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Homogenization and fluctuation for eigenvalues of lattice Anderson Hamiltonians

(Anderson 模型の固有値の homogenization と揺らぎについて)

Ryoki Fukushima
RIMS, Kyoto University
(Based on a joint work with Marek Biskup and Wolfgang König)

1 introduction

In this talk, we discussed a homogenization problem for the random Schrödinger operator. The purpose of this article is to present the results in the simplest setting and to compare an analytic approach by Bal [1] with our probabilistic one [2, 3]. For the background and related works, we refer the reader to [2]. Let us start by introducing the notation to describe the setting.

- $D\subset\mathbb{R}^{d}$: a bounded domain with smooth boundary;
- $D_{\varepsilon}=D\cap\varepsilon\mathbb{Z}^{d}$: a natural discretization with mesh size $\varepsilon>0$;
- $\Delta_{\varepsilon}f(x)=\varepsilon^{-2}\sum_{|y-x|=\varepsilon}(f(y)-f(x))$ for $f:\varepsilon\mathbb{Z}^{d}\to\mathbb{R}$;
- $(\{\xi(x)\}_{x\in D_{\varepsilon}}, \mathbb{P})$: $\mathbb{R}$-valued independent and identically distributed random variables.

We are interested in the operator of the form

$$H_{\varepsilon,\xi}=-\Delta_{\varepsilon}+\xi$$

with the Dirichlet boundary condition imposed outside D_{ε}. Let $\{\lambda_{\varepsilon,\xi}^{(k)}\}_{k\geq 1}$ be the eigenvalues of this operator (matrix) ordered increasingly.

Assumptions 1. *Assume* $\mathbb{E}[\xi(x)]=0$, $\mathbb{E}[\xi(x)^{2}]=1$ *and for some* $K>2\vee\frac{d}{2}$, $\mathbb{E}\left[|\xi(x)|^{K}\right]<\infty$. *We also assume that* ξ *is truncated as* $\max_{x\in D_{\varepsilon}}|\xi(x)|\leq\varepsilon^{-\kappa}$ *for some* $\kappa\in(d/K, 2\wedge d/2)$.

Remark 1. The latter assumption holds with high probability under the first one but we do assume this, see Remark 2 below. In [3], both the mean and variance are allowed to depend on x in a continuous way. Here we consider the simplest i.i.d. case for simplicity.

As ξ varies rapidly in x for small ε, it is natural to expect that some averaging occurs in the limit $\varepsilon\to 0$. Then the limiting object should be the k-th smallest eigenvalue $\lambda_{D}^{(k)}$ of $-\Delta$ on $H_{0}^{2}(D)$ and the following result shows that it is the case.

Theorem 1 (homogenization). *Under Assumption 1,*

$$\lambda_{\varepsilon,\xi}^{(k)}\to\lambda_{D}^{(k)}\quad as\quad\varepsilon\downarrow 0$$

in probability for each $k\geq 1$.

We further found a Gaussian fluctuation of the eigenvalues around the mean. The covariances are described in terms of L^2-normalized eigenfunction $\varphi_D^{(k)}$ corresponding to $\lambda_D^{(k)}$.

Theorem 2 (fluctuation). *Suppose Assumption 1 holds and $\lambda_D^{(k_1)}, \dots, \lambda_D^{(k_n)}$ are distinct simple eigenvalues. Then,*

$$\varepsilon^{-d/2}\big(\lambda_{\varepsilon,\xi}^{(k_1)} - \mathbb{E}\lambda_{\varepsilon,\xi}^{(k_1)}, \dots, \lambda_{\varepsilon,\xi}^{(k_n)} - \mathbb{E}\lambda_{\varepsilon,\xi}^{(k_n)}\big) \xrightarrow{\varepsilon\downarrow 0} \mathscr{N}(0,\sigma)$$

in law, where σ is the covariance matrix with elements

$$\sigma_{ij}^2 := \|\varphi_D^{(k_i)}\varphi_D^{(k_j)}\|_2^2.$$

Remark 2. The truncation made in Assumption 1 does not affect $\lambda_{\varepsilon,\xi}^{(k)}$ with high probability. However, it may affect the mean value $\mathbb{E}\lambda_{\varepsilon,\xi}^{(k)}$.

