Title	Appendix : On a class of generalized Sturm-Liouville operators (Spectra of Random Operators and Related Topics)
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Citation	数理解析研究所講究録 (2015), 1970: 96-101
Issue Date	2015-11
URL	http://hdl.handle.net/2433/224299
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録
第 1970 巻 2015 年 96-101

Appendix: On a class of generalized Sturm-Liouville operators

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1 Introduction

In [1], Dumitriu and Edelman constructed a random real symmetric tri-diagonal matrix, and proved that its spectrum is exactly the finite point process called the beta-ensemble. Later, Ramírez, Rider and Virág [7] proved that as the size of this random matrix tends to infinity, its spectrum, if suitably scaled, converges in distribution to the spectrum of the random "differential operator " (the stochastic Airy operator)

$$\mathcal{H}=-\frac{d^{2}}{dt^{2}}+t+\frac{2}{\sqrt{\beta}}B_{\omega}'(t)\ ,\ t\geq 0,$$

considered under the Dirichlet boundary condition at $t=0$. Here $B_{\omega}'(t)$ is the formal derivative of the standard Brownian motion namely the white noise. In [7], the stochastic Airy operator $\mathcal{H}$ is interpreted as a random linear mapping sending absolutely continuous functions into the space of Schwartz distributions. The present author showed that the operator $\mathcal{H}$ can be realized as a generalized Sturm-Liouville operator, which is symmetric in the Hilbert space $L^{2}(0,\infty)$, and is self-adjoint with probability one [5].

Now a real tri-diagonal operators is regarded as second order difference operator, which in turn is a special case of generalized Sturm-Liouville operators (see e.g. [2], [3] or [6]). Hence, if we could set up a class of generalized Sturm-Liouville operators which is vast enough to include both tri-diagonal matrices and the stochastic Airy operator, in such a way that a suitable convergence theorem for spectral measures holds in that class, then we would obtain a natural interpretation of the "continuum limit theorem" due to Ramírez, Rider and Virág. This note is an intermediate report of a still ongoing work toward this goal.

2 A class of generalized Sturm-Liouville operatrors

Let the following objects are given:

(i) $m = m(x)$ is a non-decreasing, right-continuous function on $[0, \infty]$ with values in $[0, \infty]$, such that $m(0) = 0$ and $m(\infty) = \infty$. We call $l(m) := \sup\{x;\ m(x) < \infty\}$ the endpoint of m. (Such an m is called a 'string'by M.G. Krein.)

(ii) $Q = Q(x)$ is a real-valued function defined on $[0, l(m))$ with $Q(0) = 0$ which is right-continuous and has limits from the left, and such that $Q(x)$ is constant on every subintervals of the set $[0, l(m)) \setminus \{\mathrm{supp}(dm)\}$. Here dm is the Lebesgue-Stieltjes measure on $[0, l(m))$ corresponding to the function m.

Given a pair (m, Q) as above, we define the function space $\mathcal{C}(m, Q)$ and the generalized Sturm-Liouville operator $L_{m,Q}$ in the following manner:

Definition 1. A function $u = u(t)$, defined on $[0, l(m))$ belongs to $\mathcal{C}(m, Q)$ if and only if it is absolutely continuous and differentiable from the right, and if there exists a function $v \in L^1_{loc}([0, l(m)); dm)$ such that the equation

$$u^+(t) = u^+(0) + Q(t)u(t) - \int_0^t Q(y)u^+(y)dy - \int_{(0,t]} v(y)dm(y)$$

holds. Here $u^+(t)$ is the right-derivative of $u(\cdot)$ at t. The function $v(\cdot)$ is uniquely determined from u up to on a set of dm-measur zero. We define the generalized Sturm-Liouville operator $L_{m,Q}$, defined on $\mathcal{C}(m, Q)$, by letting

$$v = L_{m,Q}u \ .$$

From the assumption on Q, it is easy to see that every function u belonging to $\mathcal{C}(m, Q)$ has constant slope on every subinterval of $[0, l(m)) \setminus \{\mathrm{supp}(dm)\}$.

Kotani [2] considered the Sturm-Liouville operator of the type

$$L\varphi = -\frac{d\varphi^+ + \varphi dQ}{dm} \ ,$$

where m is as above, but Q is of bounded variation on every compact interval. In the case of $Q = 0$, the spectral theory of L was thoroughly studied by M.G. Krein. See [3] for a summary of Krein's theory.

