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<th>$L_{p}$-estimates for a viscous compressible fluid in an infinite time interval (model problems) (Mathematical Analysis of Viscous Incompressible Fluid)</th>
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Abstract. The paper contains analysis of model problems arising in the proof of maximal regularity $L_{p}$-estimates for linearized free boundary problem governing the motion of two viscous compressible fluids. The proof is based on convenient representation formulas for the Fourier-Laplace transformation of solutions of model problems and on the Marcinkiewicz-Mikhlin-Lizorkin theorem.

1 Construction of solutions of model problems.

The motion of two viscous compressible fluids contained in a bounded vessel and separated by a free interface is governed by the evolution free boundary problem

$$
\begin{cases}
\rho(v + (v \cdot \nabla)v) - \nabla \cdot T(v) + \nabla p(\rho) = 0, \\
\rho_t + \nabla \cdot \rho v = 0, \quad x \in \Omega_t^+ \cup \Omega_t^-,
\end{cases}
$$

(1.1)

where $v(x, t) = v^\pm(x, t)$, $\rho(x, t) = \rho^\pm(x, t)$ for $x \in \Omega_t^\pm$, $\Omega_t^+$ and $\Omega_t^-$ are bounded domains occupied by the fluids and separated by a free interface $\Gamma_t = \partial\Omega_t^+$. It is given for $t = 0$ and should be found for $t > 0$. The domain $\Omega = \Omega_t^+ \cup \Gamma_t \cup \Omega_t^-$ is fixed; the surface $S = \partial\Omega$ is bounded away from $\Gamma_t$. By $T(v) \equiv T^\pm(v)$ we mean the viscous part of the stress tensor:

$$
T^\pm(v) = \mu^\pm S(v^\pm) + \mu^\pm_1 I \nabla \cdot v^\pm, \quad x \in \Omega_t^\pm,
$$

$S(v) = (\nabla v) + (\nabla v)^T$ is the doubled rate-of-strain tensor, $\mu = \mu^\pm, \mu_1 = \mu^\pm_1$ are positive constants, $p^\pm(p^\pm)$ are positive strictly increasing functions of positive arguments, $n$ is the normal to $\Gamma_t$ exterior with respect to $\Omega_t^+$, $V_n$ is the velocity of evolution of $\Gamma_t$ in the direction $n$, $[u] = u^+ - u^-$ is the jump of the function $u$ on $\Gamma_t$.

By the rest state we mean the solution of (1.1) with $v^\pm(x, t) = 0$. Then the domains $\Omega_t^\pm$ are independent of $t$ and $\rho^\pm(x, t) = \text{const} = \overline{\rho}^\pm = M^\pm/|\Omega^\pm|$, where $M^\pm$ are total masses of the fluids and $|\Omega^\pm| = \text{mes}\Omega^\pm$. The jump conditions on $\Gamma_t$ reduce to

$$
p^+(\overline{\rho}^+) = p^-(\overline{\rho}^-).
$$

The rest state is stable, if solutions of (1.1) with small $v_0^\pm, \rho_0^\pm$ close to $\overline{\rho}^\pm$, $\Omega_0^\pm$ close to $\Omega^\pm$ are defined for $t > 0$ and tend to $(v = 0, \rho = \overline{\rho}^\pm, \Omega^\pm)$ as $t \to \infty$. In the paper [1] this was
proved under the assumptions
\[ \overline{\rho}^+ = \overline{\rho} = \overline{\rho} = M/|\Omega|, \quad M = M^+ + M^- , \]