Note that the fluctuation *not* around the "homogenized eigenvalues" but around their mean is found. The following result due to Bal [1] tells us that the mean $\mathbb{E}\lambda_{\varepsilon,\xi}^{(k)}$ can be replaced by $\lambda_D^{(k)}$ for $d \le 3$ under the existence of fourth moment.

Theorem 3 (fluctuation in low dimensions: Bal [1]). *Let $d \le 3$ and suppose that $\{\xi(x)\}_{x\in D_\varepsilon}$ are independent and identically distributed with $\mathbb{E}[\xi(x)^4] < \infty$. Then*

$$\lambda_{\varepsilon,\xi}^{(k)} \to \lambda_D^{(k)} \quad \text{as} \quad \varepsilon \downarrow 0$$

in probability for each $k \ge 1$. Moreover, if $\lambda_D^{(k_1)}, \dots, \lambda_D^{(k_n)}$ are distinct simple eigenvalues, then

$$\varepsilon^{-d/2}\big(\lambda_{\varepsilon,\xi}^{(k_1)} - \lambda_D^{(k_1)}, \dots, \lambda_{\varepsilon,\xi}^{(k_n)} - \lambda_D^{(k_n)}\big) \xrightarrow{\varepsilon\downarrow 0} \mathscr{N}(0,\sigma)$$

in law, where the covariance σ is the same as above.

Remark 3. Bal [1] established the above central limit theorem not only for i.i.d. case but also for sufficiently mixing case. The above setting is in fact slightly different from the original one which studies an operator without any discretization.

Notation: For a function $f : D_\varepsilon \to \mathbb{R}$, we write $\langle\cdot,\cdot\rangle$ and $\|\cdot\|_2$ for the ℓ^2 inner product and corresponding norm with respect to the counting measure multiplied by ε^d.

2 The argument of Bal in the i.i.d. case

We present the argument of Bal [1] in a simplified i.i.d. setting in this section. It is based on a perturbation expansion and in order to control the reminder terms, we need that the Green's function for $-\Delta$ (with the boundary condition) is in $L^{2+\delta}(D)$ for some $\delta > 0$, which requires $d \le 3$. We shall focus on the first eigenvalue $\lambda_{\varepsilon,\xi}$, with the superscript $^{(1)}$ dropped, and write $\varphi_{\varepsilon,\xi}$ for the first eigenfunction. Let G_ξ and G be the resolvent operators for $H_{\varepsilon,\xi}$ and $-\Delta_\varepsilon$ with the Dirichlet boundary condition outside D_ε, respectively. As G_ξ is well-defined with high probability, we assume it always exists for simplicity.

The key to the proof of Theorem 3 is the following lemma.

Lemma 1.

$$\lim_{M\to\infty}\limsup_{\varepsilon\downarrow 0}\mathbb{P}(\max\{\|G\xi G\|_{2\to 2}, \|G\xi G\xi\|_{2\to 2}, \|G_\xi - G\|_{2\to 2}\} \ge M\varepsilon^{d/2}) = 0. \quad (1)$$

Moreover for any $\delta > 0$,

$$\lim_{\varepsilon\downarrow 0}\mathbb{P}(\|G\xi G\xi G\|_{2\to 2} \ge \delta\varepsilon^{d/2}) = 0. \quad (2)$$

Proof. For any $f : D_\varepsilon \to \mathbb{R}$, we use the Cauchy-Schwarz inequality to obtain

$$\begin{aligned}\|G\xi G\xi f\|_2^2 &= \sum_{x\in D_\varepsilon}\varepsilon^d\left|\sum_{y\in D_\varepsilon}\sum_{z\in D_\varepsilon}\varepsilon^{2d}g(x,y)\xi(y)g(y,z)\xi(z)f(z)\right|^2\\ &\le \|f\|_2^2\sum_{x\in D_\varepsilon}\sum_{z\in D_\varepsilon}\varepsilon^{2d}\left|\sum_{y\in D_\varepsilon}\varepsilon^d g(x,y)\xi(y)g(y,z)\xi(z)\right|^2\\ &= \|f\|_2^2\sum_{x,y_1,y_2,z\in D_\varepsilon}\varepsilon^{4d}g(x,y_1)\xi(y_1)g(y_1,z)g(x,y_2)\xi(y_2)g(y_2,z)\xi(z)^2.\end{aligned}$$

