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Time periodic flows of an incompressible viscous fluid in perturbed channels

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1 The time periodic Poiseuille flow

In this section, for a straight channel in \( \mathbb{R}^n (n = 2, 3) \), which is parallel to the \( x_1 \)-axis, let us consider a time periodic flow of an incompressible viscous fluid which is also parallel to the \( x_1 \)-axis.

In the case \( n = 2 \), for \( a > 0 \) we suppose \( \Sigma := (-a, a) \). In the case \( n = 3 \), we suppose that \( \Sigma \) is a bounded smooth simply connected domain in \( \mathbb{R}^2 \). We write

\[
\omega = \mathbb{R} \times \Sigma.
\]

\( \Sigma \) is a cross section of the channel \( \omega \).

In \( \omega \), we consider the nonstationary Navier-Stokes equations

\[
\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = 0 \quad \text{in} \quad \mathbb{R} \times \omega, \tag{1.1}
\]

\[
\text{div} \mathbf{u} = 0 \quad \text{in} \quad \mathbb{R} \times \omega, \tag{1.2}
\]

\[
\mathbf{u} = 0 \quad \text{on} \quad \mathbb{R} \times \partial \omega \tag{1.3}
\]

with the time periodic condition and the flux condition

\[
\mathbf{u}(t \pm T) \quad \text{in} \quad \omega \tag{1.4}
\]

\[
\int_\Sigma \mathbf{u}(t) \cdot \mathbf{n} dS = \alpha(t) \quad (t \in \mathbb{R}), \tag{1.5}
\]

where \( \mathbf{u} = \mathbf{u}(t, x) \) and \( p = p(t, x) \) are the unknown velocity and the unknown pressure of the fluid motion in \( \omega \), respectively, \( \nu \) is the given viscosity constant, \( T > 0 \) is a given constant, \( \mathbf{n} \) is the unit parallel vector to the \( x_1 \)-axis and \( \alpha(t) \) is a given \( T \)-periodic real function.

Since we look for a solution parallel to the \( x_1 \)-axis, we may assume that

\[
\mathbf{u}(t, x) = (v(t, x), 0) \quad (n = 2),
\]

\[
\mathbf{u}(t, x) = (v(t, x), 0, 0) \quad (n = 3).
\]

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Then it follows that $v$ does not depend on $x_1$ from (1.2), $(u \cdot \nabla)u = 0$ and $p$ depends only on $t$ and $x_1$ from (1.1). Therefore we obtain the equation

$$\frac{\partial v}{\partial t} - \nu \Delta v = -\frac{\partial p}{\partial x_1} \quad \text{in} \quad \mathbb{R} \times \Sigma,$$

(1.6)

where $\Delta = \partial^2/\partial x_1^2$ ($n = 2$), $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$ ($n = 3$). It is easy to see that $v$ does not depend on $x_1$ and $p$ depends only on $t$ and $x_1$. Therefore it follows from the equation (1.6) that $\partial v/\partial t - \nu \Delta v$ and $\partial p/\partial x_1$ depends only on $t$. Integrating (1.6) on $\Sigma$, we obtain

$$p(t, x_1) = -\frac{1}{|\Sigma|} \left( \alpha'(t) - \nu \int_{\Sigma} \Delta v(t) dS \right),$$

where $|\Sigma|$ is the Lebesgue measure of $\Sigma$. Therefore there exists a time periodic solution $u$ of the Navier-Stokes equations (1.1)–(1.5) in $\omega$, with the form $u = (v, 0)$ or $u = (v, 0, 0)$, if and only if $v$ is a solution of the problem

$$v' + \nu Av - \frac{\nu}{|\Sigma|} (Av, e)e = \frac{\alpha'}{|\Sigma|} e$$

(1.7)

with the time periodic condition and the flux condition

$$v(t) = v(t + T) \quad (t \in \mathbb{R}),$$

$$v(t), e) = \alpha(t) \quad (t \in \mathbb{R}),$$

(1.8)

(1.9)

where $e(y) = 1$ ($y \in \Sigma$), $A = -\Delta$ with the domain $D(A) = H^2(\Sigma) \cap H_0^1(\Sigma)$, $(v, e) = \int_\Sigma v e dS$.

