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Partial regularity of $p(x)$-harmonic maps

Maria Alessandra Ragusa Atsushi Tachikawa *

1 Introduction

This note is concerned with the partial regularity of local minimizers of functionals which satisfies the so-called $p(x)$-*growth* condition.

Let $\Omega \subset \mathbb{R}^m$ $(m \geq 2)$ be a bounded open set, and $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn} \to \mathbb{R}$ a Carathéodory function satisfying

$$\lambda|\xi|^p \leq f(x,u,\xi) \leq \Lambda\big(1+|\xi|^q\big) \quad \text{for all } (x,u,\xi) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn}, \tag{1.1}$$

for some constants $\Lambda \geq \lambda > 0$, $q \geq p > 1$. For $u : \Omega \to \mathbb{R}^n$, we consider the functional defined by

$$\mathcal{F}(u;\Omega) = \int_\Omega f(x,u,Du)dx. \tag{1.2}$$

The functional $\mathcal{F}$ is said to be of *standard growth* if $q = p$. When $q > p$, it is said to be of *non-standard growth* or, more precisely, of (p,q)-*growth*.

As a particular case of non-standard growth, we consider the following $p(x)$-*growth* condition.

$$\lambda|\xi|^{p(x)} \leq f(x,u,\xi) \leq \Lambda\big(1+|\xi|^{p(x)}\big), \quad \text{for all } (x,u,\xi) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn}, \tag{1.3}$$

where $p(x)$ is a function defined on Ω. For $p(x)$ we assume always that $p(x) > 1$. In this note, by a technical reason, we treat only the case that $p(x) \geq 2$.

In recent years, functionals and problems with $p(x)$-growth became of increasing interest. They appear in some problems of mathematical physics. For example, Zhikov [26] treated thermistor problems using functionals with $p(x)$-growth, Rajagopal and Růžička (see also [22]) proposed some models of of electrorheological fluid using equations with $p(x)$-growth term, and Acerbi and Mingione [3] treated stationary electrorheological fluid and obtained some regularity results.

In this note, we treat regularity problem for vector valued $(n \geq 2)$ minimizers of functional with $p(x)$-growth.

For the scalar valued case (n=1), see [17, 4, 8, 9, 10, 11] and the references therein.

About constant p-growth functionals defined for $u : \Omega \subset \mathbb{R}^m \to \mathbb{R}^n$ with general $m, n \geq 2$, roughly speaking, known regularity results differ from each

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other by the structures of functionals. Let us consider the following 3 types of functionals.

$$\mathcal{F}_1(u) = \int_\Omega a(|Du|)dx, \tag{1.4}$$

$$\mathcal{F}_2(u) = \int_\Omega a\big(x, u, g^{\alpha\beta}(x,u)h_{ij}(x,u)D_\alpha u^i D_\beta u^j\big)dx, \tag{1.5}$$

$$\mathcal{F}_3(u) = \int_\Omega A(x, u, Du)dx, \tag{1.6}$$

where $(g^{\alpha\beta})$, (h_{ij}) and the Hessian matrix of $A(x,u,\xi)$ with respect to ξ are uniformly positive definite, and $da(x,u,t)/dt \geq 0$. Moreover, we assume the following growth conditions on $a : [0,\infty) \to [0,\infty)$ and $A : \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn} \to [0,\infty)$.

$$\begin{aligned} \lambda t^p \leq a(x,u,t) &\leq \Lambda(1+t^p), \quad \text{for all } \ t \in [0,\infty), \\ \lambda|\xi|^p \leq A(x,u,\xi) &\leq \Lambda(1+|\xi|^p), \quad \text{for all } \ (x,u,\xi) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn}, \end{aligned}$$

where Λ, λ are positive constants, and p is a constant or continuous function on Ω with $p \geq 2$. We have the following results for minimizers of the above types of functionals.

(I) : (Uhlenbeck [25]) Let u be a minimizer of $\mathcal{F}_1$, then $u \in C^{1,\alpha}(\Omega)$.

