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Time periodic flows of an incompressible viscous fluid in perturbed channels (Mathematical Analysis of Viscous Incompressible Fluid)

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Time periodic flows of an incompressible viscous fluid in perturbed channels

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1 The time periodic Poiseuille flow

In this section, for a straight channel in $\mathbb{R}^n (n = 2, 3)$, which is parallel to the $x_1$-axis, let us consider a time periodic flow of an incompressible viscous fluid which is also parallel to the $x_1$-axis.

In the case $n = 2$, for $a > 0$ we suppose $\Sigma := (-a, a)$. In the case $n = 3$, we suppose that $\Sigma$ is a bounded smooth simply connected domain in $\mathbb{R}^2$. We write

$$\omega = \mathbb{R} \times \Sigma.$$ 

$\Sigma$ is a cross section of the channel $\omega$.

In $\omega$, we consider the nonstationary Navier-Stokes equations

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = 0 \quad \text{in} \quad \mathbb{R} \times \omega, \quad (1.1)$$

$$\text{div} \, u = 0 \quad \text{in} \quad \mathbb{R} \times \omega, \quad (1.2)$$

$$u = 0 \quad \text{on} \quad \mathbb{R} \times \partial \omega \quad (1.3)$$

with the time periodic condition and the flux condition

$$u(t) = u(t + T) \quad \text{in} \quad \omega \quad (1.4)$$

$$\int_{\Sigma} u(t) \cdot n \, dS = \alpha(t) \quad (t \in \mathbb{R}), \quad (1.5)$$

where $u = u(t, x)$ and $p = p(t, x)$ are the unknown velocity and the unknown pressure of the fluid motion in $\omega$, respectively, $\nu$ is the given viscosity constant, $T(> 0)$ is a given constant, $n$ is the unit parallel vector to the $x_1$-axis and $\alpha(t)$ is a given $T$-periodic real function.

Since we look for a solution parallel to the $x_1$-axis, we may assume that

$$u(t, x) = (v(t, x), 0) \quad (n = 2),$$

$$u(t, x) = (v(t, x), 0, 0) \quad (n = 3).$$

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Then it follows that \(v\) does not depend on \(x_1\) from (1.2), \((u \cdot \nabla)u = 0\) and \(p\) depends only on \(t\) and \(x_1\) from (1.1). Therefore we obtain the equation

\[
\frac{\partial v}{\partial t} - \nu \Delta v = -\frac{\partial p}{\partial x_1} \quad \text{in } \mathbb{R} \times \Sigma, \tag{1.6}
\]

where \(\Delta = \partial^2 / \partial x_2^2\) (\(n = 2\)), \(\Delta = \partial^2 / \partial x_2^2 + \partial^2 / \partial x_3^2\) (\(n = 3\)). It is easy to see that \(v\) does not depend on \(x_1\) and \(p\) depends only on \(t\) and \(x_1\). Therefore it follows from the equation (1.6) that \(\partial v / \partial t - \nu \Delta v\) and \(\partial p / \partial x_1\) depends only on \(t\). Integrating (1.6) on \(\Sigma\), we obtain

\[
p(t, x_1) = -\frac{1}{|\Sigma|} \left( \alpha'(t) - \nu \int_{\Sigma} \Delta v(t)dS \right),
\]

where \(|\Sigma|\) is the Lebesgue measure of \(\Sigma\). Therefore there exists a time periodic solution \(u\) of the Navier-Stokes equations (1.1)–(1.5) in \(\omega\), with the form \(u = (v, 0)\) or \(u = (v, 0, 0)\), if and only if \(v\) is a solution of the problem

\[
v' + \nu Av - \frac{\nu}{|\Sigma|}(Av, e)e = \frac{\alpha'}{|\Sigma|}e \tag{1.7}
\]

with the time periodic condition and the flux condition

\[
\begin{align*}
v(t) &= v(t + T) \quad (t \in \mathbb{R}), \\
(v(t), e) &= \alpha(t) \quad (t \in \mathbb{R}),
\end{align*} \tag{1.8}
\]

where \(e(y) = 1\) \((y \in \Sigma)\), \(A = -\Delta\) with the domain \(D(A) = H^2(\Sigma) \cap H_0^1(\Sigma)\), \((v, e) = \int_{\Sigma} vedS\).