Before stating the time periodic result, we introduce the function space. Let $X$ be a Banach space. We set

$$H_n^1(\mathbb{R}) = \{ \varphi \in H_{1oc}^1(\mathbb{R}); \varphi(t) = \varphi(t + T) \text{ a.e. } t \in \mathbb{R} \},$$

$$L_n^2(\mathbb{R}; X) = \{ \varphi \in L_{1oc}^2(\mathbb{R}; X); \varphi(t) = \varphi(t + T) \text{ in } X \text{ for a.e. } t \in \mathbb{R} \},$$

$$C_n(\mathbb{R}; X) = \{ \varphi \in C(\mathbb{R}; X); \varphi(t) = \varphi(t + T) \text{ in } X \text{ for } t \in \mathbb{R} \}.$$

Beirão da Veiga [4] proved that for $n \geq 2$ if a flux $\alpha \in H_n^1(\mathbb{R})$ is given, then there exists a unique time periodic solution $v^\alpha$ of this problem (1.7)–(1.9) satisfying

$$v^\alpha \in L_n^2(\mathbb{R}; H_0^1(\Sigma) \cap H^2(\Sigma)) \cap C_n(\mathbb{R}; H_0^1(\Sigma)),$$

$$(v^\alpha)' \in L_n^2(\mathbb{R}; L^2(\Sigma)).$$

Set

$$V_n^\alpha(t, x) = (v^\alpha(t, x), 0) \quad (n = 2),$$

$$V_n^\alpha(t, x) = (v^\alpha(t, x), 0, 0) \quad (n = 3).$$

Let us call $V^\alpha$ “the time periodic Poiseuille flow”.  

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2 Problem in a perturbed channel

Let $\Omega$ be a smooth and unbounded domain in $\mathbb{R}^n$ ($n = 2, 3$) and $\partial \Omega$ be the boundary of the domain $\Omega$. A domain $\Omega$ is called a perturbed channel if $\Omega$ satisfies

$$\Omega \backslash B(0, R) = \omega \backslash B(0, R)(=: \omega_0),$$

where $B(0, R) = \{x \in \mathbb{R}^n; |x| < R\}$. $\omega_0$ is a perturbed and bounded part, $\omega_L$ is channel parts. The boundary $\partial \Omega$ of $\Omega$ has connected components $\Gamma_0, \Gamma_1, \ldots, \Gamma_J$ of $C^\infty$-surface such that $\Gamma_1, \ldots, \Gamma_J$ lie inside of $\Gamma_0$ with $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$, and such that $\partial \Omega = \bigcup_{j=0}^{J} \Gamma_{j}$. Let us call the domain $\Omega$ "a perturbed channel".

In the domain $\Omega$, we consider the nonstationary Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \nabla \cdot \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in} \quad (0, T) \times \Omega,$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in} \quad (0, T) \times \Omega,$$  \hspace{1cm} (2.1)

with the boundary condition

$$\mathbf{u} = \mathbf{\beta} \quad \text{on} \quad (0, T) \times \partial \Omega,$$  \hspace{1cm} (2.2)

and the time periodic condition

$$\mathbf{u}(0) = \mathbf{u}(T) \quad \text{in} \quad \Omega,$$  \hspace{1cm} (2.3)

where $\mathbf{u} = \mathbf{u}(t, x)$ and $p = p(t, x)$ are the unknown velocity and the unknown pressure of an incompressible viscous fluid in $\Omega$ respectively, while $\nu > 0$ is the kinematic viscosity, $\mathbf{f} = \mathbf{f}(t, x)$ is the given external force and $\mathbf{\beta} = \mathbf{\beta}(t, x)$ is the given function on $(0, T) \times \partial \Omega$ with compact support. Since the solution $\mathbf{u}(t)$ satisfies $\nabla \cdot \mathbf{u}(t) = 0$ in $\Omega$ for a fixed $t \in (0, T)$, the given boundary data $\mathbf{\beta}(t)$ on $\partial \Omega$ is required to fulfill the compatibility condition which is called "General Outflow Condition" (GOC)

$$\int_{\partial \Omega} \mathbf{\beta}(t) \cdot \mathbf{n} \, d\sigma = 0,
$$  \hspace{1cm} (2.4)

where $\mathbf{n}$ is the unit outer normal to $\partial \Omega$. The purpose is that if the given boundary date $\mathbf{\beta}$ satisfies (GOC), we will seek a solution of (2.1)-(2.5).