(II) : (Giaquinta-Modica [15], Fusco-Hutchinson [12] ($a(x,u,t) = t^{p/2}$)) Let u be a minimizer of $\mathcal{F}_2$, then $u \in C^{1,\alpha}(\Omega_0)$, where Ω_0 is an open subset of Ω with $\mathcal{H}^{m-p-\varepsilon}(\Omega \setminus \Omega_0) = 0$ for some $\varepsilon > 0$. Here, $\mathcal{H}^q$ denotes the q-dimensional Hausdorff measure.

(III) : (Giaquinta-Giusti [14]($p = 2$), Fusco-Hutchinson [12] ($p \geq 2$)) Let u be a bounded minimizer of $\mathcal{F}_2$ with $a(x,u,t) = t^{p/2}$ and $g^{\alpha\beta}(x,u) = g^{\alpha\beta}(x)$. Then $u \in C^{1,\alpha}(\Omega_0)$ with $\mathcal{H}^{m-[p]-1}(\Omega \setminus \Omega_0) = 0$. Here, $[p]$ stands for the integer part of p.

(IV) : (Giaquinta-Giusti [13]) Let u be a bounded minimizer of $\mathcal{F}_3$, then $u \in C^{1,\alpha}(\Omega_0)$, where Ω_0 is an open subset of Ω with $|\Omega \setminus \Omega_0| = 0$. Here, for a measurable set $D \subset \mathbb{R}^m$, $|D|$ denotes the Lebesgue measure of D.

On the other hand, for $p(x)$-growth cases, the results of Coscia-Mingione [5] and of Acerbi-Mingione [2] correspond to the above results (I) and (IV) respectively. In this note we present the regularity results of [21] that correspond to a part of (II).

Remark 1.1. *For the sake of simplicity, we are restricting ourselves to consider only the case that* $p \geq 2$. *There are also regularity results for* $1 < p$ *(constant)* ≤ 2 *(eg. [1]). Moreover, the results in [5] and [2] are valid for* $p(x) > 1$.

2 Some definitions

In the following we write

$$B(x,R) := \{y \in \mathbb{R}^m \ ; \ |x-y| \leq R\}$$

For $f \in L^1(\Omega)$ we set the integral mean $f_{x,R}$ by

$$f_{x,R} = \fint_{\Omega\cap B(x,R)} f(y)dy = \frac{1}{|\Omega \cap B(x,R)|} \int\limits_{\Omega\cap B(x,R)} f(y)dy$$

where $|\Omega \cap B(x,R)|$ is the Lebesgue measure of $\Omega \cap B(x,R)$.

If we are not interested in specifying which the center is, we only set f_R.

Definition 2.1. *For a bounded open set $\Omega \subset \mathbb{R}^m$ and a function $p : \Omega \to [1, +\infty)$, we define $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ as follows:*

$$L^{p(x)} := \{u \in L^1(\Omega) \; ; \; \int_\Omega |u|^{p(x)} dx < +\infty\}.$$
$$W^{1,p(x)} := \{u \in L^{p(x)} \cap W^{1,1}(\Omega) \; ; \; Du \in L^{p(x)}(\Omega)\}.$$

We also define $L^{p(x)}_{\mathrm{loc}}(\Omega)$ and $W^{1,p(x)}_{\mathrm{loc}}(\Omega)$ similarly.

As mentioned in [6], if $p(x)$ is uniformly continuous and $\partial\Omega$ satisfies uniform cone property, then

$$W^{1,p(x)}(\Omega) = \{u \in W^{1,1}(\Omega) \; ; Du \in L^{p(x)}(\Omega)\}.$$

In any case, if $p(x)$ is continuous in Ω we have

$$W^{1,p(x)}_{\mathrm{loc}}(\Omega) = \{u \in L^1_{\mathrm{loc}}(\Omega) \, ; |Du|^{p(x)} \in W^{1,1}_{\mathrm{loc}}(\Omega)\}.$$

Definition 2.2. *We also define*

$$W^{1,p(x)}_0(\Omega) := \{u \in W^{1,1}_0(\Omega) \; ; \; \int_\Omega |Du|^{p(x)} dx < \infty\},$$

and for a given map φ

$$\varphi + W^{1,p(x)}_0(\Omega) := \{u \in W^{1,p(x)}(\Omega) \; ; \; u - \varphi \in W^{1,p(x)}_0(\Omega)\}.$$

A map $u \in W^{1,p(x)}_{\mathrm{loc}}(\Omega)$ is called to be a *local minimizer* of $\mathcal{F}$ if it satisfies

$$\mathcal{F}(u; \mathrm{supp}\varphi) \le \mathcal{F}(u + \varphi; \mathrm{supp}\varphi),$$

for any $\varphi \in W^{1,p(x)}_0(\Omega)$ with compact support in Ω.

