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# TOWARD THE CLASSIFICATION OF THE STRUCTURE OF THE POSITIVE SOLUTIONS FOR SUPERCRITICAL ELLIPTIC EQUATIONS IN A BALL

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## 1. Introduction and main results

This article is an announcement of the paper [5]. The proofs of Theorems A, B, and C in this article are in the paper [5].

We study the global bifurcation diagram of the semilinear elliptic Dirichlet problem

$$\begin{cases} \Delta u + \lambda f(u) = 0 & \text{in } B, \\ u > 0 & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{1.1}$$

where $B$ is the unit ball in $\mathbb{R}^N$ $(N \geq 3)$,

$$f(u) = u^p + g(u), \tag{1.2}$$

$$p > p_S := \frac{N+2}{N-2}, \tag{1.3}$$

$g(u)$ is a lower order term, and $\lambda$ is a non-negative constant. Specifically, we assume the following three conditions:

(f1) $\quad f \in C^1([0,\infty))$ and $f(t) > 0$ in $[0,\infty)$,

(f2) $\quad f(u) = u^p + g(u)$ $(p > p_S)$, where there are $u_0 > 0$, $\delta > 0$, and $C_0 > 0$ such that $|g(u)| \leq C_0 u^{p-\delta}$ for $u > u_0$,

(f3) $\quad f(u) = u^p + g(u)$ , where there are $u_0 > 0$, $\delta > 0$, and $C_0 > 0$ such that $|g'(u)| \leq C_0 u^{p-1-\delta}$ for $u > u_0$.

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The exponent $p_S$ is called the Sobolev critical exponent. Because $p > p_S$, the Sobolev embedding $H^1(B) \hookrightarrow L^{p+1}(B)$ does not hold. Hence, it is difficult to use a variational method in the function space $H^1(B)$. By the symmetry result of Gidas-Ni-Nirenberg [2], every positive solution $u$ is radially symmetric and $\|u\|_\infty = u(0)$. This enables us to use ODE techniques. Then there is an unbounded branch $\{(\lambda, u)\}$ consisting of positive radial solutions of (1.1) such that the branch emanates from $(0,0)$.

We mention the existence of the singular solution of (1.1).

**Proposition 1.1.** *Suppose that (f1)–(f3) hold. Then (1.1) has a singular positive solution* $(\lambda^*, u^*)$ *such that*

$$u^*(r) = A(p,N)(\sqrt{\lambda^*}r)^{-\theta}(1 + O(r^{\delta\theta})) \quad \text{as} \quad r \to 0, \tag{1.4}$$

*where* $\delta > 0$ *is the constant in (f2).*

**Corollary 1.2.** *Let* $(\lambda^*, u^*)$ *be a singular solution given in Proposition 1.1. If* $p > p_S$, *then* $u^* \in H^1(B)$.

The proof of Proposition 1.1 is essentially the same as one of [4, Theorem 1.1], and Corollary 1.2 is an immediate consequence of Proposition 1.1. The singular solution plays an important role in the study of the global bifurcation diagram.

We are interested in the classification of the bifurcation diagram. We call the bifurcation diagram *Type I* if there is $\lambda^* > 0$ such that the branch has infinitely many turning points near $\lambda^*$ and that a singular solution exists at $\lambda^*$. The first main theorem is the following:

**Theorem A.** *Suppose that (f1)–(f3) hold. If* $p_S < p < p_{JL}$, *then the bifurcation diagram of (1.1) is of Type I and the extremal solution is regular. In particular, if* $3 \le N \le 10$, *then the bifurcation diagram is always of Type I. Moreover,* $m(u^*) = \infty$, *where* $u^*$ *is the singular solution given in Proposition 1.1 and* $m(u^*)$ *is the Morse index of* $u^*$.

Neither the monotonicity of $f$ nor the convexity of $f$ is assumed in Theorem A.

We consider the case where $p > p_{JL}$. Brezis-Vázques [1] studied (1.1) when

(1.5) $f$ is a continuous, positive, increasing, and convex function on $[0, \infty)$ such that $f(t)/t \to \infty$ as $t \to \infty$.