Before stating the time periodic result, we introduce the function space. Let \(X\) be a Banach space. We set

\[
\begin{align*}
H^1_{\pi}(\mathbb{R}) &= \{ \varphi \in H^1_{\text{loc}}(\mathbb{R}); \varphi(t) = \varphi(t + T) \text{ a.e. } t \in \mathbb{R} \}, \\
L^2_t(\mathbb{R}; X) &= \{ \varphi \in L^2_{\text{loc}}(\mathbb{R}; X); \varphi(t) = \varphi(t + T) \text{ in } X \text{ for a.e. } t \in \mathbb{R} \}, \\
C_\pi(\mathbb{R}; X) &= \{ \varphi \in C(\mathbb{R}; X); \varphi(t) = \varphi(t + T) \text{ in } X \text{ for } t \in \mathbb{R} \}.
\end{align*}
\]

Beirão da Veiga [4] proved that for \(n \geq 2\) if a flux \(\alpha \in H^1_{\pi}(\mathbb{R})\) is given, then there exists a unique time periodic solution \(v^\alpha\) of this problem (1.7)–(1.9) satisfying

\[
\begin{align*}
v^\alpha &\in L^2_t(\mathbb{R}; H^1_0(\Sigma) \cap H^2(\Sigma)) \cap C_\pi(\mathbb{R}; H^1_0(\Sigma)), \\
(v^\alpha)' &\in L^2_t(\mathbb{R}; L^2(\Sigma)).
\end{align*}
\]

Set

\[
\begin{align*}
V^\alpha(t, x) &= (v^\alpha(t, x), 0) \quad (n = 2), \\
V^\alpha(t, x) &= (v^\alpha(t, x), 0, 0) \quad (n = 3).
\end{align*}
\]

Let us call \(V^\alpha\) “the time periodic Poiseuille flow”. 
2 Problem in a perturbed channel

Let $\Omega$ be a smooth and unbounded domain in $\mathbb{R}^n$ ($n = 2, 3$) and $\partial \Omega$ be the boundary of the domain $\Omega$. A domain $\Omega$ is called a perturbed channel if $\Omega$ satisfies

$$\Omega \setminus B(0, R) = \omega \setminus B(0, R)(=: \omega_0),$$

where $B(0, R) = \{x \in \mathbb{R}^n; |x| < R\}$. $\omega_0$ is a perturbed and bounded part, $\omega_L$ is channel parts. The boundary $\partial \Omega$ of $\Omega$ has connected components $\Gamma_0, \Gamma_1, \ldots, \Gamma_J$ of $C^\infty$-surface such that $\Gamma_1, \ldots, \Gamma_J$ lie inside of $\Gamma_0$ with $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$, and such that $\partial \Omega = \bigcup_{j=0}^{J} \Gamma_j$. Let us call the domain $\Omega$ "a perturbed channel".

In the domain $\Omega$, we consider the nonstationary Navier-Stokes equations

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p = f \quad \text{in} \quad (0, T) \times \Omega, \quad (2.1)$$

with the boundary condition

$$u = \beta \quad \text{on} \quad (0, T) \times \partial \Omega, \quad (2.3)$$

and the time periodic condition

$$u(0) = u(T) \quad \text{in} \quad \Omega, \quad (2.5)$$

where $u = u(t, x)$ and $p = p(t, x)$ are the unknown velocity and the unknown pressure of an incompressible viscous fluid in $\Omega$ respectively, while $\nu > 0$ is the kinematic viscosity, $f = f(t, x)$ is the given external force and $\beta = \beta(t, x)$ is the given function on $(0, T) \times \partial \Omega$ with compact support. Since the solution $u(t)$ satisfies $\text{div} u(t) = 0$ in $\Omega$ for a fixed $t \in (0, T)$, the given boundary data $\beta(t)$ on $\partial \Omega$ is required to fulfill the compatibility condition which is called "General Outflow Condition" (GOC)

$$\int_{\partial \Omega} \beta(t) \cdot n \, d\sigma = 0, \quad (2.6)$$

where $n$ is the unit outer normal to $\partial \Omega$. The purpose is that if the given boundary date $\beta$ satisfies (GOC), we will seek a solution of (2.1)-(2.5).