It should be mentioned that in [21] the continuity of the coefficients $g^{\alpha\beta}$ is not assumed to get continuity of a minimizer. Under the condition that $g^{\alpha\beta}$ is in the class so-called VMO, the partial $C^{0,\alpha}$-regularity of a minimizer u is shown. (About regularity results for standard growth problems with VMO-coefficients, see, for example, [7, 19, 18, 20].)

VMO is given as a particular subclass of BMO. Let us now give the definition of BMO and VMO. The function space BMO (bounded mean oscillation) has been first appeared in the article by John and Nirenberg [16].

Definition 2.3. *Let $f \in L^1_{\mathrm{loc}}(\Omega)$. We say that f belongs to $BMO(\Omega)$ if*

$$\|f\|_* \equiv \sup_{B(x,R)} \frac{1}{\Omega \cap |B(x,R)|} \int\limits_{\Omega\cap B(x,R)} |f(y) - f_{x,R}| dy < \infty.$$

VMO (vanishing mean oscillation) is given at first by Sarason in [23].

Definition 2.4. *Let $f \in BMO(\Omega)$ and put*

$$\eta(f, R) := \sup_{\rho \leq R} \sup_{B(x,\rho)} \frac{1}{|\Omega \cap B(x,\rho)|} \int_{\Omega \cap B(x,\rho)} |f(y) - f_\rho| dy$$

where $B(x, \rho)$ ranges over the class of the balls of $\mathbb{R}^m$ of radius ρ. We say that $f \in VMO(\Omega)$ if

$$\lim_{R \to 0} \eta(f, R) = 0.$$

Let us mention that $C^0 \subsetneq VMO$. For vector valued case, in general, we can not expect to get regularity of weak solutions for elliptic systems with discontinuous coefficients. So, it should be interesting to consider the regularity problems for systems with VMO-coefficients.

3 Partial regularity results

In [21], we obtained partial regularity for minimizers of the $p(x)$*-energy functional* defined as

$$\mathcal{E}(u; \Omega) := \int_\Omega \left(g^{\alpha\beta}(x) h_{ij}(u) D_\alpha u^i D_\beta u^j\right)^{p(x)/2} dx. \tag{3.1}$$

On the above functional we consider the following conditions.

(H-1) There exist constants $\lambda_0, \Lambda_0, \lambda_1, \Lambda_1$ $(0 < \lambda_i < \Lambda_i,\ i = 0, 1)$ such that

$$\lambda_0 |\zeta|^2 \leq g^{\alpha\beta}(x) \zeta_\alpha \zeta_\beta \leq \Lambda_0 |\zeta|^2, \quad \lambda_1 |\eta|^2 \leq h_{ij}(u) \eta^i \eta^j \leq \Lambda_1 |\eta|^2$$

for all $x \in \Omega$, $u, \zeta \in \mathbb{R}^m$ and $\eta \in \mathbb{R}^n$.

(H-2) For every $u, v \in \mathbb{R}^n$

$$|h_{ij}(u) - h_{ij}(v)| \leq \omega_0(|u - v|^2),$$

where ω_h is some monotone increasing concave function with $\omega_h(0) = 0$.

(H-3) $g^{\alpha\beta}$ are in the class $L^\infty \cap VMO(\Omega)$.

In the sequel, we put

$$\eta(g, R) := \max_{1 \leq \alpha, \beta \leq m} \eta(g^{\alpha\beta}, R).$$

Moreover, we assume the following conditions on the exponent $p(x)$.