When (1.5) holds, there is a maximal or extremal value of $\lambda > 0$ such that (1.1) has a solution which is minimal. In [1] the authors studied the corresponding extremal solution when it is unbounded, i.e., the singular solution. We call the bifurcation diagram *Type II* if there is $\lambda^* > 0$ such that the branch consists only of minimal solutions for $\lambda \in (0, \lambda^*)$ and that a singular solution exists at $\lambda^*$. They have shown that

**Proposition 1.3** (Brezis-Vazquez [1, Theorem 3.1]). *Suppose that (1.5) holds. If $(\lambda^*, u^*)$ is a singular solution of (1.1), if $u^* \in H^1(B)$, and if $u^*$ is stable in the sense where*

$$\int_B \left(|\nabla \phi|^2 - \lambda^* f'(u^*)\phi^2\right) dx \geq 0 \quad \text{for all} \quad \phi \in C_0^1(B), \tag{1.6}$$

*then $(\lambda^*, u^*)$ is the extremal solution which indicates that the bifurcation diagram of (1.1) is of Type II.*

Roughly speaking, Proposition 1.3 says that if $u^* \in H^1(B)$ and if $m(u^*) = 0$, then the bifurcation diagram is of Type II.

The second main theorem is the following:

**Theorem B.** *Suppose that (f1)-(f3) hold. If $p > p_{JL}$, then $m(u^*) < \infty$.*

We are interested in the case $1 \leq m(u^*) < \infty$. We call the branch *Type III* when the branch has at least one but finitely many turning points. We conjecture the following:

**Conjecture 1.4.** *Suppose that (f1)-(f3) hold. If $1 \leq m(u^*) < \infty$, then the bifurcation diagram is of Type III. Moreover, for a certain class of nonlinearities, the bifurcation diagram of (1.1) has exactly $m(u^*)$ turning point(s).*

If $f$ is analytic, then the set of the turning points do not have an accumulation points. It is enough to prove the nondegeneracy of large solutions of (1.1) in order to prove the first statement of Conjecture 1.4 for analytic nonlinearities. However, it is difficult to prove the non-degeneracy, because (1.1) is supercritical. We give one example of Type III.

**Theorem C.** *Let $f(u) := (u + \varepsilon) + (u + \varepsilon)^p$. If $p > p_{JL}$ is large, and if $\varepsilon > 0$ is small, then the bifurcation diagram of (1.1) is of Type III. Moreover, every solution is nondegenerate if $\|u\|_\infty$ is large.*

Theorem C indicates that the bifurcation diagram cannot be classified by $p$ if $p > p_{JL}$. The information of the whole graph of $f$ is needed in order to determine the type of the bifurcation diagram.

The relations of Theorems A, B, and C, Proposition 1.3 and Conjecture 1.4 are shown as follows:

$$\begin{cases} p_S < p < p_{JL} & \underset{\text{Theorem A}}{\Longrightarrow} & m(u^*) = \infty & \underset{\text{Theorem A}}{\Longrightarrow} & \text{Type I} \\ p > p_{JL} & \underset{\text{Theorem B}}{\Longrightarrow} & \begin{cases} m(u^*) = 0 \\ \text{or} \\ 1 \le m(u^*) < \infty \end{cases} & \begin{matrix} \underset{\text{Proposition 1.3}}{\Longrightarrow} \\ \\ \underset{\substack{\text{Conjecture 1.4} \\ \text{Theorem C}}}{\Longrightarrow} \end{matrix} & \begin{matrix} \text{Type II} \\ \\ \text{Type III ?} \end{matrix} \end{cases}$$

When $p = p_{JL}$, a more detailed asymptotics is needed to determine the type. However, we need assumptions of $g$. We do not pursue the case $p = p_{JL}$ in this article.

Joseph-Lundgren [3] studied the positive radial branch of the problem

$$\begin{cases} \Delta u + \lambda(1+u)^p = 0 & \text{in } B, \\ u > 0 & \text{in } B, \\ u = 0 & \text{on } \partial B. \end{cases} \tag{1.7}$$

In [3] the authors have shown that the bifurcation diagram of (1.7) is of Type I if $p_S < p < p_{JL}$ and that it is of Type II if $p \geq p_{JL}$. This example is a prototype of our study. In Subsection 2.1 we shall study this equation by our theory.

This article consists of two sections. In Section 2 we give two examples: $f(u) = (u+1)^p$ and (2.1). In Subsection 2.1 we classify the bifurcation diagrams of the equation $\Delta u + \lambda(u+1)^p = 0$ by Theorem A and Proposition 1.3. We obtain the same results as above. In the case of the second equation we cannot expect a special change of variables. We see in Corollary 2.2 that Theorem A and Proposition 1.3 determine the structure of the solutions of (2.1).

