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Appendix: On a class of generalized Sturm-Liouville operators

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1 Introduction

In [1], Dumitriu and Edelman constructed a random real symmetric tri-diagonal matrix, and proved that its spectrum is exactly the finite point process called the beta-ensemble. Later, Ramírez, Rider and Virág [7] proved that as the size of this random matrix tends to infinity, its spectrum, if suitably scaled, converges in distribution to the spectrum of the random "differential operator "(the stochastic Airy operator)

$$\mathcal{H}=-\frac{d^{2}}{dt^{2}}+t+\frac{2}{\sqrt{\beta}}B_{\omega}'(t)\ ,\ t\geq 0,$$

considered under the Dirichlet boundary condition at $t=0$. Here $B_{\omega}'(t)$ is the formal derivative of the standard Brownian motion namely the white noise. In [7], the stochastic Airy operator $\mathcal{H}$ is interpreted as a random linear mapping sending absolutely continuous functions into the space of Schwartz distributions. The present author showed that the operator $\mathcal{H}$ can be realized as a generalized Sturm-Liouville operator, which is symmetric in the Hilbert space $L^{2}(0,\infty)$, and is self-adjoint with probability one [5].

Now a real tri-diagonal operators is regarded as second order difference operator, which in turn is a special case of generalized Sturm-Liouville operators (see e.g. [2], [3] or [6]). Hence, if we could set up a class of generalized Sturm-Liouville operators which is vast enough to include both tri-diagonal matrices and the stochastic Airy operator, in such a way that a suitable convergence theorem for spectral measures holds in that class, then we would obtain a natural interpretation of the "continuum limit theorem"due to Ramírez, Rider and Virág. This note is an intermediate report of a still ongoing work toward this goal.

2 A class of generalized Sturm-Liouville operatrors

Let the following objects are given:

(i) $m = m(x)$ is a non-decreasing, right-continuous function on $[0, \infty]$ with values in $[0, \infty]$, such that $m(0) = 0$ and $m(\infty) = \infty$. We call $l(m) := \sup\{x;\ m(x) < \infty\}$ the endpoint of m. (Such an m is called a 'string'by M.G. Krein.)

(ii) $Q = Q(x)$ is a real-valued function defined on $[0, l(m))$ with $Q(0) = 0$ which is right-continuous and has limits from the left, and such that $Q(x)$ is constant on every subintervals of the set $[0, l(m)) \setminus \{\mathrm{supp}(dm)\}$. Here dm is the Lebesgue-Stieltjes measure on $[0, l(m))$ corresponding to the function m.

Given a pair (m, Q) as above, we define the function space $\mathcal{C}(m, Q)$ and the generalized Sturm-Liouville operator $L_{m,Q}$ in the following manner:

Definition 1. A function $u = u(t)$, defined on $[0, l(m))$ belongs to $\mathcal{C}(m, Q)$ if and only if it is absolutely continuous and differentiable from the right, and if there exists a function $v \in L^1_{loc}([0, l(m)); dm)$ such that the equation

$$u^+(t) = u^+(0) + Q(t)u(t) - \int_0^t Q(y)u^+(y)dy - \int_{(0,t]} v(y)dm(y)$$

holds. Here $u^+(t)$ is the right-derivative of $u(\cdot)$ at t. The function $v(\cdot)$ is uniquely determined from u up to on a set of dm-measur zero. We define the generalized Sturm-Liouville operator $L_{m,Q}$, defined on $\mathcal{C}(m, Q)$, by letting

$$v = L_{m,Q}u\ .$$

From the assumption on Q, it is easy to see that every function u belonging to $\mathcal{C}(m, Q)$ has constant slope on every subinterval of $[0, l(m)) \setminus \{\mathrm{supp}(dm)\}$.

Kotani [2] considered the Sturm-Liouville operator of the type

$$L\varphi = -\frac{d\varphi^+ + \varphi dQ}{dm}\ ,$$

where m is as above, but Q is of bounded variation on every compact interval. In the case of $Q = 0$, the spectral theory of L was thoroughly studied by M.G. Krein. See [3] for a summary of Krein's theory.

3 Examples.

When $m(t) = t$ and $Q(t) = (t^2/2) + (2/\sqrt{\beta})B_\omega(t)$, then $L_{m,Q}$ is the stochastic Airy operator.

