) = \exp(-A_t^\mu(\omega))\mathbb{P}_x(d\omega),$$

where ζ is the lifetime of $\mathbb{M}^\mu$. Then $G^\mu(x,y)$ is the 0-resolvent of $\mathbb{M}^\mu$. $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is a regular Dirichlet form generated by $\mathbb{M}^\mu$ ([18, Theorem 6.1.1, Theorem 6.1.2]).

We now introduce a class $\mathcal{K}_\infty(G^\mu)$ associated with the Green kernel G^μ: $\nu \in \mathcal{K}$ is said to be in $\mathcal{K}_\infty(G^\mu)$ if

$$\lim_{R\to\infty} \sup_{x\in\mathbb{R}^1} \int_{|y|\geq R} G^\mu(x,y)|\nu|(dy) = 0. \tag{6.4}$$

We call a measure ν in $\mathcal{K}_\infty(G^\mu)$ *G^μ-Green tight* measure. Since $\mathbb{M}^\mu$ has the strong Feller property ([1, Theorem 7.5]) and

$$\lim_{t\to 0} \sup_{x\in\mathbb{R}^1} \mathbb{E}_x^\mu[A_t^\nu] \leq \lim_{t\to 0} \sup_{x\in\mathbb{R}^1} \mathbb{E}_x[A_t^\nu] = 0,$$

$\mathcal{K}_\infty(G^\mu)$ is contained in the class introduced in [8, Definition 2.2] ([21]). It is known in [8, Proposition 2.2] that a measure ν in $\mathcal{K}_\infty(G^\mu)$ is *G^μ-Green bounded* :

$$\sup_{x\in\mathbb{R}^1} G^\mu(\nu)(x) = \sup_{x\in\mathbb{R}^1} \mathbb{E}_x^\mu[A_\infty^\nu] < \infty. \tag{6.5}$$

Let $\mu = \mu^+ - \mu^- \in \mathcal{K} - \mathcal{K}_\infty(G^{\mu^+})$. The Schrödinger operator $\mathcal{H}^\mu$ is said to be *critical* (resp. *subcritical*) if $\lambda(\mu) = 1$ (resp. $\lambda(\mu) > 1$). Define

$$\beta_p(\mu) = \sup_{t>0} \|e^{-t\mathcal{H}^\mu}\|_{p,p}.$$

We see from the symmetry and interpolation that

$$\|e^{-t\mathcal{H}^\mu}\|_{2,2} \leq \|e^{-t\mathcal{H}^\mu}\|_{p,p} \leq \|e^{-t\mathcal{H}^\mu}\|_{\infty,\infty}, \ 1 \leq p \leq \infty.$$

Hence

$$\beta_2(\mu) \leq \beta_p(\mu) \leq \beta_\infty(\mu), \ \ 1 \leq p \leq \infty. \tag{6.6}$$

In Simon [25], $\mathcal{H}^\mu$ is said to be critical if $\lambda_\infty(\mu) = 0$ but $\lambda_\infty((1+\epsilon)\mu) < 0$ for all $\epsilon > 0$ and is said to be subcritical if $\lambda_\infty((1+\epsilon)\mu) = 0$ for some $\epsilon > 0$. We see that if $\mu = \mu^+ - \mu^- \in \mathcal{K}_\infty - \mathcal{K}_\infty$, then these two definitions are equivalent. Here $G_1(x,y)$ is the 1-resolvent density of $\mathbf{M}^\alpha$; in fact, first note that for $\mu \in \mathcal{K}_\infty$

$$\begin{aligned}
\mathcal{E}^{\mu^+}(u,u) =& \mathcal{E}^{(\alpha)}(u,u) + \int_{\mathbb{R}^1} u^2 1_{B(R)} d\mu^+ + \int_{\mathbb{R}^1} u^2 1_{B(R)^c} d\mu^+ \\
\leq& \mathcal{E}^{(\alpha)}(u,u) + \int_{\mathbb{R}^1} u^2 1_{B(R)} d\mu^+ + \|G_1(1_{B(R)^c}\mu^+)\|_\infty \cdot \mathcal{E}_1^{(\alpha)}(u,u).
\end{aligned}$$