## 2. Two examples

2.1. **First example.** This case was studied by Joseph-Lundgren [3]. They used a special change of variables. Then the equation can be reduced to an autonomous system in the phase plane. Hence, the phase plane analysis can be done. In this subsection we will see that Theorem A and Proposition 1.3 are applicable and that the classification of the bifurcation diagrams can be done by Theorem A and Proposition 1.3.

Let $f(u) := (u+1)^p$. Then $g(u) := (u+1)^p - u^p$. (1.1) has a singular solution $(\lambda^*, u^*) = (A^{p-1}, r^{-\theta} - 1)$, where $\theta = 2/(p-1)$ and $u^* \in H^1(B)$

provided that $p > p_S$. We can easily check (f1). There is a large $C_0 > 0$ such that

$$|g(u)| = |(u+1)^p - u^p| \le p(u+1)^{p-1} \le C_0 u^{p-1}.$$

Hence, (f2) holds. Since

$$\begin{aligned} |g'(u)| &= p(u+1)^{p-1} - pu^{p-1} \\ &\le \begin{cases} p(u+1)^{p-2} & (p \ge 2), \\ pu^{p-2} & (1 < p < 2). \end{cases} \end{aligned}$$

If $\delta \in (0, p-1)$ is small, then there are constants $C_1 > 0$, $u_1 > 0$ such that

$$|g'(u)| \le \max\{p(u+1)^{p-2}, pu^{p-2}\} \le C_1 u^{p-1-\delta} \quad (u > u_1)$$

Thus (f3) holds. Applying Theorem A, we see that if $p_S < p < p_{JL}$, then the bifurcation diagrams is of Type I and $m(u^*) = \infty$. Next, we consider the case $p \ge p_{JL}$. It is easy to see that $f$ satisfies (1.5). By direct calculation we can show that if $p \ge p_{JL}$, then $pA^{p-1} \le (N-2)^2/4$. We have

$$\begin{aligned} \int_B (|\nabla\phi|^2 - \lambda^* f'(u^*)\phi^2)\, dx &= \int_B \left(|\nabla\phi|^2 - \frac{pA^{p-1}}{r^2}\phi^2\right) dx \\ &\ge \int_B \left(|\nabla\phi|^2 - \frac{(N-2)^2}{4r^2}\phi^2\right) dx \ge 0, \end{aligned}$$

where we use Hardy's inequality and $(N-2)^2/4$ is its best constant. Applying Proposition 1.3, we see that if $p \ge p_{JL}$, then the bifurcation diagram is of Type II and $m(u^*) = 0$. We have the following:

**Corollary 2.1.** *Let $f(u) := (u+1)^p$. Then (1.1) has the singular solution $(\lambda^*, u^*) = (A^{p-1}, r^{-\theta} - 1)$ and*

$$\text{the bifurcation diagram is of} \begin{cases} \text{Type I and } m(u^*) = \infty \text{ if } p_S < p < p_{JL}, \\ \text{Type II and } m(u^*) = 0 \text{ if } p \ge p_{JL}. \end{cases}$$

*In particular, the Type III bifurcation diagram does not appear.*

2.2. **Second example.** Let $\varepsilon > 0$ be small, and let

$$a := \frac{1}{2}\sqrt{(u+1-\varepsilon)^2 + 4\varepsilon} + \frac{1}{2}(u+1-\varepsilon)$$

and $b := (u+1-\varepsilon)^2 + 4\varepsilon$. We define

$$f(u) := a^p + \varepsilon\frac{N-2+\theta}{N-2-\theta}a^{p-2}. \tag{2.1}$$

Then (1.1) has a singular solution

$$(2.2)\qquad (\lambda^*, u^*) := (\theta(N-2-\theta), r^{-\theta} - 1 - \varepsilon(r^{\theta} - 1)).$$

Note that $N - 2 - \theta > 0$. If $\varepsilon = 0$, then $f(u) = (u+1)^p$ and $u^* = r^{-\theta} - 1$. We can expect that the same property as in Subsection 2.1 holds provided that $\varepsilon > 0$ is small.