Noting that $\mathbb{E}[\xi(y_1)\xi(y_2)\xi(z)^2] \le \delta_{y_1,y_2}\mathbb{E}[\xi(z)^4]$, we find

$$\mathbb{E}[\|G\xi G\xi\|_{2\to 2}^2] \le \text{const.}\varepsilon^d\sum_{x,y,z\in D_\varepsilon}\varepsilon^{2d}g(x,y)^2 g(y,z)^2.$$

The sum on the right-hand side is bounded for $d \le 3$, due to the square integrability of the continuum Green's function, and thus

$$\lim_{M\to\infty}\mathbb{P}(\|G\xi G\xi\|_{2\to 2} \ge M\varepsilon^{d/2}) = 0$$

follows by the Chebyshev inequality. The estimate for $\|G\xi G\|_{2\to 2}$ is essentially the same and simpler. As for (2), it is routine to find

$$\begin{aligned}\|G\xi G\xi G\|_{2\to 2}^2 \le \sum_{x,y_1,y_2,z_1,z_2,w\in D_\varepsilon}&\varepsilon^{6d}g(x,y_1)\xi(y_1)g(y_1,z_1)\xi(z_1)g(z_1,w)\\ &g(x,y_2)\xi(y_2)g(y_2,z_2)\xi(z_2)g(z_2,w)\end{aligned}$$

as above. Taking expectation and using

$$\mathbb{E}[\xi(y_1)\xi(y_2)\xi(z_1)\xi(z_2)] \le (\delta_{y_1,y_2}\delta_{z_1,z_2} + \delta_{y_1,z_1}\delta_{y_2,z_2} + \delta_{y_1,z_2}\delta_{y_2,z_1})\mathbb{E}[\xi(z)^4],$$

we get

$$\begin{aligned}
&\mathbb{E}[\xi(x)^4]^{-1}\mathbb{E}[\|G\xi G\xi G\|_{2\to2}^2] \\
&\le \sum_{x,y_1,z_1,w\in D_\varepsilon} \varepsilon^{6d} g(x,y_1)^2 g(y_1,z_1)^2 g(z_1,w)^2 \\
&\quad + \sum_{x,y_1,y_2,w\in D_\varepsilon} \varepsilon^{6d} g(x,y_1)g(y_1,y_1)g(y_1,w)g(x,y_2)g(y_2,y_2)g(y_2,w) \qquad (3)\\
&\quad + \sum_{x,y_1,y_2,w\in D_\varepsilon} \varepsilon^{6d} g(x,y_1)g(y_1,y_2)g(y_2,w)g(x,y_2)g(y_2,y_1)g(y_1,w).
\end{aligned}$$

The first term on the right-hand side is $O(\varepsilon^{2d})$ when $d \le 3$. Next, using $2ab \le a^2 + b^2$ one can bound the second term by

$$\begin{aligned}
&\sum_{x,y_1,y_2,w\in D_\varepsilon} \varepsilon^{6d}[g(x,y_1)^2 + g(x,y_2)^2]g(y_1,y_1)g(y_2,y_2)[g(y_1,w)^2 + g(y_2,w)^2] \qquad (4)\\
&\le \text{const.}\varepsilon^{2d} \max_{u\in D_\varepsilon} g(u,u)^2 = o(\varepsilon^d)
\end{aligned}$$

since $\varepsilon^d \max_{u\in D_\varepsilon} g(u,u) = o(\varepsilon^{d/2})$ for $d \le 3$. As the third term can be estimated in the same way, we obtain

$$\mathbb{E}[\|G\xi G\xi G\|_{2\to2}^2] = o(\varepsilon^d)$$

and (2) follows.