On the other extreme, let $m(x)$ be a step function

$$m(x) = \sum_{j=1}^n m_j 1_{[x_j,\infty)}(x) + \infty \cdot 1_{\{\infty\}}(x)\ ,$$

where $m_j > 0$ and $0 < x_1 < \cdots < x_n < l(m) = \infty$. If we consider $L_{m,Q}$ under the boundary conditions $u(0) = 0$ and $u^+(x_n) = 0$, then by Definition 1, we have

$$u^+(x_j) - u^+(x_{j-1}) = \Delta Q(x_j)u(x_j) - m_j v(x_j)\ ,\ j = 1, \ldots, n,$$

where we let $\Delta Q(x) = Q(x) - Q(x-0)$. By setting $x_0 = 0$, $x_{n+1} = \infty$, we get for $j = 1, \ldots, n$,

$$u^+(x_{j-1}) = \frac{u(x_j) - u(x_{j-1})}{x_j - x_{j-1}}\ ;\ u^+(x_j) = \frac{u(x_{j+1}) - u(x_j)}{x_{j+1} - x_j}\ .$$

Hence for $v = L_{m,Q}u$, we have

$$v(x_j) = \frac{1}{m_j}\left[\left\{\Delta Q(x_j) + \frac{1}{x_{j+1} - x_j} + \frac{1}{x_j - x_{j-1}}\right\}u(x_j) - \frac{u(x_{j-1})}{x_j - x_{j-1}} - \frac{u(x_{j+1})}{x_{j+1} - x_j}\right]\ .$$

In this case, $L_{m,Q}$ reduces to the tridiagonal matrix

$$\tilde{H} = \begin{bmatrix} a_1 & b_1 & 0 & \cdots & \cdots & \cdots & 0 \\ c_2 & a_2 & b_2 & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & b_{n-2} & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & a_{n-1} & b_{n-1} \\ 0 & \cdots & \cdots & \cdots & \cdots & c_n & a_n \end{bmatrix},$$

where

$$a_j = \frac{1}{m_j}\left\{\Delta Q(x_j) + \frac{1}{x_{j+1} - x_j} + \frac{1}{x_j - x_{j-1}}\right\}\ ;$$

$$b_j = -\frac{1}{m_j}\frac{1}{x_{j+1} - x_j}\ ;\quad c_j = -\frac{1}{m_j}\frac{1}{x_j - x_{j-1}}\ .$$

The matrix $\tilde{H}$ is symmetric with respect to the weight $\{m_j\}$ in the sense that $m_j c_j = m_{j-1} b_{j-1}$ for $j = 2, \ldots, n$.

4 Limit point vs limit circle.

Let $\varphi_\lambda(x)$ and $\psi_\lambda(x)$ be solutions of $L_{m,Q}u = \lambda u$ such that $\varphi_\lambda(0) = \psi_\lambda^+(0) = 1$, $\varphi_\lambda^+(0) = \psi_\lambda(0) = 0$. For each $\lambda \in \mathbf{C}$ and $b \in [0, l(m))$, we consider the linear fractional transformation

$$l_{b,\lambda}(z) = -\frac{\varphi_\lambda(b)z + \varphi_\lambda^+(b)}{\psi_\lambda(b)z + \psi_\lambda^+(b)}\ .$$

If $\mathrm{Im}\lambda \neq 0$, $l_{b,\lambda}(\cdot)$ maps $\mathbf{R}$ into a circle $C_b(\lambda)$ of finite radius

$$r_b(\lambda) := \left[2|\mathrm{Im}\lambda| \int_{(0,b]} |\psi_\lambda(x)|^2 dm(x) \right]^{-1} .$$

We shall say that the endpoint $l(m)$ is of *limit point type* if for some λ with $\mathrm{Im}\lambda \neq 0$, the intersection of circles $\bigcap_{0<b<l(m)} C_b(\lambda)$ shrinks to a singleton, in which case

(i) $\displaystyle\int_{(0,l(m))} |\psi_\lambda(x)|^2 dm(x) = \infty$;

(ii) $L_{m,Q}$ with Dirichlet boundary condition at $t = 0$ defines a self-adjoint operator in $L^2((0,l(m)); dm)$. Hence $l(m)$ being of limit point type does not depend on the choice of $\lambda \in \mathbf{C} \setminus \mathbf{R}$.