We can easily check (f1). We show that (f2) holds. Since $\sqrt{b} \le u + 1 + \varepsilon$, there are a small $\delta > 0$ and a large $u_0 > 0$ such that

$$\begin{aligned} |g(u)| &\le \left(\frac{u+1+\varepsilon}{2} + \frac{u+1-\varepsilon}{2}\right)^p - u^p + \varepsilon\frac{N-2+\theta}{N-2-\theta}a^{p-2} \\ &\le (u+1)^p - u^p + C_0 u^{p-\delta} \quad (u \ge u_0). \end{aligned}$$

Since $u_0 > 0$ is large, there is $C_1 > 0$ such that $(u+1)^p - u^p \le p(u+1)^{p-1} \le C_1 u^{p-\delta}$ $(u \ge u_0)$. Therefore, $|g(u)| \le (C_0 + C_1)u^{p-\delta}$ $(u \ge u_0)$ and (f2) holds. We show that (f3) holds. We can easily show that $p(u+1-\varepsilon)^{p-1} - pu^{p-1} \le C_2 u^{p-1-\delta}$ for large $u$. Using this inequality, we see that there is $u_1 > 0$ such that

$$\begin{aligned} |g'(u)| &= p\frac{a^p}{\sqrt{b}} - pu^{p-1} + \varepsilon(p-2)\frac{N-2+\theta}{N-2-\theta}\frac{a^{p-2}}{\sqrt{b}} \\ &\le p\frac{(u+1+\varepsilon)^p}{u+1-\varepsilon} - pu^{p-1} + C_3 u^{p-1-\delta} \\ &= p(u+1-\varepsilon)^{p-1} - pu^{p-1} + 2\varepsilon p\frac{(u+1+\varepsilon)^{p-1}}{u+1-\varepsilon} + C_3 u^{p-1-\delta} \\ &\le (C_2 + C_4 + C_3)u^{p-1-\delta} \quad (u \ge u_1). \end{aligned}$$

Thus, (f3) holds. We can apply Theorem A. We obtain the Type I bifurcation diagram when $p_S < p < p_{JL}$.

Next, we check (1.5) when $p \ge p_{JL}$ and $N \ge 11$. In particular, we prove $f'(u) > 0$ $(u > 0)$ and $f''(u) > 0$ $(u > 0)$.

First, we show that $f'(u) > 0$ $(u > 0)$. Since

$$f'(u) = \frac{a^{p-2}}{\sqrt{b}}\left\{pa^2 + \varepsilon(p-2)\frac{N-2+\theta}{N-2-\theta}\right\},$$

$f'(u) > 0$ $(u > 0)$ provided that $p \ge 2$. We consider the case $p_{JL} \le p < 2$. If $p_{JL} < 2$, then $N \ge 16$. This case appears when $N \ge 16$. Since $a \ge 1$, it is enough to show that

$$(2.3)\qquad \varepsilon < \frac{p(N-2-\theta)}{(2-p)(N-2+\theta)} (=: y_N(p)).$$

By elemental calculation we can show that $y_N(p)$ is increasing in $p \in [p_{JL}, 2)$. We have

$$y_N(p_{JL}) = \frac{(N-2\sqrt{N-1})(N-2+2\sqrt{N-1})}{(N-8-2\sqrt{N-1})(3N-6-2\sqrt{N-1})} > 0 \quad (N \geq 16)$$

and $\lim_{N\to\infty} y_N(p_{JL}) = 1/3$. Therefore, $\inf_{N\geq 16} y_N(p_{JL}) > 0$. This means that if $\varepsilon > 0$ is small, then (2.3) holds for all $p \in [p_{JL}, 2)$. Thus, $f'(u) > 0$ $(u > 0)$ if $p_{JL} \leq p < 2$. Combining the cases $p \geq 2$ and $p_{JL} < p < 2$, we have shown that $f'(u) > 0$ $(u > 0)$ for all $p \geq p_{JL}$.

Second, we show that $f''(u) > 0$ $(u > 0)$. We have

$$\text{(2.4)} \quad f''(u) = \frac{pa^p}{b}\left(a - \frac{u+1-\varepsilon}{\sqrt{b}}\right) + \varepsilon(p-2)\frac{N-2+\theta}{N-2-\theta}\frac{a^{p-2}}{\sqrt{b}}\left(p-2-\frac{u+1-\varepsilon}{\sqrt{b}}\right).$$

Since $(u+1-\varepsilon)/\sqrt{b} < 1$, we easily see that if $p \geq 3$, then $f''(u) > 0$ $(u > 0)$. When $1 < p \leq 2$, the second term of (2.4) is positive, hence $f''(u) > 0$ $(u > 0)$. All we have to do is to study the case $2 < p < 3$. Using

$$\begin{aligned} p^2a^2\sqrt{b} - pa^2(u+1-\varepsilon) &= pa^2\left\{\sqrt{b} - (u+1-\varepsilon)\right\} + p(p-1)a^2\sqrt{b} \\ &\geq p(p-1)a^2\sqrt{b} \\ &\geq p(p-1)a(u+1-\varepsilon)\sqrt{b}, \end{aligned}$$