In order to bound $\|G_\xi - G\|_{2\to2}$, we recast the equation $(-\Delta_\varepsilon + \xi)G_\xi f = f$ as

$$G_\xi f = G(f - \xi G_\xi f) = Gf - G\xi Gf + G\xi G\xi G_\xi f,$$

where in the second equality, we have used the first equality to rewrite $G_\xi f$ in the middle. We further rearrange the above to get rid of G_ξ from the right-hand side and arrive at

$$(\mathrm{id} - G\xi G\xi)(G_\xi - G)f = -G\xi Gf + G\xi G\xi Gf. \qquad (5)$$

We know from the above argument that $\|G\xi G\|_{2\to2}$, $\|G\xi G\xi\|_{2\to2}$ and $\|G\xi G\xi G\|_{2\to2}$ are of order $O(\varepsilon^{d/2})$ with high probability. For such a ξ, we conclude from (5) that

$$\|(G_\xi - G)f\|_2 \le \text{const.}M\varepsilon^{d/2}\|f\|_2$$

for sufficiently small ε. □

This lemma and a simple bound

$$|\lambda_{\varepsilon,\xi}^{-1} - \lambda_D^{-1}| \le \|G_\xi - G\|_{2\to2}$$

yield $\lambda_{\varepsilon,\xi} - \lambda_D = O(\varepsilon^{d/2})$ with high probability, that is, the first part of Theorem 3 with an error control.

Let us turn to the proof of fluctuation result. Note that the argument so far applies to the second eigenvalue as well. Thus we may assume that both the first and second eigenvalues are close to the homogenized eigenvalues. Then, it is not hard to verify, by using the Rayleigh-Ritz variational formula and the well-known fact $\lambda_D^{(1)} < \lambda_D^{(2)}$, that the first eigenfunctions are close to each other:

$$\|\varphi_{\varepsilon,\xi} - \varphi_D\|_2 \to 0 \quad \text{as } \varepsilon \downarrow 0, \text{ in probability.} \tag{6}$$

The starting point of the argument is the following perturbative representation of the eigenvalue difference:

$$\begin{aligned} \lambda_{\varepsilon,\xi}^{-1} - \lambda_D^{-1} &= \langle \varphi_D, (G_\xi - G)\varphi_D \rangle \\ &\quad + \left\langle \varphi_{\varepsilon,\xi} - \varphi_D, [(G_\xi - \lambda_{\varepsilon,\xi}^{-1}) - (G - \lambda_D^{-1})]\varphi_D \right\rangle . \end{aligned} \tag{7}$$

Thanks to Lemma 1, the second term is bounded (in modulus) by

$$\|\varphi_{\varepsilon,\xi} - \varphi_D\|_2(\|G_\xi - G\|_{2\to 2} + |\lambda_{\varepsilon,\xi}^{-1} - \lambda_D^{-1}|) = o(\varepsilon^{d/2}),$$

which is of smaller order than the expected fluctuation. On the other hand, by using Lemma 1 in (5), we find that the first term is approximated by

$$\langle \varphi_D, -G\xi G\varphi_D \rangle = \sum_{x,y,z\in D_\varepsilon} \varepsilon^{3d} \varphi_D(x) g(x,y)\xi(y) g(y,z)\varphi_D(z)$$

up to an $o(\varepsilon^{d/2})$ error. This right-hand side is nothing but a sum of i.i.d. random variables ξ with the weight

$$\sum_{x,z\in D_\varepsilon} \varepsilon^{2d} \varphi_D(x) g(x,\cdot) g(\cdot,z)\varphi_D(z) = \lambda_D^{-2}\varphi_D(\cdot)^2$$

by the symmetry of g and the eigen-equation. It is well-known that such a sum satisfies the central limit theorem with the variance $\lambda_D^{-4}\|\varphi_D^2\|_2^2$. As we know

$$\lambda_{\varepsilon,\xi}^{-1} - \lambda_D^{-1} \sim -\frac{\lambda_{\varepsilon,\xi} - \lambda_D}{\lambda_D^2} \quad \text{as } \varepsilon \downarrow 0$$

from the first part of Theorem 3, the proof of the second part is completed.

3 The probabilistic argument

In this section, we review key points of the probabilistic methods in [2, 3] which covers the general dimensions with an optimal moment condition. The argument is probabilistic compared with the perturbative one in the previous section in that it heavily uses concentration inequalities and a martingale central limit theorem. However, the reader will notice that it also relies on some analytic inputs in an essential way. We will focus only on the first eigenvalue as before.