We introduce some function spaces. $\mathcal{C}^\infty_{0, \sigma}(\Omega)$ is the set of all real smooth vector functions with compact support in $\Omega$ and $\nabla \varphi = 0$. $\mathbb{L}^2_2(\Omega)$ is the closure of $\mathcal{C}^\infty_{0, \sigma}(\Omega)$ for the usual $\mathbb{L}^2(\Omega)$ norm. The $\mathbb{L}^2$ inner product and norm on $\Omega$ are denoted as $(\cdot, \cdot)_{\Omega}$ and $\|\cdot\|_{2, \Omega}$ respectively. $\mathbb{H}^1_0(\Omega)$ and $\mathbb{H}^1_{0, \sigma}(\Omega)$ are the closures of $\mathcal{C}^\infty(\Omega)$ and $\mathcal{C}^\infty_{0, \sigma}(\Omega)$ for the usual Dirichlet norm $\|\nabla \cdot \|_{2, \Omega}$, respectively. $\mathbb{H}^1_0(\Omega)$ is the set of all $\mathbb{H}^1(\Omega)$ functions with $\nabla \varphi = 0$. Let $X$ be a Banach space. $C_\varepsilon([0, T]; X)$ and $H^1_\varepsilon((0, T); X)$ are the set of all the $C([0, T]; X)$ and $H^1((0, T); X)$ functions satisfying the time periodic condition $\mathbf{u}(0) = \mathbf{u}(T)$ in $X$.

3 Result

Our definition of a time periodic weak solution of the Navier-Stokes equations (2.1), (2.2), (2.3), (2.4), (2.5) is as follows.
Definition 3.1 A measurable function \( u = u(t, x) \) on \((0, T) \times \Omega\) is called a time periodic weak solution of the Navier-Stokes equations (2.1), (2.2), (2.3), (2.4), (2.5) if \( u \) satisfies the following condition. 

1. \( v := u - \hat{V}^\alpha - b \in L^2((0, T); \mathbb{H}_{0,\sigma}^1(\Omega)) \cap L^\infty((0, T); \mathbb{L}_{\sigma}^2(\Omega)) \). 
2. \( u \) satisfies 
   \[
   \frac{d}{dt}(u, \varphi) + v(\nabla u, \nabla\varphi) + ((u \cdot \nabla)u, \varphi) = (\mathbb{H}_{0,\sigma}^1)'(f, \varphi)_{\mathbb{H}_{0,\sigma}^1} \quad (\varphi \in \mathbb{H}_{0,\sigma}^1(\Omega)).
   \]
3. \( v(0) = v(T) \in L^2(\Omega) \),

where the function \( \hat{V}^\alpha \) and \( b \) are to be such that

\[
\text{div} \, \hat{V}^\alpha = 0 \quad \text{in} \quad \Omega,
\]
\[
\hat{V}^\alpha = 0 \quad \text{on} \quad \partial\Omega,
\]
\[
\hat{V}^\alpha = V^\alpha \quad \text{in} \quad \omega_L,
\]

and

\[
\text{div} \, b = 0 \quad \text{in} \quad \Omega,
\]
\[
b = \beta \quad \text{on} \quad \partial\Omega.
\]

\( V^\alpha \) is “the extended time periodic Poiseuille flow” and \( b \) is “the boundary extension”.

Before stating our result, we define a constant concerning the time periodic Poiseuille flow.

Definition 3.2 We set

\[
\begin{align*}
\gamma^\alpha(t) &= \sup_{\varphi \in \mathbb{H}_{0,\sigma}^1(\omega)} \frac{((\varphi \cdot \nabla)\varphi, V^\alpha(t))_\omega}{\|\nabla\varphi\|_{2,\omega}^2} \quad (t \in [0, T]), \\
\hat{\gamma}^\alpha &= \sup_{t \in [0, T]} \gamma^\alpha(t).
\end{align*}
\]

We have the following result.