By passing to the Lagrangian coordinates \( \{y\} \) and linearizing (1.1) about the rest state we arrive at the linear problem

\[
\begin{aligned}
\left\{ \begin{array}{l}
\overline{\rho}v_t - \nabla \cdot T(v) + p'(\overline{\rho})\nabla \theta = f(y, t), \\
\theta_t + \overline{\rho} \nabla \cdot v = h(y, t), \\
|v| = 0, \quad [-p'(\overline{\rho})\nabla \theta + T(v)n] = b(y, t), \\
v^-(y, t) = 0, \quad y \in S, \\
v(y, 0) = v_0(y), \quad \theta(y, 0) = \theta_0(y), \\
\end{array} \right.
\end{aligned}
\tag{1.2}
\]

where \( f, h, b, v_0, \theta_0 \) are some given functions. In the present paper we study model problems arising in the analysis of (1.2), in particular, we obtain maximal regularity weighted \( L_p \)-estimates of solutions of these problems. In combination with the Schauder localization method, our results enable us to estimate the higher order Sobolev norms of the solution of (1.2) by the data and by the weighted \( L_p \)-norms of \( v \) and \( \theta \).

Main attention is given to the problem

\[
\begin{aligned}
\left\{ \begin{array}{l}
\overline{\rho}^\pm w^\pm_t - \nabla \cdot T^\pm(w^\pm) + p^{\pm'}(\overline{\rho}^\pm)\nabla \theta^\pm = f^\pm, \\
\theta^\pm_t + \overline{\rho}^\pm \nabla \cdot w^\pm = h^\pm, \\
[w] = 0, \quad [T_{03}(w)] = b_\alpha, \quad \alpha = 1, 2, \\
[-p'(\overline{\rho})\theta + T_{33}(w)] = b_3, \quad y_3 = 0, \\
w^\pm(y, 0) = 0, \quad \theta^\pm(y, 0) = 0, \quad \pm y_3 > 0,
\end{array} \right.
\end{aligned}
\tag{1.3}
\]

related to the estimates of the solution of (1.2) near the interface \( \Gamma_0 \).

Having in mind the application of the localization method, we assume that \( \text{supp} w \) and \( \text{supp} \theta \) are contained in the closure of the domain \( \mathcal{Q} = \mathcal{Q}^+ \cup \mathcal{Q}^- \), where \( \mathcal{Q}^\pm = \mathcal{Q}' \times I^\pm, \mathcal{Q}' = \{ |x_j| \leq d_0, j = 1, 2, I^\pm = \{ \pm x_3 \in (0, d_0) \} \}. \) Moreover, we assume for a while that \( f = 0 \) and \( h = 0 \).

We expand \( b_j(y', t) \) in the Fourier series in \( \mathcal{Q}' \):

\[
b_j(y', t) = \frac{1}{(2d_0)^2} \sum_{k \in \mathbb{Z}^2} \tilde{b}_j(\xi, t)e^{i\xi y'}, \quad \xi = (\frac{\pi}{d_0}k_1, \frac{\pi}{d_0}k_2),
\]

where \( \tilde{b}_j(\xi, t) = \int_{\mathcal{Q}'} e^{-i\xi x'}b_j(y', t)dy' \) and make the Fourier-Laplace transform

\[
\begin{aligned}
\tilde{w}^\pm(\xi, s, y_3) &= \int_0^\infty e^{-st}dt \int_{\mathcal{Q}'} w^\pm(y, t)e^{-i\xi y'}dy' = \mathcal{F}w, \\
\tilde{\theta}^\pm(\xi, s, y_3) &= \int_0^\infty e^{-st}dt \int_{\mathcal{Q}'} \theta^\pm(y, t)e^{-i\xi y'}dy' = \mathcal{F}\theta.
\end{aligned}
\tag{1.4}
\]
This converts problem (1.3) into

\[
\begin{align*}
\bar{\rho} s \tilde{w}_\alpha - \left( \sum_{\beta=1,2} i \xi_\beta \tilde{T}_{\beta\alpha}(\tilde{w}) + \frac{d}{dy_3} \tilde{T}_{3\alpha}(\tilde{w}) \right) + p'(\bar{\rho}) i \xi_\alpha \tilde{\vartheta} = 0, & \quad \alpha = 1, 2, \\
\bar{\rho} s \tilde{w}_3 - \left( \sum_{\beta=1,2} i \xi_\beta \tilde{T}_{\beta3}(\tilde{w}) + \frac{d}{dy_3} \tilde{T}_{33}(\tilde{w}) \right) + p'(\bar{\rho}) \frac{d}{dy_3} \tilde{\vartheta} = 0, \\
n \tilde{\vartheta} + \bar{\rho} \left( \sum_{\beta=1,2} i \xi_\beta \tilde{w}_\beta + \frac{d}{dy_3} \tilde{w}_3 \right) = 0, & \quad \pm y_3 > 0,
\end{align*}
\]