Noting the bottom of spectrum $(-d^2/dx^2)^{\alpha/2}$ equals 0, we can take a sequence $\varphi_n \in C_0^\infty(\mathbb{R}^1)$, $n = 1, 2, \ldots$ such that $\lim_{n\to\infty} \mathcal{E}^{(\alpha)}(\varphi_n, \varphi_n) = 0$ and $\int_{\mathbb{R}^1} \varphi_n^2 dx = 1$. Furthermore, since $\mathcal{E}^{(\alpha)}$ is spatially homogeneous, we may suppose that the support of every φ_n is contained in the complement of $B(R)$. Hence we see that

$$\inf\left\{\mathcal{E}^{\mu^+}(u,u) : \int_{\mathbb{R}^1} u^2 dx = 1\right\} \leq \|G_1(1_{B(R)^c}\mu^+)\|_\infty \longrightarrow 0$$

as $R \to \infty$, and thus $\lambda_2(\mu) \leq 0$ for $\mu = \mu^+ - \mu^- \in \mathcal{K}_\infty - \mathcal{K}_\infty$. We then know that $\lambda_p(\mu)$ is independent of $1 \leq p \leq \infty$, because the independence is equivalent with $\lambda_2(\mu) \leq 0$ by [33, Example 4.2] (for recent results on the L^p-independence, see [10]). Define

$$F(\theta) = \inf\left\{\mathcal{E}(u,u) + \theta\int_{\mathbb{R}^1} u^2 d\mu : \int_{\mathbb{R}^1} u^2 dx = 1\right\},\ \theta \geq 0$$

and

$$G(\theta) = \inf\left\{\mathcal{E}(u,u) + \theta\int_{\mathbb{R}^1} u^2 d\mu^+ : \theta\int_{\mathbb{R}^1} u^2 d\mu^- = 1\right\},\ \theta \geq 0.$$

As shown above, if $\mu \in \mathcal{K}_\infty - \mathcal{K}_\infty$ then $F(\theta) \leq 0$. Put

$$\theta_0 = \sup\{\theta \geq 0 : F(\theta) = 0\}.$$

We see that θ_0 is a unique solution of $G(\theta) = 1$ and $G(\theta) \geq 1$ if and only if $0 \leq \theta \leq \theta_0$. Note $\lambda_2(\mu) = F(1)$. We then see that $\mathcal{H}^\mu$ is critical in the sense of Simon [25] if and only if $\lambda(\mu)(:= G(1)) = 1$ ($\Longleftrightarrow \theta_0 = 1$). Therefore, we have the next lemma.

Lemma 6.1. *Let $\mu = \mu^+ - \mu^- \in \mathcal{K}_\infty - \mathcal{K}_\infty$. Then $\mathcal{H}^\mu$ is critical in the sense of Simon if and only if $\lambda(\mu) = 1$.*

For the argument above, the L^p-independence of $\lambda_p(\mu)$ is crucial. We here give another proof of Theorem A.12 in [25] which is relevant to the L^p-independence.

Theorem 6.2. ([37]) *Let $\mu = \mu^+ - \mu^- \in \mathcal{K}_\infty - \mathcal{K}_\infty$. Let $f \in \mathcal{B}_b(\mathbb{R}^1)$ with $f \geq 0$ a.e. and $m(\{f(x) > 0\}) > 0$. Then for any $x \in \mathbb{R}^1$*

$$\alpha_f(x) := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x\left[\exp(-A_t^\mu) f(X_t)\right]$$

exists. Moreover, the limit is equal to $-\lambda_2(\mu)$, in particular, independent of f and x.