3.1 Homogenization

We first give an outline proof of the convergence in probability of the random eigenvalue. Our starting point is the Rayleigh-Ritz formula:

$$\lambda_{\varepsilon,\xi} = \inf\left\{\|\nabla_\varepsilon g\|_2^2 + \langle \xi, g^2\rangle : \|g\|_2 = 1 \text{ and } g = 0 \text{ outside } D_\varepsilon\right\},$$
$$\lambda_D = \inf\left\{\|\nabla \psi\|_2^2 + \langle U, \psi^2\rangle : \psi \in H_0^1(D), \|\psi\|_2 = 1\right\}$$

which are minimized by $\varphi_{\varepsilon,\xi}$ and φ_D. Roughly speaking, we prove

- $\lambda_{\varepsilon,\xi} \lesssim \lambda_D$ by taking $g = \varphi_D$ in the first formula and
- $\lambda_{\varepsilon,\xi} \gtrsim \lambda_D$ by taking $\psi = \varphi_{\varepsilon,\xi}$ in the second formula.

The first step

$$\lambda_{\varepsilon,\xi} \le \|\nabla_\varepsilon \varphi_D\|_2^2 + \langle \xi, \varphi_D^2\rangle \xrightarrow{\varepsilon\downarrow 0} \|\nabla \varphi_D\|_2^2 = \lambda_D$$

is nothing but the weak law of large numbers. On the other hand, the second step

$$\lambda_D \le \underbrace{\|\nabla \varphi_{\varepsilon,\xi}\|_2^2}_{\text{need an interpolation to give a sense}} \overset{?}{\sim} \|\nabla_\varepsilon \varphi_{\varepsilon,\xi}\|_2^2 + \underbrace{\langle \xi, \varphi_{\varepsilon,\xi}^2\rangle}_{\text{randomly weighted sum}}$$

is more problematic. We use the following two tools:

Lemma 2. *There exists a piecewise affine interpolation $\widetilde{\varphi_{\varepsilon,\xi}} \in H_0^1(D)$ of $\varphi_{\varepsilon,\xi}$ such that $\|\nabla_\varepsilon \varphi_{\varepsilon,\xi}\|_2 = \|\nabla \widetilde{\varphi_{\varepsilon,\xi}}\|_2$.*

Lemma 3. *For any $p \in (2\wedge d/2, K)$, ξ is bounded in $\ell^p(D_\varepsilon)$ with high probability. On such a event, $\|\varphi_{\varepsilon,\xi}\|_q$ and $\|\nabla_\varepsilon \varphi_{\varepsilon,\xi}\|_2$ are bounded for any $q > 1$.*

Lemma 2 is a well-known scheme in the "finite element method" in numerics and it solves the problem around the gradient. The first half of Lemma 3 is a consequence of the moment assumption and the second one follows by the so-called Moser iteration in the elliptic regularity theory. This H^1-boundedness combined with the Poincaré inequality allows us to approximate $\varphi_{\varepsilon,\xi}$ by a step function with large ($\gg \varepsilon$) plateaus. Then we can use the weak law of large numbers (with a tail bound) plateau-wise to show that $\lim_{\varepsilon\downarrow 0}\langle \xi, \varphi_{\varepsilon,\xi}^2\rangle = 0$ in probability. This verifies the second step and thus completes the proof.

Remark 4. We need a tail bound in the above argument since we use the union bound to ensure that weak LLN holds for all plateaus simultaneously. The first part of Assumption 1 would not give a sufficiently good bound but the second part enables us to get it by using the Hoeffding inequality.

3.2 Gaussian fluctuation

To show the fluctuation result in Theorem 2, we use a martingale central limit theorem due to Brown [4]. Let $D_\varepsilon = \{x_1, \dots, x_n\}$ ($n = \# D_\varepsilon$) and $\xi_m := \xi(x_m)$ and define the filtration $\mathcal{F}_m = \sigma[\xi(x_1), \dots, \xi(x_m)]$. We decompose the fluctuation around the mean as the sum of martingale differences as

$$\begin{aligned}\lambda_{\varepsilon,\xi} - \mathbb{E}[\lambda_{\varepsilon,\xi}] &= \sum_{m=1}^{n} \mathbb{E}[\lambda_{\varepsilon,\xi} | \mathcal{F}_m] - \mathbb{E}[\lambda_{\varepsilon,\xi} | \mathcal{F}_{m-1}] \\ &=: \sum_{m=1}^{n} Z_m.\end{aligned}$$