(1.5)

where \( \tilde{T}^\pm(\tilde{w}^\pm) = \mathcal{F} T^\pm(w^\pm), \tilde{S}^\pm(\tilde{w}^\pm) = \mathcal{F} S^\pm(w^\pm), \tilde{w} = \tilde{w}^\pm, \tilde{\theta} = \tilde{\theta}^\pm \) for \( \pm y_3 > 0. \)

Along with (1.3), (1.5), we study similar problems for the Lamé system:

\[
\begin{align*}
\overline{\rho}^\pm \omega_\alpha - \nabla \cdot T^\pm(w^\pm) &= f^\pm, & \pm y_3 > 0, \\
[w] &= 0, & [T_{3\alpha}(w)] = b_\alpha, & i = 1, 2, 3, & y_3 = 0, \\
[w]^\pm(y, 0) &= 0, & \pm y_3 > 0.
\end{align*}
\]

(1.6)

and

\[
\begin{align*}
\overline{\rho} s \tilde{w}_\alpha - \left( \sum_{\beta=1,2} i \xi_\beta \tilde{T}_{\beta\alpha}(\tilde{w}) + \frac{d}{dy_3} \tilde{T}_{3\alpha}(\tilde{w}) \right) &= 0, & \alpha = 1, 2, \\
\overline{\rho} s \tilde{w}_3 - \left( \sum_{\beta=1,2} i \xi_\beta \tilde{T}_{\beta3}(\tilde{w}) + \frac{d}{dy_3} \tilde{T}_{33}(\tilde{w}) \right) &= 0, & \pm y_3 > 0, \\
[w] &= 0, & [\tilde{T}_{3\alpha}(\tilde{w})] = \tilde{b}_\alpha, & i = 1, 2, 3, \\
\tilde{\omega}, \tilde{\vartheta} &\to 0, & |y_3| &\to \infty.
\end{align*}
\]

(1.7)

We proceed with constructing the solution of problem (1.7) in the explicit form, assuming for a while that \(|k| \geq 1, |\xi| \geq \pi/d_0\). We follow the same scheme as in [2,3], see also [4]. The general form of the solution of (1.7) is

\[
\begin{align*}
\tilde{\omega}^+(\xi, s, y_3) &= C_1^+ \begin{pmatrix} r^+ \\ i \xi_1 \end{pmatrix} e^{-r^+ y_3} + C_2^+ \begin{pmatrix} 0 \\ i \xi_2 \end{pmatrix} e^{-r^+ y_3} + C_3^+ \begin{pmatrix} i \xi_1 \\ -r^+ \end{pmatrix} e^{-r^+ y_3}, & y_3 > 0, \\
\tilde{\omega}^-(\xi, s, y_3) &= C_1^- \begin{pmatrix} r^- \end{pmatrix} e^{-r^- y_3} + C_2^- \begin{pmatrix} 0 \\ i \xi_2 \end{pmatrix} e^{-r^- y_3} + C_3^- \begin{pmatrix} i \xi_1 \\ r^- \end{pmatrix} e^{r^- y_3}, & y_3 < 0,
\end{align*}
\]

(1.8)

where \( r^\pm = \sqrt{\frac{s}{\nu^\pm} + |\xi|^2}, r^1_\pm = \sqrt{\frac{s}{2\nu^\pm + \nu_{1\pm}} + |\xi|^2}, \nu^\pm = \mu^\pm/\bar{\rho}^\pm, \nu^1_\pm = \mu_{1\pm}/\bar{\rho}^\pm, \) i.e.,

\[
\begin{align*}
\tilde{\omega}^+(\xi, s, y_3) &= \left( C_1^+ r^+ + i \xi_1 C_3^+ \right) e^{-r^+ y_3} + C_2^+ \begin{pmatrix} i \xi_2 \\ -r^+ \end{pmatrix} (e^{-r^+ y_3} - e^{-r^- y_3}), & y_3 > 0,
\end{align*}
\]