Proof. Define $g(x) = \mathbb{E}_x\left[\exp(-A_1^\mu) f(X_1)\right]$. The continuity of g follows from the strong Feller property of p_t^μ ([1, Theorem 7.5]). Since $\mathbb{E}_x[f(X_1)] > 0$ by the assumption on f and $\exp(-A_1^\mu) > 0$, $\mathbb{P}_x$-a.s., the function g is strictly positive and continuous. Put $m_R = \inf_{x\in B(R)} g(x) > 0$. Then by the Markov property

$$\begin{aligned}\mathbb{E}_x\left[\exp(-A_t^\mu) f(X_t)\right] &= \mathbb{E}_x\left[\exp(-A_{t-1}^\mu) g(X_{t-1})\right] \\ &\geq m_R \cdot \mathbb{E}_x\left[\exp(-A_{t-1}^\mu); t-1 < \tau_{B(R)}\right],\ t > 1.\end{aligned}$$

Hence Theorem 1.1 in [34] tells us that for $x \in B(R)$

$$\begin{aligned}
&\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x \left[\exp(-A_t^\mu) f(X_t)\right] \\
&\geq \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x \left[\exp(-A_{t-1}^\mu); t-1 < \tau_{B(R)}\right] \\
&\geq -\lambda_R \left(:= -\inf\left\{ \mathcal{E}^\mu(u,u) : u \in C_0^\infty(B(R)),\ \int_{\mathbb{R}^1} u^2 dx = 1 \right\}\right).
\end{aligned}$$

Noting $\lambda_R \downarrow \lambda_2(\mu)$ as $R \uparrow \infty$, we have

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x[\exp(-A_t^\mu) f(X_t)] \geq -\lambda_2(\mu).$$

Since

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x \left[\exp(-A_t^\mu) f(X_t)\right] \\
&\leq \limsup_{t\to\infty} \frac{1}{t} \log \left(\|f\|_\infty \cdot \sup_{x\in\mathbb{R}^1} \mathbb{E}_x \left[\exp(-A_t^\mu)\right]\right) = -\lambda_\infty(\mu),
\end{aligned}$$

the L^p-independence of λ_p leads us to this theorem. □

The condition $\lambda(\mu) > 1$ gives the following probabilistic meaning, so called, *gaugeability* of μ^- with respect to $\mathbb{M}^{\mu^+}$.

Theorem 6.3. ([8]) *It holds that*

$$\lambda(\mu) > 1 \iff \sup_{x\in\mathbb{R}^1} \mathbb{E}_x^{\mu^+}\left[\exp(A_\zeta^{\mu^-})\right] < \infty.$$

We define an $\mathcal{H}^\mu$-harmonic functions probabilistically as follows:

Definition 6.4. A bounded finely continuous function h on $\mathbb{R}^1$ is said to be $\mathcal{H}^\mu$-harmonic, if for any relatively compact domain $D \subset \mathbb{R}^1$,

$$h(x) = \mathbb{E}_x\left[\exp(-A_{\tau_D}^\mu) h(X_{\tau_D})\right], \quad x \in D \tag{6.7}$$

where τ_D is the first exit time from D.

Lemma 6.5. *Suppose that $\mathcal{H}^\mu$ is critical, $\lambda(\mu) = 1$. If μ^- has a compact support, then there exists a bounded $\mathcal{H}^\mu$-harmonic function. If, in addition, μ^+ has a compact support, then there exists an $\mathcal{H}^\mu$-harmonic function uniformly lower-bounded by a positive constant.*

Proof. Fiest note that there exists a ground state h ([37]):

$$\mathcal{E}^{\mu^+}(h,h) = \inf\left\{ \mathcal{E}^{\mu^+}(u,u) :\ u \in \mathcal{D}_e(\mathcal{E}^{\mu^+}),\ \int_{\mathbb{R}^1} u^2 d\mu^- = 1\right\}. \tag{6.8}$$

Then the function h satisfies

$$h(x) = \mathbb{E}_x^{\mu^+}\left[h(X_{\sigma_F})\right] = \mathbb{E}_x\left[\exp(-A_{\sigma_F}^{\mu^+}) h(X_{\sigma_F})\right],$$

where F is the fine support of μ^-. Put $M = \sup_{x\in F} h(x)$. Noting that $0 < M < \infty$ by the continuity of h, we have $h(x) \leq M$.