We introduce some function spaces. $C^\infty_{0,\sigma}(\Omega)$ is the set of all real smooth vector functions with compact support in $\Omega$ and $\text{div} \varphi = 0$. $L^2_{\sigma}(\Omega)$ is the closure of $C^\infty_{0,\sigma}(\Omega)$ for the usual $L^2(\Omega)$ norm. The $L^2$ inner product and norm on $\Omega$ are denoted as $(\cdot, \cdot)_{\Omega}$ and $\| \cdot \|_{L^2, \Omega}$ respectively. $H^1_0(\Omega)$ and $H^1_{0,\sigma}(\Omega)$ are the closures of $C^\infty(\Omega)$ and $C^\infty_{0,\sigma}(\Omega)$ for the usual Dirichlet norm $\| \nabla \cdot \|_{L^2, \Omega}$, respectively. $H^1_\sigma(\Omega)$ is the set of all $H^1(\Omega)$ functions with $\text{div} \varphi = 0$. Let $X$ be a Banach space. $C([0, T]; X)$ and $H^1([0, T]; X)$ are the set of all the $C([0, T]; X)$ and $H^1([0, T]; X)$ functions satisfying the time periodic condition $u(0) = u(T)$ in $X$.

3 Result

Our definition of a time periodic weak solution of the Navier-Stokes equations (2.1), (2.2), (2.3), (2.4), (2.5) is as follows.
Definition 3.1 A measurable function \( u = u(t, x) \) on \((0, T) \times \Omega \) is called a time periodic weak solution of the Navier-Stokes equations (2.1), (2.2), (2.3), (2.4), (2.5) if \( u \) satisfies the following condition.

1. \( v := u - \hat{V}^\alpha - b \in L^2((0, T); \mathbb{H}^{1,0}_{0,\sigma}(\Omega)) \cap L^\infty((0, T); \mathbb{L}^{2}_{\sigma}(\Omega)) \).
2. \( u \) satisfies \( \frac{d}{dt}(u, \varphi) + v(\nabla u, \nabla \varphi) + ((u \cdot \nabla)u, \varphi) = (\mathbb{H}^{1,0}_{0,\sigma})'(f, \varphi)_{\mathbb{H}^{1,0}_{0,\sigma}} \) (\( \varphi \in \mathbb{H}^{1,0}_{0,\sigma}(\Omega) \)).
3. \( v(0) = v(T) \in L^2(\Omega) \),

where the function \( \hat{V}^\alpha \) and \( b \) are to be such that

\[
\begin{align*}
\text{div} \hat{V}^\alpha &= 0 \quad \text{in} \quad \Omega, \\
\hat{V}^\alpha &= 0 \quad \text{on} \quad \partial \Omega, \\
\hat{V}^\alpha &= V^\alpha \quad \text{in} \quad \omega_L,
\end{align*}
\]

and

\[
\begin{align*}
\text{div} b &= 0 \quad \text{in} \quad \Omega, \\
b &= \beta \quad \text{on} \quad \partial \Omega.
\end{align*}
\]

\( V^\alpha \) is "the extended time periodic Poiseuille flow" and \( b \) is "the boundary extension".

Before stating our result, we define a constant concerning the time periodic Poiseuille flow.

Definition 3.2 We set

\[
\gamma^\alpha(t) = \sup_{\varphi \in \mathbb{H}^{1,0}_{0,\omega}(\omega)} \frac{((\varphi \cdot \nabla)\varphi, V^\alpha(t))_{\omega}}{\|\nabla \varphi\|_{2,\omega}^{2}} \quad (t \in [0, T]),
\]

(3.1)

\[
\hat{\gamma}^\alpha := \sup_{t \in [0, T]} \gamma^\alpha(t).
\]

(3.2)

We have the following result.