(1.9)
\[ \tilde{w}^{-}(\xi, s, y_3) = \begin{pmatrix} -C_1^{-}r^{-} & +i\xi_1C_3^{-} \\ -C_2^{-}r^{-} & +i\xi_2C_3^{-} \\ C_3^{-} & +r_1^{-}C_3^{-} \end{pmatrix} e^{r^{-}y_3} + C_3^{-} \begin{pmatrix} i\xi_1 \\ i\xi_2 \\ r_1^{-} \end{pmatrix} (e^{r_1^{+}y_3} - e^{r^{-}y_3}), \quad y_3 < 0, \quad (1.10) \]

where \( C^{\pm} = i\xi_1C_1^{\pm} + i\xi_2C_2^{\pm} \). Substituting (1.9) and (1.10) into the jump conditions in (1.7), we obtain

\[
\begin{align*}
&\mu^{+}(-r^{+2}C^{+}_\alpha - i\xi_\alpha r^{+}_1 C^{+}_3 + i\xi_\alpha (C^{+} - r^{+}_1 C^{+}_3)) \\
- \mu^{-}(-r^{-2}C^{-}_\alpha + i\xi_\alpha r^{-}_1 C^{-}_3 + i\xi_\alpha (C^{-} + r^{-}_1 C^{-}_3)) = \tilde{b}_\alpha, \quad \alpha = 1, 2, \\
(2\mu^{+} + \mu^{+}_1)(-r^{+}C^{+} + r^{+}_1 C^{+}_3) - (2\mu^{-} + \mu^{-}_1)(r^{-}C^{-} + r^{-}_1 C^{-}_3) \\
+ \mu^{+}_1(C^{+}r^{+} - |\xi|^2C^{+}_3) - \mu^{-}_1(-C^{-}r^{-} - |\xi|^2C^{-}_3) = \tilde{b}_3, \\
C^{+}r^{+}_1 C^{+}_3 = -C^{-}r^{-}_1 C^{-}_3, \quad \alpha = 1, 2, \\
\end{align*}
\]

(1.11)

and, as a consequence,

\[
\begin{align*}
C^{+}r^{+} - |\xi|^2C^{+}_3 &= -C^{-}r^{-} - |\xi|^2C^{-}_3, \\
\mu^{+}(-r^{+2}C^{+} + |\xi|^2r^{+}_1 C^{+}_3) - \mu^{-}(-r^{-2}C^{-} - r^{-}_1 |\xi|^2C^{-}_3) \\
- (\mu^{+} - \mu^{-})(|\xi|^2C^{+} - r^{+}_1 C^{+}_3)) &= \sum_{\beta=1,2} i\xi_\beta \tilde{b}_\beta \equiv \tilde{B}. \\
\end{align*}
\]

(1.12)

At first we find \( C^{\pm}_3 \). We replace \( C^{\pm} \) with \((C^{+} - r^{+}_1 C^{+}_3) + r^{+}_1 C^{+}_3, \) \( C^{-} \) with \((C^{-} + r^{-}_1 C^{-}_3) - r^{-}_1 C^{-}_3 \) in (1.11), (1.12) and make use of the formulas

\[
\begin{align*}
&\frac{r^{\pm 2} - |\xi|^2}{\nu^{\pm}} = \frac{s}{\mu^{\pm}}, \quad \frac{r^{\pm}_1 - |\xi|^2}{\nu^{\pm}_1} = \frac{\overline{\rho}^{\pm}s}{2\mu^{\pm} + \mu^{\pm}_1}, \\
&\frac{r^{\pm}_1 r^{\pm} - |\xi|^2}{\nu^{\pm}} = \frac{s}{\nu^{\pm}} = \frac{s}{\nu^{\pm}_1} = \frac{s}{\nu^{\pm}_1} R^{\pm} = \frac{s}{\nu^{\pm}} R^{\pm}, \\
\end{align*}
\]

where

\[
R^{\pm} = \frac{r^{\pm}_1 + x^{\pm} r^{\pm}}{r^{\pm}_1 + r^{\pm}_1}, \quad x^{\pm} = \frac{\nu^{\pm}}{2\nu^{\pm} + \nu^{\pm}_1}. \\
\]