(H-4) The exponent $p(x)$ is bounded and satisfies $p(x) \geq 2$. In the following we put

$$p_1 := \inf_\Omega p(x)\ (\geq 2), \quad p_2 := \sup_\Omega p(x). \tag{3.2}$$

(H-5) For some constants $L > 0$ and $\sigma \in (0, 1)$

$$|p(x) - p(y)| \leq L|x - y|^\sigma := \omega_p(|x - y|) \quad \text{for all } x, y \in \Omega \tag{3.3}$$

In the following we use the following notation:

$$\rho_1(y,r) = \inf_{x\in B(y,r)} p(x), \quad \rho_2(y,r) = \sup_{B(y,r)} p(x) \quad \text{for } B(y,r) \subset \Omega. \tag{3.4}$$

Moreover, when there is no doubt of confusion, we omit the center y.

Theorem 3.1 ([21, Theorem 2.4]). *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$. Assume that $g^{\alpha\beta}(x)$ and $h_{ij}(u)$ satisfy the conditions (H-1)–(H-3). Let $u \in W^{1,p(x)}(\Omega)$ be a local minimizer of the $p(x)$-energy functional defined by (3.1), where $p(x) : \Omega \to [2,\infty)$ satisfies (H-4) and (H-5). Then $u \in C^{0,\alpha}(\Omega_0)$ for some $\alpha \in (0,1)$, where Ω_0 is an open subset of Ω with $\mathcal{H}^{m-p_1}(\Omega \setminus \Omega_0) = 0$.*

Moreover, if $g^{\alpha\beta}(x)$ and $h_{ij}(u)$ are Hölder continuous, $u \in C^{1,\alpha'}(\Omega_0)$ for some $\alpha' \in (0,1)$

For the purpose of getting regularity results for minimizers, we frequently use the following theorem.

Theorem [Morrey's theorem on the growth of the Dirichlet integral] *Let u be in $W^{1,q}(\Omega)$ and suppose that*

$$\rho^{-m+q}\int_{B(x,\rho)} |Du|^q dy \le C\rho^{q\alpha} \quad \text{for all} \quad \rho < \operatorname{dist}(x,\Omega).$$

Then $u \in C^{0,\alpha}_{\text{loc}}(\Omega)$.

For p-growth, we estimate

$$r^{-m+p}\int_{B(x,r)} |Du|^p dy,$$

to Hölder continuity of the minimizers. For $p(x)$-growth the question "What quantity shall we employ?" arises. In earlier literatures on $p(x)$-growth problems, [5, 3] etc., taking $R > 0$ sufficiently small, the authors used the quantity

$$r^{-m+\rho_2(x,R)}\int_{B(x,r)} |Du|^{\rho_2(x,R)} dy, \quad (B(x,r) \subset B(x_0,R)).$$

On the other hand, in [21], we employed another quantity,

$$r^{-m+\rho_2(x,r)}\int_{B(x,r)} |Du|^{\rho_2(x,r)} dy, \tag{3.5}$$

which enables us to use the iteration argument.

A brief sketch of the proof of Theorem 3.1.
First, we mention that, as standard p-growth problems, we have higher integrability result for $p(x)$-growth case also: there exists a positive constant δ such that $u \in W^{1,(1+2\delta)p(x)}$.

For a fixed x_0, choose $R_1 > 0$ sufficiently small so that

$$(1+2\delta)\rho_1(x_0,R_1) \ge (1+\delta)\rho_2(x_0,R_1).$$

Then, we have

$$u \in W^{1,(1+2\delta)p(x)}(B(x_0, R_1)) \subset W^{1,(1+\delta)p_2(x_0,R_1)}(B(x_0, R_1)).$$

In order to show the assertion, we employ so-called "direct approach". Namely, we consider *frozen functionals* which are sufficiently near to the original "wild" functional and sufficiently "tame" to get the regularity, and compare the minimizer of the original functional with those of frozen functionals.