When the support μ^+ is also compact, we take $R > 0$ such that $\bar{B}(R) \supset F \cup \mathrm{supp}[\mu^+]$. Since $\sigma_F = \sigma_{\bar{B}(R)} + \sigma_F(\theta_{\sigma_{\bar{B}(R)}})$ and $A^{\mu^+}_{\sigma_F} = A^{\mu^+}_{\sigma_{\bar{B}(R)}} + A^{\mu^+}_{\sigma_F}(\theta_{\sigma_{\bar{B}(R)}})$,

$$\begin{aligned} h(x) &= \mathbb{E}_x\left[\exp(-A^{\mu^+}_{\sigma_{\bar{B}(R)}})\mathbb{E}_{X_{\sigma_{\bar{B}(R)}}}\left[\exp(-A^{\mu^+}_{\sigma_F})h(X_{\sigma_F})\right]\right] \\ &= \mathbb{E}_x\left[\exp(-A^{\mu^+}_{\sigma_{\bar{B}(R)}})h(X_{\sigma_{\bar{B}(R)}})\right] \end{aligned}$$

by the strong Markov property. Since $\bar{B}(R) \supset \mathrm{supp}[\mu^+]$, we have $A^{\mu^+}_{\sigma_{\bar{B}(R)}.} = 0$. Note $\mathbb{P}_x(\sigma_{\bar{B}(R)} < \infty) = 1$ by the recurrence of $\mathbb{M}^\alpha$. Hence

$$h(x) = \mathbb{E}_x\left[h(X_{\sigma_{\bar{B}(R)}})\right] \geq \inf_{x \in \bar{B}(R)} h(x) > 0$$

by the continuity of h. □

Lemma 6.6. *Suppose μ has a compact support. Then the function h in Proposition 6.5 is p^μ_t-excessive.*

Proof. Since h is bounded continuous, $\lim_{t\to 0} p^\mu_t h(x) = h(x)$.

Let $x \in B(m)$. By Definition 6.4, h satisfies

$$h(x) = \mathbb{E}_x\left[\exp(-A^\mu_{\tau_n})h(X_{\tau_n})\right]$$

for any $n > m$. Here τ_n is the first exit time from $B(n)$. It follows from the Markov property that

$$\begin{aligned} &\mathbb{E}_x\left[\exp(-A^\mu_t)h(X_t); t < \tau_m\right] \\ =&\mathbb{E}_x\left[\exp(-A^\mu_t)\mathbb{E}_{X_t}[\exp(-A^\mu_{\tau_n})h(X_{\tau_n})]; t < \tau_m\right] \\ =&\mathbb{E}_x\left[\exp(-A^\mu_t)\exp(-A^\mu_{\tau_n}\circ\theta_t)h(X_{\tau_n}\circ\theta_t); t < \tau_m\right] \\ =&\mathbb{E}_x\left[\exp(-A^\mu_{\tau_n})h(X_{\tau_n}); t < \tau_m\right] \leq h(x). \end{aligned}$$

Hence we have

$$p^\mu_t h(x) = \lim_{m\to\infty} \mathbb{E}_x\left[\exp(-A^\mu_t)h(X_t); t < \tau_m\right] \leq h(x).$$

□

Theorem 6.7. ([37]) *Suppose μ has a compact support. If $\lambda_\infty(\mu) = 0$, then $\beta_\infty(\mu) < \infty$.*

Proof. If $\lambda_\infty(\mu) = 0$, then $\lambda_2(\mu) \leq \lambda_\infty(\mu) = 0$ by (6.6). We easily see that $\lambda_2(\mu) > 0$ is equivalent to $\lambda(\mu) < 1$, and thus $\lambda_2(\mu) \leq 0$ is equivalent to $\lambda(\mu) \geq 1$.