This leads to the system

\[
\begin{align*}
A_1(C^{+} - r^{+}_1 C^{+}_3) - r^{+}_1 \overline{\rho}^{+} s C^{+}_3 - r^{-}_1 \overline{\rho}^{-} s C^{-}_3 &= \tilde{B}, \\
A_2(C^{+} - r^{+}_1 C^{+}_3) + \overline{\rho}^{+} s (1 - 2R^{+}) C^{+}_3 - \overline{\rho}^{-} s (1 - 2R^{-}) C^{-}_3 &= \tilde{b}_3, \\
A_3(C^{+} - r^{+}_1 C^{+}_3) + \overline{\rho}^{+} 2R^{+} C^{+}_3 - \overline{\rho}^{-} 2R^{-} C^{-}_3 &= 0, \\
\end{align*}
\]

(1.13)

where

\[
A_1 = -\mu^{+}(r^{+2} + |\xi|^2) + \mu^{-}(r^{-2} + |\xi|^2), \quad A_2 = -2\mu^{+} r^{+} - 2\mu^{-} r^{-}, \quad A_3 = r^{+} + r^{-}. \\
\]

It follows that \((\overline{\rho}^{+} s C^{+}_3, \overline{\rho}^{-} s C^{-}_3) \) satisfy

\[
\mathcal{L}(\overline{\rho}^{+} s C^{+}_3, \overline{\rho}^{-} s C^{-}_3)^T = (\tilde{B}, \tilde{b}_3)^T \\
\]

(1.14)
with
\[ \mathcal{L} = \begin{pmatrix} - \frac{A_3 R^+ - r_1^+}{\mu^+} & \frac{A_3 R^- - r_1^-}{\mu^-} \\ - \frac{A_3 R^+}{\mu^+} + 1 - 2R^+ & \frac{A_3 R^-}{\mu^-} - (1 - 2R^-) \end{pmatrix}. \]

Hence
\[ C_3^+ = \frac{1}{\rho^+ s D_0} [(A_2 \frac{R^-}{\mu^-} - A_3 (1 - 2R^-)) \tilde{B} - (A_1 \frac{R^-}{\mu^-} - A_3 r_1^-) \tilde{b}_3], \]
\[ C_3^- = \frac{1}{\rho^- s D_0} [A_2 \frac{R^+}{\mu^+} - A_3 (1 - 2R^+)] \tilde{B} - (A_1 \frac{R^+}{\mu^+} + A_3 r_1^+) \tilde{b}_3, \]
\[ D_0 = A_3 det \mathcal{L} = (-\mu^+(r^2 + |\xi|^2) - \mu^{-}(r^2 + |\xi|^2))(\frac{R^-}{\mu^-}(1 - 2R^+) + \frac{R^+}{\mu^+}(1 - 2R^-)) + 2(\mu^+r^+ + \mu^-r^-)(\frac{r_1^+ R^-}{\mu^-} + \frac{r_1^- R^+}{\mu^+}) + (r^+ + r^-)(r_1^+ (1 - 2R^-) + r_1^- (1 - 2R^+)). \]

Next, we find \( \tilde{\omega}_{\alpha} = C_{\alpha}^+ r^+ + i\xi_{\alpha} C_{3}^+ = -C_{\overline{\alpha}}^+ r^- + i\xi_{\alpha} C_{3}^- \) and \( \tilde{\omega}_3 = C^+ - r_1^+ C_{3}^+ \). By (1.11),
\[- \mu^+r^+ (C_{\alpha}^+ r^+ + i\xi_{\alpha} C_{3}^+) - \mu^-r^- (-C_{\overline{\alpha}} r^- + i\xi_{\alpha} C_{3}^-) + \mu^+(r^+ - r_1^+)i\xi_{\alpha} C_{3}^+ + \mu^- (r^- - r_1^-)i\xi_{\alpha} C_{3}^- = \tilde{b}_\alpha, \]
which implies
\[ \omega_{\alpha} = \tilde{\omega}_{\alpha} = \frac{-\tilde{b}_\alpha}{\mu^+ r^+ + \mu^- r^-} + \frac{i\xi_{\alpha} C_{3}^+ (r^+ - r_1^+)}{\mu^+ r^+ + \mu^- r^-}. \]