For $x_1 \in B(x_0, R_1)$, choosing $R > 0$ so that $B_{2R} := B(x_1, 2R) \subset B(x_0, R_1)$, we define two *"frozen functionals"* as

$$f_1(\xi) := \left(g_R^{\alpha\beta} h_{ij}(u_R)\xi_\alpha^i \xi_\beta^j\right)^{p(x)/2}, \qquad f_2(\xi) := \left(g_R^{\alpha\beta} h_{ij}(u_R)\xi_\alpha^i \xi_\beta^j\right)^{p_2(2R)/2},$$

$$\mathcal{E}_1(v) := \int_{B_R} f_1(Dv)dx, \qquad \mathcal{E}_2(v) := \int_{B_R} f_2(Dv)dx.$$

Let v be a minimizer of $\mathcal{E}_1$ in the class $u + W_0^{1,p(x)}(B_R)$. By virtue of the regularity result by Cosia-Mingione [5], we see that for every $\beta \in (0,1)$ there exists a positive constant c such that

$$\int_{B_s} |Dv|^{p_2(2R)}dx \le c\left(\frac{s}{R}\right)^{m-\beta}\left[\int_{B_R} |Dv|^{p_2(2R)}dx + R^{m-\beta}\right] \tag{3.6}$$

holds for any $s \in [0, R)$.

In order to get a similar type of decay estimate for Du, we estimate $|Du - Dv|$. By Taylor's theorem, we have

$$\begin{aligned} f_2(Du) = f_2(Dv) &+ \frac{\partial f_2}{\partial \xi_\alpha^i}(Dv)(D_\alpha u^i - D_\beta v^i) \\ &+ \int_0^1 (1-s)\frac{\partial^2 f_2}{\partial \xi_\alpha^i \partial \xi_\beta^j}(Du + s(Dv - Du)) \\ &\quad \cdot (D_\alpha u^i - D_\alpha v^i)(D_\beta u^j - D_\beta v^j)ds. \end{aligned}$$

Here, we should mention that v is a minimizer of $\mathcal{E}_1$, not of $\mathcal{E}_2$, so the second term of the right-hand side of the above equality does not vanish. However, estimating the difference between the Euler-Lagrange equations of $\mathcal{E}_1$ and $\mathcal{E}_2$, we can obtain

$$\begin{aligned} &\int_{B_R} |Du - Dv|^{p_2(2R)}dx \\ \le& c(\mathcal{E}_2(u) - \mathcal{E}_2(v)) + C(\varepsilon)R^\sigma \int_{B_R} (1 + |Dv|^2)^{(1+\varepsilon)p_2(2R)/2}dx. \end{aligned} \tag{3.7}$$

Here, we also used the the fact that for any $\varepsilon > 0$ there exists a positive constant $C(\varepsilon)$ such that for all $t > 0$ and $s \ge r > 0$

$$|t^r - t^s| \le C(\varepsilon)(s - r)\left(1 + t^{(1+\varepsilon)s}\right). \tag{3.8}$$

Adding to and subtracting from (3.7) the terms $\mathcal{E}_1(u), \mathcal{E}(v), \mathcal{E}(v)$ and $\mathcal{E}_1(v)$, and using the minimality of $\mathcal{E}(u)$, we get

$$\begin{aligned} &\int_{B_R} |Du - Dv|^{p_2(2R)}dx \\ \le\ & c\big(\mathcal{E}_2(u) - \mathcal{E}_1(u) + \mathcal{E}_1(u) - \mathcal{E}(u) + \mathcal{E}(v) - \mathcal{E}_1(v) \\ &+ \mathcal{E}_1(v) - \mathcal{E}_2(v)\big) + cR^\sigma \int_{B_R} (1 + |Dv|^2)^{(1+\varepsilon)p_2/(2R)2}dx \end{aligned}$$

By estimating $|\mathcal{E}_2(\cdot)-\mathcal{E}_1(\cdot)|$ and $|\mathcal{E}_1(\cdot)-\mathcal{E}(\cdot)|$, we see that, for some $\delta \in (0, \sigma/m)$ and $q > 1$,

$$\int_{B_R} |Du - Dv|^{\rho_2(2R)} dx \\ \leq c\Big[R^{\sigma - m\delta} + \omega_0^{1/q}\big(c_1 R^{2-m} \int_{B_{2R}} |Dv|^2 dx\big)\Big] \int_{B_{2R}} (1 + |Du|^2)^{\rho_2(2R)/2} dx.$$