(iii) The function $h(\lambda) := -\lim_{b \uparrow l(m)} \varphi_\lambda(b)/\psi_\lambda(b)$ is holomorphic on $\mathbf{C} \setminus \mathbf{R}$, and the integral kernel with respect to dm of $(L_{m,Q} - \lambda)^{-1}$ is given by

$$G_\lambda(x,y) = \psi_\lambda(x \wedge y)\{\varphi_\lambda(x \vee y) + h(\lambda)\psi_\lambda(x \vee y)\} .$$

Otherwise, $l(m)$ is said to be of limit circle type.

5 A continuity theorem.

In order to formulate the "continuum limit"suggested in the introduction, we need to define a suitable topology in the space of (m, Q). The following definition is still provisional.

Definition 2. A sequence (m_n, Q_n) converges to (m_∞, Q_∞) if and only if

(a) $l(m_n) \uparrow l(m_\infty)$;

(b) $m_n(x) \to m_\infty(x)$ for every continuity point x of $m_\infty(\cdot)$;

(c) $Q_n(x) \to Q_\infty(x)$ uniformly on every compact subinterval of $[0, l(m_\infty))$.

Let L_{m_n,Q_n} and L_{m_∞,Q_∞} be the generalized Sturm-Liouville operators obtained from (m_n, Q_n) and (m_∞, Q_∞) respectively, and let $\varphi_{n,\lambda}(x)$, $\psi_{n,\lambda}(x)$, $\varphi_{\infty,\lambda}(x)$ and $\psi_{\infty,\lambda}(x)$ be the solutions of $L_{m_n,Q_n}u = \lambda u$ and $L_{m_\infty,Q_\infty}u = \lambda u$ with the same initial conditions as before. If $(m_n, Q_n) \to (m_\infty, Q_\infty)$ in the sense just described, then for each $\lambda \in \mathbf{C}$, the convergences $\varphi_{n,\lambda}(x) \to \varphi_{\infty,\lambda}(x)$ and $\psi_{n,\lambda}(x) \to \psi_{\infty,\lambda}(x)$ hold uniformly on every compact subintervals of $[0, l(m_\infty))$.

For the time being, we have only the following partial result, which is analogous to Lemma 3 of [4].

Propsition 1. Suppose that $l(m_n) = l(m_\infty) = \infty$ are of limit point type, and that $(m_n, Q_n) \to (m_\infty, Q_\infty)$. Then for any $\lambda \in \mathbf{C}\setminus\mathbf{R}$ and for any $u \in C_0([0,\infty))$, one has

$$\left((L_{m_n,Q_n} - \lambda)^{-1}u, u\right)_{m_n} \to \left((L_{m_\infty,Q_\infty} - \lambda)^{-1}u, u\right)_{m_\infty},$$

where $(\cdot,\cdot)_m$ denotes the inner product in $L^2([0,l(m)); dm)$.

Proof. It suffices to show that for each $\lambda \in \mathbf{C}\setminus\mathbf{R}$, one has $h(\lambda; m_n, Q_n) \to h(\lambda; m_\infty, Q_\infty)$. On the other hand, this assertion is equivalent to saying that for each $\lambda \in \mathbf{C}\setminus\mathbf{R}$, the function $h(\lambda;\cdot)$ is continuous on the compact set

$$K := \{(m_n.Q_n);\ n = 1, 2, \ldots, \infty\} .$$

Now when $l(m) = \infty$ is of limit point type, then $h(\lambda; m, Q)$ is the limit of $-\varphi_\lambda(b)/\psi_\lambda(b)$ as $b \uparrow \infty$, which is, for fixed $\lambda \in \mathbf{C}\setminus\mathbf{R}$ and $b > 0$, a continuous function of $(m.Q)$ on K. Moreover

$$\left|-\frac{\varphi_\lambda(b)}{\psi_\lambda(b)} - h(\lambda; m, Q)\right|$$

is bounded by $r_b(\lambda; m_n, Q_n)$, which is also a continuous function of (m, Q) on K, and tends to 0 monotonically as $b \uparrow \infty$. Hence by Dini's lemma, $h(\lambda; m, Q)$ is a uniform limit of $-\varphi(b)/\psi_\lambda(b)$ on K, and is continuous on K.

The tri-diagonal matrices considered in §3, viewed as generalized Sturm-Liouville operators, are not of limit point type. Hence, unfortunately, Proposition 1 cannot be applied to the question of continuum limit of beta-ensemble mentioned in the introduction.

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