Then [4] tells us that the desired convergence in law follows form the two conditions:

(1) $\varepsilon^{-d} \sum_m \mathbb{E}[Z_m^2 | \mathcal{F}_{m-1}] \xrightarrow{\varepsilon \downarrow 0} \int_D \varphi_D(x)^4 \mathrm{d}x$ in probability and

(2) $\varepsilon^{-d} \sum_m \mathbb{E}[Z_m^2 1_{\{|Z_m| > \delta \varepsilon^{d/2}\}} | \mathcal{F}_{m-1}] \xrightarrow{\varepsilon \downarrow 0} 0$ in probability.

The second condition can be checked by using a rather simple L^∞ bound on the eigenfunction and we omit the detail. To check the first condition, we will rewrite the martingale differences. Note first that by independence, the conditional expectation $\mathbb{E}[\cdot | \mathcal{F}_m]$ is just an integration over the variables $(\xi_{m+1}, \dots, \xi_n)$ and hence

$$\begin{aligned}Z_m &= \mathbb{E}[\lambda_{\varepsilon,\xi} | \mathcal{F}_m] - \mathbb{E}[\lambda_{\varepsilon,\xi} | \mathcal{F}_{m-1}] \\ &= \widehat{\mathbb{E}}\left[\lambda_{D_\varepsilon, \xi_{\le m}, \widehat{\xi}_{>m}} - \lambda_{\varepsilon, \xi_{<m}, \widehat{\xi}_{\ge m}}\right],\end{aligned}$$

where $(\widehat{\xi}, \widehat{\mathbb{P}})$ is an independent copy of $(\xi, \mathbb{P})$ and $\xi_{\le m} = (\xi_1, \dots, x_m)$ etc. We can further rewrite the right hand side by using Hadamard's first variation formula $\partial_{\xi_m} \lambda_{\varepsilon,\xi} = \varepsilon^d \varphi_{\varepsilon,\xi}(x_m)^2$ as

$$\widehat{\mathbb{E}}\left[\int_{\widehat{\xi}_m}^{\xi_m} \partial_{\xi_m} \lambda_{\varepsilon, \xi_{<m}, \widetilde{\xi}_m, \widehat{\xi}_{>m}} \mathrm{d}\widetilde{\xi}_m\right] = \widehat{\mathbb{E}}\left[\int_{\widehat{\xi}_m}^{\xi_m} \varepsilon^d \varphi^2_{\varepsilon, \xi_{<m}, \widetilde{\xi}_m, \widehat{\xi}_{>m}}(x_m) \mathrm{d}\widetilde{\xi}_m\right].$$

Now, as in the argument of Bal, we can show that the eigenfunction $\varphi_{\varepsilon,\xi}$ converges in $\|\cdot\|_2$ to φ_D in probability, and then in fact the convergence holds in $\|\cdot\|_q$ for any $q < \infty$ by Lemma 3. Then it is natural to expect that

$$\begin{aligned}\varepsilon^{-d} \sum_{m=1}^{n} \mathbb{E}[Z_m^2 | \mathcal{F}_{m-1}] &= \sum_{m=1}^{n} \varepsilon^d \int \mathbb{P}(\mathrm{d}\xi_m) \widehat{\mathbb{E}}\left[\int_{\widehat{\xi}_m}^{\xi_m} \varphi^2_{\varepsilon, \xi_{<m}, \widetilde{\xi}_m, \widehat{\xi}_{>m}}(x_m) \mathrm{d}\widetilde{\xi}_m\right]^2 \\ &\overset{?}{\sim} \sum_{m=1}^{n} \varepsilon^d \int \mathbb{P}(\mathrm{d}\xi_m) \widehat{\mathbb{E}}\left[\int_{\widehat{\xi}_m}^{\xi_m} \varphi_D^2(x_m) \mathrm{d}\widetilde{\xi}_m\right]^2 \\ &= \sum_{m=1}^{n} \varepsilon^d \varphi_D(x_m)^4 \\ &\sim \int_D \varphi_D(x)^4 \mathrm{d}x.\end{aligned}$$

However, there is a subtle issue arising from the integration with respect to the dummy variable $\widetilde{\xi}_m$. Indeed, if ξ obeys the Bernoulli distribution for example, the configuration $(\xi_{<m}, \widetilde{\xi}_m, \widehat{\xi}_{>m})$ is typically not in the support of the law of ξ and thus the above mentioned convergence of the eigenfunction *in probability* is useless to verify $\overset{?}{\sim}$.