(See [21, pp.16–19].) We can estimate the quantity in $\omega_0(\cdot)$ as follows. First, we see that

$$R^{2-m} \int_{B_R} |Dv|^2 dx \\ \leq c\big(R^{\rho_2(2R)-m} \int_{B_R} |Dv|^{\rho_2(2R)} dx\big)^{2/\rho_2(2R)} \\ \leq c\big(R^{\rho_2(2R)-m} \int_{B_R} (1 + |Dv|^2)^{(1+\omega_p(2R))p(x)/2} dx\big)^{2/\rho_2(2R)} \\ \leq c\big(R^{\rho_2(2R)-m} \int_{B_R} (1 + |Du|^2)^{(1+\omega_p(2R))p(x)/2} dx\big)^{2/\rho_2(2R)} \\ \leq c\Big(R^{\rho_2(2R)-m-\omega_p(2R)m}\Big\{\int_{B_{2R}} (1 + |Du|^2)^{p(x)/2} dx\Big\}^{1+\omega_p(2R)}\Big)^{2/\rho_2(2R)}.$$

For the last inequality we used so-called *reverse Hölder inequality with increasing support* which is valid for the minimizers of certain $p(x)$-growth functionals (see [21, Lemma 3.2]). Since u is a local minimizer, we can assume that

$$\int_{B_{2R}} (1 + |Du|^2)^{p(x)/2} dx$$

is bounded. Moreover, by an assumption on ω_p, we see that there exists a positive constant M such that

$$R^{-\omega_p(2R)} = R^{-CR^\sigma} < M.$$

So, we have

$$R^{2-m} \int_{B_R} |Dv|^2 dx \\ \leq c\Big(R^{\rho_2(2R)-m} \int_{B_{2R}} (1 + |Du|^2)^{p(x)/2} dx\Big)^{2/\rho_2(2R)} \\ \leq c\Big(R^{\rho_2(2R)-m} \int_{B_{2R}} (1 + |Du|^2)^{\rho_2(2R)/2} dx\Big)^{2/\rho_2(2R)}, \qquad (3.9)$$

for some positive constant c.

Now, combining (3.6), (3.7) and (3.9), and putting $r = 2R$, we obtain

$$\begin{aligned}
&\int_{B_s} (1+|Du|^2)^{\rho_2(r)/2} dx \\
\le\ &\int_{B_s} (1+|Dv|^2)^{\rho_2(r)/2} dx + \int_{B_s} |Du - Dv|^{\rho_2(r)} dx \\
\le\ &c_0 \Big[\Big(\frac{s}{r}\Big)^{m-\beta} + \omega_0^{1/q} \big(c_1' \big(r^{\rho_2(r)-m} \int_{B_r} (1+|Du|^2)^{\rho_2(r)/2} dx \big)^{2/\rho_2(r)} \big) \\
&+ r^{\sigma - m\delta} \Big] \cdot \int_{B_r} (1+|Du|^2)^{\rho_2(r)/2} dx + c_2 s^{m-\beta}. \qquad (3.10)
\end{aligned}$$

Now, we are in the position to use the iteration argument to get a decay estimate for the quantity defined by (3.5). Let us put

$$\Psi(r) := r \Big(⨍_{B_r} (1+|Du|^2)^{\rho_2(r)/2} dx \Big)^{1/\rho_2(r)}.$$

Then, putting $s = \tau r \quad (\tau \in (0,1))$ in (3.10), we get

$$\begin{aligned}
\Psi(\tau r) &\le c(m, p_1, p_2) \Big(r^{\rho_2(r)-m} \int_{B_{\tau r}} (1+|Du|^2)^{\rho_2(r)/2} dx \Big)^{1/\rho_2(r)} \\
&\le c_3 \tau^\gamma \big[1 + \tau^{(\beta-m)/p_1} \{ r^{(\sigma - m\delta)/p_2} + \tilde{\omega}_0 (c_4 \Psi(r)) \} \big] \Psi(r) + c_5 (\tau r)^\alpha,
\end{aligned}$$