Theorem 3.1 (T. Kobayashi[13])

Suppose that \( \hat{\gamma}^\alpha < \nu \), \( f \in L^2((0, T); (\mathbb{H}^{1,0}_{0,\sigma}(\Omega)))' \) and \( \beta = 0 \). Then there exists a time periodic weak solution.

This result is not the problem of \((GOC)\) because \( \beta = 0 \). We need the following assumption.

Assumption 3.1 \( \Omega \) is a two dimensional symmetric domain with respect to the \( x_1 \)-axis and all the inner boundaries \( \Gamma_j(1 \leq j \leq J) \) intersect the \( x_1 \)-axis.

Theorem 3.2 (T. Kobayashi[14])

We assume that the domain \( \Omega \) satisfies Assumption 3.1. We suppose that \( \hat{\gamma}^\alpha < \nu \), \( f \in L^2((0, T); (\mathbb{H}^{1,0}_{0,\sigma}(\Omega)))' \), \( \beta \in H^1(0, T); (\mathbb{H}^{1,0}_{0,\sigma}(\Omega))' \) with compact support, \((GOC)\) and

\[
\int_{\Gamma_0^+} \beta \cdot nd\sigma = \int_{\Gamma_0^-} \beta \cdot nd\sigma = 0 \quad \text{on} \quad [0, T].
\]

Then there exists a time periodic weak solution of the Navier-Stokes equations.
We need an appropriate extension of the given boundary data $\beta$.

**Proposition 3.1** We assume that a domain $\Omega$ satisfies Assumption 3.1. Suppose that $\beta \in H^1_1((0, T); \mathbb{H}^{1,5}(\partial \Omega))$ satisfies (GOC), the support of $\beta$ is compact and

$$\int_{\Gamma^+_0} \beta \cdot n d\sigma = \int_{\Gamma^-_0} \beta \cdot n d\sigma = 0 \quad \text{on} \quad [0, T].$$

Then for any $\varepsilon > 0$ there exists an extension $b_\varepsilon \in H^1_1((0, T); \mathbb{H}^{1,5}(\Omega))$ of $\beta$ such that $b_\varepsilon$ has compact support and the inequality

$$|((v \cdot \nabla)v, b_\varepsilon(t))| < \varepsilon \|\nabla v\|_{2, \Omega}^2 \quad (v \in \mathbb{H}^{1,5}_{0,\sigma}(\Omega), t \in [0, T])$$

(3.3)

holds true.

The estimate (3.3) is “Leray’s inequality”. The estimate (3.3) is its symmetric version in an unbounded perturbed channel.

**Remark 3.1** In this paper, the domain $\Omega$ has two outlets. We can solve $K$ ($K \geq 3$) outlets problem. We consider a straight channel $\omega_i$ ($i = 1, \cdots, K$), where $\Sigma_i$ is a cross section of $\omega_i$ as Section 1 and the center line of $\omega_i$ may not be parallel to the $x_1$-axis. We assume that a given flux function $\alpha_i \in H^1_1(\mathbb{R})$ ($i = 1, \cdots, K$) satisfies $\sum_{i=1}^{K} \alpha_i(t) = 0$ ($t \in \mathbb{R}$). For each $\alpha_i$, we have the time periodic Poiseuille flow $V_i^{\alpha}$ in $\omega_i$. We assume that $\Omega$ has $K$ outlets $\omega_{0i}$ ($i = 1, \cdots, K$) where $\omega_{0i}$ is a semi-infinite channel with the cross section $\Sigma_i$. In the domain $\Omega_i$, we consider a time periodic problem with the time periodic Poiseuille flow $V_i^{\alpha}$. We define constant $\hat{\gamma} = \max_{1 \leq i \leq K}\{\hat{\gamma}_i^{\alpha}\}$ as Definition 3.2. Suppose that $\hat{\gamma} < \nu$. Then there exists a time periodic weak solution in $\Omega$ with $K$ outlets.

**References**


[14] T. Kobayashi, Time periodic solutions of the Navier-Stokes equations with the time periodic Poiseuille velocity in a two and three dimensional perturbed symmetric channels, Journal of the Mathematical Society of Japan, to appears