If $\lambda(\mu) > 1$, then by Theorem 6.3

$$\begin{aligned} \|p^\mu_t\|_{\infty,\infty} &= \sup_{x\in\mathbb{R}^1}\mathbb{E}_x\left[e^{-A^\mu_t}\right] = \sup_{x\in\mathbb{R}^1}\mathbb{E}^{\mu^+}_x\left[e^{A^{\mu^-}_t}; t < \zeta\right] \\ &\leq \sup_{x\in\mathbb{R}^1}\mathbb{E}^{\mu^+}_x\left[e^{A^{\mu^-}_\zeta}\right] < \infty, \end{aligned}$$

which implies $\beta_\infty(\mu) < \infty$.

If $\lambda(\mu) = 1$, then by Proposition 6.5 there exists a bounded $\mathcal{H}^\mu$-harmonic function uniformly lower-bounded by a positive constant. Hence by Lemma 6.6

$$\|p_t^\mu\|_{\infty,\infty} \leq \mathbb{E}_x\left[e^{-A_t^\mu}\frac{h(X_t)}{\inf_{x\in\mathbb{R}^1} h(x)}\right] = \frac{1}{\inf_{x\in\mathbb{R}^1} h(x)}\mathbb{E}_x\left[e^{-A_t^\mu}h(X_t)\right]$$
$$\leq \frac{h(x)}{\inf_{x\in\mathbb{R}^1} h(x)} \leq \frac{\sup_{x\in\mathbb{R}^1} h(x)}{\inf_{x\in\mathbb{R}^1} h(x)}.$$

□

Theorem 6.8. ([37]) *Suppose that $\mathcal{H}^\mu$ is subcritical. Then there exists no bounded positive $\mathcal{H}^\mu$-harmonic function.*

Proof. Let h be a bounded positive $\mathcal{H}^\mu$-harmonic function. Since, by the Harris recurrence of $\mathbb{M}^\alpha$, $\mathbb{P}_x(\lim_{n\to\infty} A^{\mu^+}_{\tau_{B(n)}} = \infty) = 1$ as $n \to \infty$,

$$\mathbb{P}_x^{\mu^+}(\tau_{B(n)} < \zeta) = \mathbb{E}_x\left[e^{-A^{\mu^+}_{\tau_{B(n)}}}\right] \longrightarrow 0$$

as $n \to \infty$. Moreover, the subcriticality of $\mathcal{H}^\mu$ implies $e^{A_\zeta^{\mu^-}} \in L^1(\mathbb{P}_x^{\mu^+})$ by Theorem 6.3. Hence we have

$$h(x) = \mathbb{E}_x\left[e^{-A^\mu_{\tau_{B(n)}}}h(X_{\tau_{B(n)}})\right] \leq \|h\|_\infty \cdot \mathbb{E}_x^{\mu^+}\left[e^{A_\zeta^{\mu^-}}; \tau_{B(n)} < \zeta\right] \longrightarrow 0$$

as $n \to \infty$. □

Proposition 6.8 tells us that properties of $\mathcal{H}^\mu$-harmonic functions are different whether $\mathbb{M}^\alpha$ is recurrent or transient. If $\mathbb{M}^\alpha$ is transient and $\mathcal{H}^\mu$ is subcritical, the function $\mathbb{E}_x[\exp(A^\mu_\infty)]$ is a strictly positive, bounded $\mathcal{H}^\mu$-harmonic function. Moreover, if $\mathcal{H}^\mu$ is critical, there exists no $\mathcal{H}^\mu$-harmonic function uniformly lower-bounded by a positive constant ([40]).