Theorem 3.1 (T. Kobayashi[13])

Suppose that \( \hat{\gamma}^\alpha < \nu \), \( f \in L^2((0, T); (\mathbb{H}_{0,\sigma}^1(\Omega))') \) and \( \beta = 0 \). Then there exists a time periodic weak solution.

This result is not the problem of \((GOC)\) because \( \beta = 0 \). We need the following assumption.

Assumption 3.1 \( \Omega \) is a two dimensional symmetric domain with respect to the \( x_1 \)-axis and all the inner boundaries \( \Gamma_j \) (1 \( \leq j \leq J \)) intersect the \( x_1 \)-axis.

Theorem 3.2 (T. Kobayashi[14])

We assume that the domain \( \Omega \) satisfies Assumption 3.1. We suppose that \( \hat{\gamma}^\alpha < \nu \), \( f \in L^2((0, T); (\mathbb{H}_{0,\sigma}^1(\Omega))') \), \( \beta \in H_{\pi}^1((0, T); \mathbb{H}_{\pi}^{\frac{1}{2},S}(\partial\Omega)) \) with compact support, \((GOC)\) and

\[
\int_{\Gamma_0^+} \beta \cdot n \, d\sigma = \int_{\Gamma_0^-} \beta \cdot n \, d\sigma = 0 \quad \text{on} \quad [0, T].
\]

Then there exists a time periodic weak solution of the Navier-Stokes equations.
We need an appropriate extension of the given boundary data $\beta$.

**Proposition 3.1** We assume that a domain $\Omega$ satisfies Assumption 3.1. Suppose that $\beta \in H_{\epsilon}^1((0, T); \mathbb{H}^{1, \xi}(\partial \Omega))$ satisfies (GOC), the support of $\beta$ is compact and

$$
\int_{\Gamma_0^+} \beta \cdot n d\sigma = \int_{\Gamma_0^-} \beta \cdot n d\sigma = 0 \quad \text{on} \quad [0, T].
$$

Then for any $\varepsilon > 0$ there exists an extension $b_\varepsilon \in L^2((0, T); \mathbb{H}^{1, \xi}(\Omega))$ of $\beta$ such that $b_\varepsilon$ has compact support and the inequality

$$
|\langle (\tau \cdot \nabla)v, b_\varepsilon(t) \rangle| < \varepsilon\|\nabla v\|_{2,\Omega}^2 \quad (v \in \mathbb{H}^{1, \xi}_{0, \sigma}(\Omega), t \in [0, T])
$$

(3.3)

holds true.

The estimate (3.3) is “Leray’s inequality”. The estimate (3.3) is its symmetric version in an unbounded perturbed channel.

**Remark 3.1** In this paper, the domain $\Omega$ has two outlets. We can solve $K$ ($K \geq 3$) outlets problem. We consider a straight channel $\omega_i$ ($i = 1, \cdots, K$), where $\Sigma_i$ is a cross section of $\omega_i$ as Section 1 and the center line of $\omega_i$ may not be parallel to the $x_1$-axis. We assume that a given flux function $\alpha_i \in H_{\epsilon}^1(\mathbb{R})$ ($i = 1, \cdots, K$) satisfies $\sum_{i=1}^{K} \alpha_i(t) = 0$ ($t \in \mathbb{R}$). For each $\alpha_i$, we have the time periodic Poiseuille flow $V_i^\alpha$ in $\omega_i$. We assume that $\Omega$ has $K$ outlets $\omega_{0i}$ ($i = 1, \cdots, K$) where $\omega_{0i}$ is a semi-infinite channel with the cross section $\Sigma_i$. In the domain $\Omega$, we consider a time periodic problem with the time periodic Poiseuille flow $V_i^\alpha$. We define constant $\hat{\gamma} = \max_{1 \leq i \leq K}\{\hat{\gamma}_i^\alpha\}$ as Definition 3.2. Suppose that $\hat{\gamma} < \nu$. Then there exists a time periodic weak solution in $\Omega$ with $K$ outlets.

**References**


[14] T. Kobayashi, Time periodic solutions of the Navier-Stokes equations with the time periodic Poiseuille velocity in a two and three dimensional perturbed symmetric channels, Journal of the Mathematical Society of Japan, to appears