Therefore, an essential part of the proof is to eliminate the dummy variable by showing

$$\varphi^2_{\varepsilon, \xi_{<m}, \widetilde{\xi}_m, \widehat{\xi}_{>m}}(x_m) \sim \varphi^2_{\varepsilon, \xi_{<m}, \xi_m, \widehat{\xi}_{>m}}(x_m), \tag{8}$$

that is, the eigenfunction is not sesitive to the value of ξ at a point. This is a consequence of the following two lemmas:

Lemma 4.

$$\begin{aligned}\partial_m \varphi_{\varepsilon,\xi}(x_m) &= \varphi_{\varepsilon,\xi}(x_m) \langle \delta_{x_m}, P_1^\perp (H_{\varepsilon,\xi} - \lambda_{\varepsilon,\xi})^{-1} P_1^\perp \delta_{x_m} \rangle \\ &= \varepsilon^d \varphi_{\varepsilon,\xi}(x_m) G_\varepsilon(x_m, x_m; \xi),\end{aligned}$$

where $P_1^\perp$ *denoted the orthogonal projection onto* $\operatorname{span}\{\varphi_{\varepsilon,\xi}\})^\perp$ *and*

$$G_\varepsilon(x_m, x_m; \xi) = \sum_{k\geq 2} \frac{1}{\lambda^{(k)}_{\varepsilon,\xi} - \lambda_{\varepsilon,\xi}} \varphi^{(k)}_{\varepsilon,\xi}(x_m)^2.$$

Lemma 5. *With high probability,*

$$\sup_{1\leq m\leq n} \sup_{|\xi_m|\leq \varepsilon^{-\kappa}} G_\varepsilon(x_m, x_m; \xi) \leq c_d \begin{cases} 1, & d = 1, \\ \log \frac{1}{\varepsilon}, & d = 2, \\ \varepsilon^{2-d}, & d \geq 3. \end{cases} \tag{9}$$

(Recall the truncation in Assumption 1.)

Remark 5. In fact, we will need (and do have) an error control in Lemma 5 which decay faster than a certain power of ε. The reader can verify that the probability bound we will prove below is in fact stretched exponential.

The proof of Lemma 4 is very simple and left to the reader. By solving the ODE, we find

$$\begin{aligned}&\varphi^2_{\varepsilon, \xi_{<m}, \widetilde{\xi}_m, \widehat{\xi}_{>m}}(x_m) \\ &\quad = \varphi^2_{\varepsilon, \xi_{<m}, \xi_m, \widehat{\xi}_{>m}}(x_m) \exp\left\{ 2 \int_{\xi_m}^{\widetilde{\xi}_m} G_\varepsilon(x_m, x_m; \xi_{<m}, \xi'_m, \widehat{\xi}_{>m}) d\xi'_m \right\}\end{aligned}$$

and with the help of Lemma 5, one can check that the exponential factor tends to 1 as $\varepsilon \downarrow 0$, which establishes (8). Note that it is the uniform control over ξ_m in (9) that eliminates the dummy variable.

Let us explain how to prove Lemma 5. For any $\lambda > 0$,

$$\begin{aligned}G_\varepsilon(x_m, x_m; \xi) &= \sum_{k\geq 2} \frac{1}{\lambda^{(k)}_{\varepsilon,\xi} - \lambda_{\varepsilon,\xi}} \varphi^{(k)}_{\varepsilon,\xi}(x_m)^2 \\ &\leq \sum_{k\geq 1} \frac{c_\lambda}{\lambda^{(k)}_{\varepsilon,\xi} + \lambda} \varphi^{(k)}_{\varepsilon,\xi}(x_m)^2 \\ &= c_\lambda (H_{\varepsilon,\xi} + \lambda)^{-1}(x_m, x_m).\end{aligned}$$