7. References

[1] S. Albeverio, P. Blanchard, and Z.-M. Ma, Feynman-Kac semigroups in terms of signed smooth measures, In *Random partial differential equations (Oberwolfach, 1989)*, volume 102 of *Internat. Ser. Numer. Math.*, pages 1–31. Birkhäuser, Basel, 1991.

[2] A. Beurling and J. Deny, Espaces de Dirichlet I, Le cas elementaire, *Acta Math.*, 99, 203–224, 1958.

[3] A. Beurling and J. Deny, Dirichlet spaces, *Proc. Nat. Acad. Sci. U.S.A.*, 45:208–215, 1959.

[4] K. Bogdan and T. Byczkowski, Potential theory of Schrödinger operator based on fractional Laplacian, Probability and Mathematical Statistics **20**, 293-335, (2000).

[5] N. Bouleau and F. Hirsch, *Dirichlet forms and analysis on Wiener space*, de Gruyter Studies in Mathematics, 14. Walter de Gruyter, Berlin, 1991.

[6] A. Budhiraja and P. Dupuis, Large deviations for the empirical measures of reflecting Brownian motion and related constrained processes in $\mathbf{R}_+$, *Electric Journal of Probability*, 8, 1–46, 2003.
[7] R. Carmona, W. C. Masters, and B. Simon, Relativistic Schrödinger operators: asymptotic behavior of the eigenfunctions, *J. Funct. Anal.*, 91, 117–142, 1990.
[8] Z.-Q. Chen, Gaugeability and conditional gaugeability, *Trans. Amer. Math. Soc.*, 354, 4639–4679 (electronic), 2002.
[9] Z.-Q. Chen, Analytic characterization of conditional gaugeability for non-local Feynman-Kac transforms, *J. Funct. Anal.*, 202, 226–246, 2003.
[10] Z.-Q. Chen, Lp-independence of spectral bounds of generalized non-local Feynman-Kac semigroups, *J. Funct. Anal.*, 262, 4120–4139, 2012.
[11] K. L. Chung, Doubly-Feller process with multiplicative functional, In *Seminar on stochastic processes, 1985 (Gainesville, Fla., 1985)*, volume 12 of *Progr. Probab. Statist.*, pages 63–78. Birkhäuser Boston, Boston, MA, 1986.
[12] E. B. Davies, *Heat kernels and spectral theory*, volume 92 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1989.
[13] G. De Leva, D. Kim, and K. Kuwae, L^p-independence of spectral bounds Feynman-Kac semigroups by continuous additive functionals, *J. Funct. Anal.*, 259, 690–730, 2010.
[14] M. D. Donsker and S. R. S. Varadhan, Asymptotic evaluation of certain Wiener integrals for large time, In *Functional integration and its applications (Proc. Internat. Conf., London, 1974)*, pages 15–33. Clarendon Press, Oxford, 1975.
[15] M. D. Donsker and S. R. S. Varadhan, Asymptotic evaluation of certain Markov process expectations for large time, I, *Comm. Pure Appl. Math.*, 28, 1–47, 1975.
[16] M. D. Donsker and S. R. S. Varadhan, Asymptotic evaluation of certain Markov process expectations for large time, III, *Comm. Pure Appl. Math.*, 29, 389–461, 1976.
[17] M. Fukushima, Dirichlet spaces and strong Markov processes, *Trans. Amer. Math. Soc.* , 162, 185–224, 1971.
[18] M. Fukushima, Y. Oshima, and M. Takeda, *Dirichlet forms and symmetric Markov processes*, volume 19 of *de Gruyter Studies in Mathematics*, Walter de Gruyter & Co., Berlin, 1994.
[19] K. Ito, *Essentials of stochastic processes*, American Mathematical Society, Providence, RI, 2006.
[20] D. Kim, Asymptotic properties for continuous and jump type's Feynman-Kac functionals, *Osaka J. Math.*, 37, 147–173, 2000.
[21] D. Kim and K. Kuwae, On analytic characterizations of gaugeability for generalized Feynman-Kac functionals, preprint.
[22] K. Kuwae, Stoshastic caluculus over symmetric Markov processes with time reversal., preprint.
[23] Murata, M.: Positive solutions and large time behaviors of Schrodinger semigroups, Simon's problem, J. Funct. Anal. **56**, 300-310, (1984).
[24] B. Simon, Brownian motion, L^p properties of Schrödinger operators and the localization of binding, *J. Funct. Anal.*, 35, 215–229, 1980.
[25] B. Simon, Large time behavior of the L^p norm of Schrödinger semigroups, J. Funct. Anal. **40**, 66-83, (1981).
[26] P. Stollmann and J. Voigt, Perturbation of Dirichlet forms by measures, *Potential Anal.*, 5, 109–138, 1996.
[27] K.-Th. Sturm, Schrödinger semigroups on manifolds, *J. Funct. Anal.*, 118, 309–350, 1993.
[28] M. Takeda, A large deviation for symmetric Markov processes with finite life time, *Stochastics and Stochastic Reports*, 59, 143–167, 1996.