Hence (1.9), (1.10) can be written in the form
\[ \tilde{v}^+ = \tilde{\omega} e^{-r^+ y_3} + C_3^+ (r_1^+ - r^+)(i\xi_1, i\xi_2, -r_1^+)^T e_1^+(y_3), \quad y_3 > 0, \]
\[ \tilde{v}^- = \tilde{\omega} e^{r^- y_3} + C_3^- (r_1^- - r^-)(i\xi_1, i\xi_2, r_1^-)^T e_1^-(y_3), \quad y_3 < 0, \]
with \( \tilde{\omega}_1 \) given in (1.17) and
\[ C_3^+ (r_1^+ - r^+) = -\frac{1}{r^+ + r_1^+} \frac{\mu^+ + \mu_1^+}{2 \mu^+ + \mu_1^+} \frac{1}{D_0} ((A_2 \frac{R^-}{\mu^-} - A_3 (1 - 2R^-)) \tilde{B} - (A_1 \frac{R^-}{\mu^-} - A_3 r_1^-) \tilde{b}_3), \]
\[ C_3^- (r_1^- - r^-) = -\frac{1}{r^- + r_1^-} \frac{\mu^- + \mu_1^-}{2 \mu^- + \mu_1^-} \frac{1}{D_0} ((A_2 \frac{R^+}{\mu^+} - A_3 (1 - 2R^+)) \tilde{B} - (A_1 \frac{R^+}{\mu^+} + A_3 r_1^+) \tilde{b}_3). \]

The solution of problem (1.7) defined by (1.18) is unique in the class of functions decaying to zero exponentially, as \( |y_3| \to \infty \). Indeed, it is easily seen that the solution of a homogeneous problem satisfies the energy relation
\[ s||\tilde{w}||^2 + \frac{1}{2} ||\sqrt{\mu} \tilde{S}(\tilde{w})||^2 + ||\sqrt{\mu_1}(\sum_{\gamma=1}^2 i\xi_{\gamma} \tilde{w}_{\gamma} + \frac{d\tilde{w}_3}{dy_3})||^2 = 0, \]
where \( \| \cdot \| \) is the norm in \( L_2(\mathbb{R}) \), \( \mu = \mu^{\pm}, \mu_1 = \mu_1^{\pm}, \tilde{w} = \tilde{w}^{\pm} \) for \( \pm y_3 > 0 \). By the analog of the Korn inequality

\[
|\xi|^2 \|\tilde{w}\|^2 + \left\| \frac{d\tilde{w}}{dx_3} \right\|^2 \leq c\|\tilde{S}(\tilde{w})\|^2,
\]

we have

\[
\text{Res}\|\tilde{w}\|^2 + c(|\xi|^2\|\tilde{w}\|^2 + \left\| \frac{d\tilde{w}}{dy_3} \right\|^2) \leq 0,
\]

which implies \( \tilde{w} = 0 \), if \( \text{Res} \geq -c|\xi|^2 \) (in particular, \( \text{Res} \) can be also negative, if \( |\xi| > 0 \)).