3 Examples.

When $m(t) = t$ and $Q(t) = (t^2/2) + (2/\sqrt{\beta})B_\omega(t)$, then $L_{m,Q}$ is the stochastic Airy operator.

On the other extreme, let $m(x)$ be a step function

$$m(x) = \sum_{j=1}^{n} m_j 1_{[x_j,\infty)}(x) + \infty \cdot 1_{\{\infty\}}(x) \ ,$$

where $m_j > 0$ and $0 < x_1 < \cdots < x_n < l(m) = \infty$. If we consider $L_{m,Q}$ under the boundary conditions $u(0) = 0$ and $u^+(x_n) = 0$, then by Definition 1, we have

$$u^+(x_j) - u^+(x_{j-1}) = \Delta Q(x_j)u(x_j) - m_j v(x_j) \ , \ j = 1, \dots, n,$$

where we let $\Delta Q(x) = Q(x) - Q(x-0)$. By setting $x_0 = 0$, $x_{n+1} = \infty$, we get for $j = 1, \dots, n$,

$$u^+(x_{j-1}) = \frac{u(x_j) - u(x_{j-1})}{x_j - x_{j-1}} \ ; \ u^+(x_j) = \frac{u(x_{j+1}) - u(x_j)}{x_{j+1} - x_j} \ .$$

Hence for $v = L_{m,Q}u$, we have

$$v(x_j) = \frac{1}{m_j}\left[\left\{\Delta Q(x_j) + \frac{1}{x_{j+1} - x_j} + \frac{1}{x_j - x_{j-1}}\right\}u(x_j) - \frac{u(x_{j-1})}{x_j - x_{j-1}} - \frac{u(x_{j+1})}{x_{j+1} - x_j}\right] \ .$$

In this case, $L_{m,Q}$ reduces to the tridiagonal matrix

$$\tilde{H} = \begin{bmatrix} a_1 & b_1 & 0 & \cdots & \cdots & \cdots & 0 \\ c_2 & a_2 & b_2 & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & b_{n-2} & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & a_{n-1} & b_{n-1} \\ 0 & \cdots & \cdots & \cdots & \cdots & c_n & a_n \end{bmatrix} ,$$

where

$$a_j = \frac{1}{m_j}\left\{\Delta Q(x_j) + \frac{1}{x_{j+1} - x_j} + \frac{1}{x_j - x_{j-1}}\right\} ;$$

$$b_j = -\frac{1}{m_j}\frac{1}{x_{j+1} - x_j} \ ; \quad c_j = -\frac{1}{m_j}\frac{1}{x_j - x_{j-1}} \ .$$

The matrix $\tilde{H}$ is symmetric with respect to the weight $\{m_j\}$ in the sense that $m_j c_j = m_{j-1} b_{j-1}$ for $j = 2, \dots, n$.

4 Limit point vs limit circle.

Let $\varphi_\lambda(x)$ and $\psi_\lambda(x)$ be solutions of $L_{m,Q}u = \lambda u$ such that $\varphi_\lambda(0) = \psi_\lambda^+(0) = 1$, $\varphi_\lambda^+(0) = \psi_\lambda(0) = 0$. For each $\lambda \in \mathbf{C}$ and $b \in [0, l(m))$, we consider the linear fractional transformation

$$l_{b,\lambda}(z) = -\frac{\varphi_\lambda(b)z + \varphi_\lambda^+(b)}{\psi_\lambda(b)z + \psi_\lambda^+(b)} \ .$$

If $\mathrm{Im}\lambda \neq 0$, $l_{b,\lambda}(\cdot)$ maps $\mathbf{R}$ into a circle $C_b(\lambda)$ of finite radius

$$r_b(\lambda) := \left[2|\mathrm{Im}\lambda| \int_{(0,b]} |\psi_\lambda(x)|^2 dm(x)\right]^{-1} .$$

We shall say that the endpoint $l(m)$ is of *limit point type* if for some λ with $\mathrm{Im}\lambda \neq 0$, the intersection of circles $\bigcap_{0<b<l(m)} C_b(\lambda)$ shrinks to a singleton, in which case

(i) $\displaystyle\int_{(0,l(m))} |\psi_\lambda(x)|^2 dm(x) = \infty;$

(ii) $L_{m,Q}$ with Dirichlet boundary condition at $t = 0$ defines a self-adjoint operator in $L^2((0,l(m)); dm)$. Hence $l(m)$ being of limit point type does not depend on the choice of $\lambda \in \mathbf{C} \setminus \mathbf{R}$.