[29] M. Takeda, Asymptotic properties of generalized Feynman-Kac functionals. *Potential Anal.*, 9, 261–291, 1998.

[30] M. Takeda, Exponential decay of lifetimes and a theorem of Kac on total occupation times, Potential Analysis **11**, 235-247, (1999).

[31] M. Takeda, L^p-independence of the spectral radius of symmetric Markov semigroups, Stochastic Processes, Physics and Geometry: New Interplays. **II**: A Volume in Honor of Sergio Albeverio, Edited by Fritz Gesztesy, et.al, (2000).

[32] M. Takeda, Conditional gaugeability and subcriticality of generalized Schrödinger operators, *J. Funct. Anal.*, 191, 343–376, 2002.

[33] M. Takeda, L^p-independence of spectral bounds of Schrodinger type semigroups, J. Funct. Anal. **252**, 550-565 (2007).

[34] M Takeda, M.: Gaussian bounds of heat kernels of Schrödinger operators on Riemannian manifolds, *Bull. London Math. Soc.* 39, 85–94, 2007.

[35] M. Takeda, L^p-independence of spectral bounds of Schrödinger type semigroups, *J. Funct. Anal.*, 252, 550–565, 2007.

[36] M. Takeda, A large deviation principle for symmetric Markov processes with Feynman-Kac functional, *J. Theoret. Probab.*, 24, 1097-1129, 2011.

[37] M. Takeda, Criticality for Schrödinger operators based on recurrent symmetric α-stable processes, preprint.

[38] M. Takeda and Y. Tawara, L^p-independence of spectral bounds of non-local Feynman-Kac semigroups, *Forum Math.*, 21, 1067-1080, 2009.

[39] M. Takeda and Y. Tawara, A Large deviation principle for symmetric Markov processes normalized by Feynman-Kac functionals, to appear in Osaka J. Math.

[40] M. Takeda and K. Tsuchida. Differentiability of spectral functions for symmetric α-stable processes. *Trans. Amer. Math. Soc.*, 359(8):4031–4054 (electronic), 2007.

[41] M. Takeda and K. Tsuchida, Large deviations for discontinuous additive functionals of symmetric stable processes, *Math. Nachr*, 284, 1148-1171, 2011.

[42] Y. Tawara, L^p-independence of spectral bounds of Schrödinger-type operators with non-local potentials, *J. Math. Soc. Japan*, 62, 767–788, 2010.

[43] Y. Tawara, L^p-independence of growth bounds of Generalized Feynman-Kac semigroups, Doctor Thesis, Tohoku University, 2009.

MATHEMATICAL INSTITUTE, TOHOKU UNIVERSITY, AOBA, SENDAI, 980-8578, JAPAN

E-mail address, Masayoshi Takeda: `takeda@math.tohoku.ac.jp`