We are going to compare the right-hand side with $(-\Delta_\varepsilon+\lambda)^{-1}(x_m,x_m)$, which is known to enjoy the bound in (9). From the measure concentration viewpoint, this would in principle follow if we know that $G_\varepsilon(x_m,x_m;\xi)$ is *insensitive* to the noise ξ. (And in this way, the uniformity in ξ_m would follow as a byproduct.) The difficulty is of course that it depends on ξ in a complicated nonlinear way. To cope with this problem, we express it as the Laplace transform of the kernel of semigroup:

$$(H_{\varepsilon,\xi}+\lambda)^{-1}(x_m,x_m)=\int_0^\infty e^{-t(H_{\varepsilon,\xi}+\lambda)}(x_m,x_m)\mathrm{d}t.$$

The point is that the semigroup kernel can be controlled in terms of a certain linear function of ξ_- (the negative part of ξ) as the following lemma shows.

Lemma 6 (Khas'minskii's lemma, taken in this form from [5].). *Suppose that there exists $\tau>0$ such that*

$$\sup_{z\in D_\varepsilon} I_{\tau,z}(\xi):=\sup_{z\in D_\varepsilon}\int_0^\tau e^{s\Delta_\varepsilon}\xi_-(z)\mathrm{d}s<1/8. \tag{10}$$

Then for some $\zeta(\tau)>0$, $e^{-tH_{\varepsilon,\xi}}(x_m,x_m)\le\zeta(\tau)e^{t\zeta(\tau)}e^{t\Delta_\varepsilon}(x_m,x_m)$ holds for all $t>0$.

The above $I_{\tau,z}(\xi)$ is just a weighted sum of ξ_- with mean $\mathbb{E}[I_{\tau,z}(\xi)]=\tau\mathbb{E}[\xi_-(x)]$ and the Lipschitz constant bounded as

$$\begin{aligned}|I_{\tau,z}(\xi)| &\le \int_0^\tau \varepsilon^{d/2}\|e^{s\Delta_\varepsilon}(z,\cdot)\|_2|\xi_-|_2\mathrm{d}s\\ &= |\xi_-|_2\int_0^\tau \varepsilon^{d/2}e^{2s\Delta_\varepsilon}(z,z)^{1/2}\mathrm{d}s\\ &\le c_d|\xi_-|_2\begin{cases}\tau^{1-d/4}\varepsilon^{d/2}, & d\le 3,\\ \varepsilon^2\log(\tau\varepsilon^{-2}), & d=4,\\ \varepsilon^2, & d\ge 5.\end{cases}\end{aligned} \tag{11}$$

Then Talagrand's inequality (Theorem 6.6 in [6]) implies that if $\tau>0$ is fixed sufficiently small, then $\mathbb{P}(I_{\tau,z}>1/8)\le c_1\exp\{-c_2\varepsilon^{-\delta}\}$ with $\delta>0$ for all small $\varepsilon>0$. Now (11) allows us to take $\sup_{|\xi_m|\le\varepsilon^{-\kappa}}$ inside the probability and also, since it is an exponential bound, we may further take $\sup_{1\le m\le n}\sup_{z\in D_\varepsilon}$. In this way, we conclude that for some fixed τ,

$$\mathbb{P}\left(\sup_{1\le m\le n}\sup_{|\xi_m|\le\varepsilon^{-\kappa}}\sup_{z\in D_\varepsilon} I_{\tau,z}>1/8\right)\le c_1\exp\{-c_2\varepsilon^{-\delta}\}.$$

Now if ξ lies in the event on the left-hand side above, then

$$\begin{aligned}(H_{\varepsilon,\xi}+\lambda)^{-1}(x_m,x_m) &\le \zeta(\tau)\int_0^\infty e^{-t(-\Delta_\varepsilon+\lambda-\zeta(\tau))}(x_m,x_m)\mathrm{d}t\\ &= \zeta(\tau)(-\Delta_\varepsilon+\lambda-\zeta(\tau))^{-1}(x_m,x_m)\end{aligned}$$

and taking $\lambda=2\zeta(\tau)$, we obtain (8). The rest of the proof of the first condition of martingale central limit theorem is a bit long and tedious and we omit the detail.

References

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