(iii) The function $h(\lambda) := -\lim_{b\uparrow l(m)} \varphi_\lambda(b)/\psi_\lambda(b)$ is holomorphic on $\mathbf{C} \setminus \mathbf{R}$, and the integral kernel with respect to dm of $(L_{m,Q} - \lambda)^{-1}$ is given by

$$G_\lambda(x,y) = \psi_\lambda(x \wedge y)\{\varphi_\lambda(x \vee y) + h(\lambda)\psi_\lambda(x \vee y)\} .$$

Otherwise, $l(m)$ is said to be of limit circle type.

5 A continuity theorem.

In order to formulate the "continuum limit"suggested in the introduction, we need to define a suitable topology in the space of (m, Q). The following definition is still provisional.

Definition 2. A sequence (m_n, Q_n) converges to (m_∞, Q_∞) if and only if

(a) $l(m_n) \uparrow l(m_\infty)$;

(b) $m_n(x) \to m_\infty(x)$ for every continuity point x of $m_\infty(\cdot)$;

(c) $Q_n(x) \to Q_\infty(x)$ uniformly on every compact subinterval of $[0, l(m_\infty))$.

Let L_{m_n,Q_n} and L_{m_∞,Q_∞} be the generalized Sturm-Liouville operators obtained from (m_n, Q_n) and (m_∞, Q_∞) respectively, and let $\varphi_{n,\lambda}(x)$, $\psi_{n,\lambda}(x)$, $\varphi_{\infty,\lambda}(x)$ and $\psi_{\infty,\lambda}(x)$ be the solutions of $L_{m_n,Q_n}u = \lambda u$ and $L_{m_\infty,Q_\infty}u = \lambda u$ with the same initial conditions as before. If $(m_n, Q_n) \to (m_\infty, Q_\infty)$ in the sense just described, then for each $\lambda \in \mathbf{C}$, the convergences $\varphi_{n,\lambda}(x) \to \varphi_{\infty,\lambda}(x)$ and $\psi_{n,\lambda}(x) \to \psi_{\infty,\lambda}(x)$ hold uniformly on every compact subintervals of $[0, l(m_\infty))$.

For the time being, we have only the following partial result, which is analogous to Lemma 3 of [4].

Propsition 1. Suppose that $l(m_n) = l(m_\infty) = \infty$ are of limit point type, and that $(m_n, Q_n) \to (m_\infty, Q_\infty)$. Then for any $\lambda \in \mathbf{C} \setminus \mathbf{R}$ and for any $u \in C_0([0, \infty))$, one has

$$\left((L_{m_n,Q_n} - \lambda)^{-1}u, u\right)_{m_n} \to \left((L_{m_\infty,Q_\infty} - \lambda)^{-1}u, u\right)_{m_\infty} ,$$

where $(\cdot, \cdot)_m$ denotes the inner product in $L^2([0, l(m)); dm)$.

Proof. It suffices to show that for each $\lambda \in \mathbf{C} \setminus \mathbf{R}$, one has $h(\lambda; m_n, Q_n) \to h(\lambda; m_\infty, Q_\infty)$. On the other hand, this assertion is equivalent to saying that for each $\lambda \in \mathbf{C} \setminus \mathbf{R}$, the function $h(\lambda; \cdot)$ is continuous on the compact set

$$K := \{(m_n.Q_n);\ n = 1, 2, \ldots, \infty\} .$$

Now when $l(m) = \infty$ is of limit point type, then $h(\lambda; m, Q)$ is the limit of $-\varphi_\lambda(b)/\psi_\lambda(b)$ as $b \uparrow \infty$, which is, for fixed $\lambda \in \mathbf{C} \setminus \mathbf{R}$ and $b > 0$, a continuous function of $(m.Q)$ on K. Moreover

$$\left| -\frac{\varphi_\lambda(b)}{\psi_\lambda(b)} - h(\lambda; m, Q) \right|$$

is bounded by $r_b(\lambda; m_n, Q_n)$, which is also a continuous function of (m, Q) on K, and tends to 0 monotonically as $b \uparrow \infty$. Hence by Dini's lemma, $h(\lambda; m, Q)$ is a uniform limit of $-\varphi(b)/\psi_\lambda(b)$ on K, and is continuous on K.

The tri-diagonal matrices considered in §3, viewed as generalized Sturm-Liouville operators, are not of limit point type. Hence, unfortunately, Proposition 1 cannot be applied to the question of continuum limit of beta-ensemble mentioned in the introduction.

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