we have

$$\begin{aligned} f''(u) &= \frac{a^{p-2}}{b^{\frac{3}{2}}}\Big[p^2a^2\sqrt{b} - pa^2(u+1-\varepsilon) \\ &\quad +\varepsilon(p-2)\frac{N-2+\theta}{N-2-\theta}\left\{(p-2)b - (u+1-\varepsilon)\sqrt{b}\right\}\Big] \\ &\geq \frac{a^{p-2}}{b^{\frac{3}{2}}}\Big\{p(p-1)a^2(u+1-\varepsilon)\sqrt{b} \\ &\quad -\varepsilon(p-2)\frac{N-2+\theta}{N-2-\theta}(u+1-\varepsilon)\sqrt{b}\Big\} \\ &= \frac{a^{p-2}}{b}(u+1-\varepsilon)\left\{p(p-1)a - \varepsilon(p-2)\frac{N-2+\theta}{N-2-\theta}\right\}. \end{aligned}$$

Since

$$p(p-1)a - \varepsilon(p-2)\frac{N-2+\theta}{N-2-\theta} > p(p-1)a - \varepsilon(p-1)\frac{N-2+\theta}{N-2-\theta},$$

it is enough to show that

$$\varepsilon < p\frac{N-2-\theta}{N-2+\theta}(=: z_N(p)) \quad (2<p<3). \tag{2.5}$$

We easily see that $z_N(p)$ is increasing in $p \in (2,3)$ when $N \geq 3$. Since $z_N(2) = 2(N-4)/N > 0$ ($N \geq 5$) and $\lim_{N\to\infty} z_N(2) = 3$, we see that $\inf_{N\geq 11} z_N(2) > 0$. This inequality means that if $\varepsilon > 0$ is small and if $N \geq 5$, then (2.5) holds for all $p \in (2,3)$. Thus, $f''(u) > 0$ ($u > 0$) for $p \in (2,3)$. Combining three cases, we have shown that $f''(u) > 0$ ($u > 0$) for all $p \geq p_{JL}$. The proof of (1.5) is complete.

We check (1.6) when $p \geq p_{JL}$ and $N \geq 11$. Since $p \geq p_{JL}$, $(N-2)^2 \geq 4p\theta(N-2-\theta)$. Then

$$\begin{aligned}(N-2)^2 &- 4\theta(p-2)(N-2+\theta)\\ &\geq 4p\theta(N-2-\theta) - 4\theta(p-2)(N-2+\theta)\\ &= 8\theta(N-4) > 0,\end{aligned}$$

because $N \geq 11$. Using this inequality, we have

$$(N-2)^2 - 4p\theta(N-2-\theta) + \varepsilon r^{2\theta}\left\{(N-2)^2 - 4\theta(p-2)(N-2+\theta)\right\} \geq 0 \tag{2.6}$$

for $0 \leq r \leq 1$. Using (2.6), we have

$$\begin{aligned}\lambda^* f(u^*) &= \frac{\lambda^* r^{-(p-2)\theta}}{r^{-\theta}+\varepsilon r^{\theta}}\left\{pr^{-2\theta} + \varepsilon(p-2)\frac{N-2+\theta}{N-2-\theta}\right\}\\ &= \frac{\theta(N-2-\theta)}{(1+\varepsilon r^{2\theta})r^2}\left\{p + \varepsilon(p-2)\frac{N-2+\theta}{N-2-\theta}r^{2\theta}\right\}\\ &\leq \frac{(N-2)^2}{4r^2} \quad (0 \leq r \leq 1).\end{aligned}$$

By Hardy's inequality we have

$$\int_B \left(|\nabla\phi|^2 - \lambda^* f'(u^*)\phi^2\right) dx \geq \int_B \left(|\nabla\phi|^2 - \frac{(N-2)^2}{4r^2}\phi^2\right) dx \geq 0$$

for all $\phi \in C_0^1(B)$. Thus, (1.6) holds, and Proposition 1.3 is applicable. We have the following:

**Corollary 2.2.** *Let $f$ be given by (2.1), and let $\varepsilon > 0$ be small. Then (1.1) has the singular solution (2.2) and*

$$\textit{the bifurcation diagram is of} \begin{cases} \textit{Type I and } m(u^*) = \infty \textit{ if } p_S < p < p_{JL}, \\ \textit{Type II and } m(u^*) = 0 \textit{ if } p \geq p_{JL}. \end{cases}$$

*In particular, the Type III bifurcation diagram does not appear.*

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