Let us prove that (1.21) holds if \( \tilde{w} \) satisfies the homogeneous jump conditions in (1.7). We have

\[
(i\xi_1)^2\tilde{w}_1 = i\xi_1(i\xi_1\tilde{w}_1) = \frac{i\xi_1}{2}\tilde{S}_{11}(\tilde{w}),
\]

\[
(i\xi_2)^2\tilde{w}_1 = i\xi_2(i\xi_2\tilde{w}_1 + i\xi_1\tilde{w}_2) - i\xi_1(i\xi_2\tilde{w}_2) = i\xi_2\tilde{S}_{23}(\tilde{w}) - \frac{i\xi_1}{2}\tilde{S}_{22}(\tilde{w}),
\]

\[
\frac{d^2\tilde{w}_1}{dy_3^2} = \frac{d}{dy_3}(\frac{d\tilde{w}_1}{dy_3} + i\xi_1\tilde{w}_3) - i\xi_1\frac{d\tilde{w}_3}{dy_3} = \frac{d}{dy_3}\tilde{S}_{13}(\tilde{w}) - \frac{i\xi_1}{2}\tilde{S}_{33}(\tilde{w}),
\]

which implies

\[
\left\| \frac{d\tilde{w}_1}{dy_3} \right\|^2 + |\xi|^2\|\tilde{w}_1\|^2 = \int_{\mathbb{R}}(\sum_{j=1,2}i\xi_j\tilde{w}_1 \cdot F_j(y_3) + \frac{d\tilde{w}_1}{dy_3}F_3(y_3))dy_3,
\]

where \( F_j \) are linear combinations of \( \tilde{S}_{km}(\tilde{w}) \), in particular, \( F_3 = \tilde{S}_{13}(\tilde{w}) \). Using the Cauchy inequality, we obtain

\[
\left\| \frac{d\tilde{w}_1}{dy_3} \right\|^2 + |\xi|^2\|\tilde{w}_1\|^2 \leq c\|F\| \leq c\|\tilde{S}(\tilde{w})\|.
\]

The function \( \tilde{w}_2 \) is estimated in a similar way, in addition, we have

\[
|\xi_j|\|\tilde{w}_3\| \leq \|\tilde{S}_{j3}(\tilde{w})\| + \frac{d\tilde{w}_j}{dy_3} \leq c\|\tilde{S}(\tilde{w})\|, \quad j = 1, 2,
\]

and finally \( \frac{d\tilde{w}_3}{dy_3} = \frac{1}{2}\tilde{S}_{33}(\tilde{w}) \). This completes the proof of (1.21).

If \( \text{Res} > 0 \), then the above arguments are true for arbitrary non-zero \( \xi \in \mathbb{R}^2 \), in particular, the solution (1.18) of the problem (1.7) is unique in the class of functions \( \tilde{w}^{\pm} \in W_2^2(\mathbb{R}^{\pm}) \). It follows that \( D_0 \) is different from zero for arbitrary \( s \) with \( \text{Res} > 0 \) and \( \xi \in \mathbb{R}^2 \). In the opposite case the homogeneous system (1.14) and problem (1.7) would have non-zero solutions, which is not possible. Since \( D_0 \) is homogeneous, i.e., \( D_0(\lambda \xi, \lambda^2 s) = \lambda^2 D_0(\xi, s), \forall \lambda > 0 \), there holds

\[
|D_0(\xi, s)| \geq c(|s| + |\xi|^2)
\]

with \( c = \inf_{|\sigma|+|\eta|^2=1, \text{Res} > 0} |D_0(\sigma, \eta)| > 0 \). Hence the same inequality (with another constant \( c \)) is satisfied, if \( |\xi| \geq \pi/d_0 \) and \( \text{Res} \) is a negative number, small in comparison with \( \pi/d_0 \).
We pass to the analysis of a homogeneous problem (1.5). Elimination of $\tilde{\vartheta}$ leads to

$$\begin{cases}
\bar{p}s \tilde{w}_\alpha - (\sum_{\beta=1,2} i\xi_\beta T_{\beta\alpha}(\tilde{w})) + \frac{d}{dy_3} T_{3\alpha}(\tilde{w}) + \frac{\tilde{p}'(\bar{p})}{s} i\xi_\alpha (\sum_{\beta=1,2} i\xi_\beta \tilde{w}_\beta + \frac{d\tilde{w}_3}{dy_3}) = 0,
\bar{p} s \tilde{w}_3 - (\sum_{\beta=1,2} i\xi_\beta T_{\beta3}(\tilde{w})) + \frac{d}{dx_3} T_{33}(\tilde{w}) + \frac{\tilde{p}'(\bar{p})}{s} \frac{d}{dy_3} (\sum_{\beta=1,2} i\xi_\beta \tilde{w}_\beta + \frac{d\tilde{w}_3}{dy_3}) = 0,
\pm y_3 > 0,
\end{cases}$$