where $\tilde{\omega}_0 := \omega_0^{1/(qp_2)}$, $p_1 := \inf p(x) \le \rho_2(r) \le \sup p(x) =: p_2$, $\gamma := 1 - (\beta/\rho_2(r))$ and $\alpha \in (0, \gamma)$. Fix $\nu \in (\alpha, \gamma)$ and take $\tau \in (0,1)$ so that $c_3 \tau^\gamma \le \tau^\nu / 5$. Choose ε_0 and $r_0 > 0$ so that

$$\tau^{(\beta-m)/p_1} r_0^{(\sigma - m\delta)/p_2} < 1, \quad \tau^{(\beta-m)/p_1} \tilde{\omega}_0 (c_4 \varepsilon_0) < 1, \quad c_5 r_0^\alpha < \frac{\varepsilon_0}{5}.$$

If $\Psi(r) < \varepsilon_0$ for some $r \in (0, r_0)$, we have

$$\Psi(\tau r) \le \frac{3}{5} \tau^\nu \Psi(r) + c_5 r^\alpha < \varepsilon_0.$$

Thus, by an iteration argument, we obtain

$$\begin{aligned}
\Psi(\tau^{k+1} r) &\le (\tau^{k+1})^\nu \Psi(r) + c_5 r^\alpha \tau^{k\alpha} \sum_{j=0}^{k} \tau^{j(\nu - \alpha)} \\
&\le (\tau^{k+1})^\nu \Psi(r) + c_6 (\tau^k r)^\alpha.
\end{aligned}$$

So, we get the following estimate which imply the Hölder continuity.

$$\Psi(s) < C_7 s^\alpha$$

Now, let

$$\Omega_0 := \Big\{ y \in \Omega \,;\, \Big(r^{\rho_2(y,r)-m} \int_{B(y,r)} (1+|Du|^2)^{\rho_2/2} dx \Big)^{1/\rho_2} \le \varepsilon_0$$
$$\text{for some } r < r_0 \text{ with } B(y,r) \in \Omega \Big\}.$$

Then, for every $x_1 \in \Omega_0$, we have

$$\begin{aligned} & s^{-m+p_1-\alpha p_1} \int_{B(x_1,s)} |Du|^{p_1} dx \\ \leq & \left[s^{-\alpha} (s^{p_1-m} \int_{B_s} (1+|Du|^2)^{p_1/2} dx)^{1/p_1} \right]^{p_1} \\ \leq & \left(s^{-\alpha} \Psi(s) \right)^{p_1} \leq C_7^{p_1}. \end{aligned}$$

So, we conclude that $u \in C^{0,\alpha}(\Omega_0)$.

By a standard argument on the Hausdorff measure, we can see that

$$\mathcal{H}^{m-p_1}(\Omega \setminus \Omega_0) = 0.$$

Once we have shown the $C^{0,\alpha}$-regularity on Ω_0, we can show the $C^{1,\beta}$-regularity on Ω_0 by standard arguments, estimating the quantity

$$\int_{B_\rho} |Du - (Du)_\rho|^{\rho_2(2R)} dx,$$

for $\rho < R$. □

It seems that many regularity results for minimizers or weak solutions to the standard p-growth problems can be generalized to for those of $p(x)$-growth problems. In fact, in preprint [24], it is shown that, when $p(x) \in C^{0,1}$, $g^{\alpha\beta}(x) \in C^{0,\tau}$ and $h_{ij}(u) \in C^{0,\tau'}$ $(0 < \tau, \tau' \leq 1)$, we can improve the estimate on the Haudorff-dimension of the singular set for bounded minimizers as in [14, 12].

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Maria Alessandra Ragusa
Dipartimento di Matematica e Informatica,
Universitá di Catania,
Viale Andrea Doria, 6-95128 Catania, Italy,
e-mail:maragusa@dmi.unict.it

Atsushi Tachikawa
Department of Mathematics, Faculty of Science and Technology,
Tokyo University of Science,
Noda, Chiba, 278-8510, Japan,
e-mail:tachikawa_atsushi@ma.noda.tus.ac.jp