$$\begin{align*}
[\tilde{w}] &= 0, \quad [\mu (\frac{d\tilde{w}_\alpha}{dy_3} + i\xi_\alpha \tilde{w}_3)] = \tilde{b}_\alpha, \quad \alpha = 1, 2, \\
(2\mu + \mu_1) + \frac{\tilde{p}'(\bar{p})}{s} \frac{d\tilde{w}_3}{dy_3} + (\mu_1 + \frac{\tilde{p}'(\bar{p})}{s}) (\sum_{\beta=1,2} i\xi_\beta \tilde{w}_\beta) &= \tilde{b}_3, \quad y_3 = 0, \\
\tilde{w}, \tilde{\vartheta} &\to 0, \quad |y_3| \to \infty,
\end{align*}$$

(1.23)

Clearly, (1.23) can be viewed as problem (1.7) with $\mu_1^\pm$ replaced by $\mu_1^\pm(s) = \mu_1^\pm + \frac{\bar{p}^\pm p'\pm(\bar{\rho}^\pm)}{s}$ (and $\nu_1^\pm$ replaced by $\nu_1^\pm(s) = \nu_1^\pm + \frac{p'\pm(\bar{\rho}^\pm)}{s}$). The expressions $1/(2\nu^\pm + \nu_1^\pm)$ go over into

$$\frac{s}{a^\pm s + b^\pm}, \quad \sqrt{\frac{s^2}{a^\pm s + b^\pm} + |\xi|^2} \equiv r_{11}^\pm(\xi, s), \quad e_{11}^\pm(y_3) = \frac{e^{\mp r_{11}^\pm y_3} - e^{\mp r^\pm y_3}}{r_{11}^\pm - r^\pm},$$

where

$$a^\pm = 2\nu^\pm + \nu_1^\pm, \quad b^\pm = p^{\pm}(\bar{\rho}^\pm) > 0, \quad r_{11}^\pm = \sqrt{\frac{s^2}{a^\pm s + b^\pm} + |\xi|^2}.$$

We notice that all these expression are meaningful for negative $\text{Re} s = s_1$, if $|\xi| > 0$.

We proceed with reducing (1.3) to a similar homogeneous problem (with $f = 0, h = 0$). We set $f^\pm = 0, h^\pm = 0$ for $\pm y_3 > d_0$ and define $f_{\ast}^\pm$ and $h_{\ast}^\pm$ as extensions of $f^\pm$ and $h^\pm$, respectively, from $Q^\pm = \mathbb{Q}^\prime \times \mathbb{R}^\pm$ into $Q = Q^+ \cup Q^-$ with preservation of class. We set $u^\pm = u_1^\pm + u_2^\pm, \sigma^\pm = \sigma_1^\pm + \sigma_2^\pm$, where $u_1^\pm, \sigma_1^\pm$ satisfy

$$\begin{cases}
\bar{p} u_1^\pm_t - \nabla \cdot T^\pm(u_1^\pm) + p'(\bar{p}) \nabla \sigma_1^\pm = f_1^\pm, \\
\sigma_1^\pm + \bar{p} \nabla \cdot u_1^\pm = h_1^\pm, \quad y \in Q, \quad t > 0, \\
u_1^\pm(y, 0) = 0, \quad \sigma_1^\pm(y, 0) = 0, \quad y \in Q,
\end{cases}$$

(1.24)

and $u_2^\pm, \sigma_2^\pm$ solve the problem

$$\begin{cases}
\bar{p} u_2^\pm_t - \nabla \cdot T(u_2^\pm) + p'(\bar{p}) \nabla \sigma_2^\pm = 0, \\
\sigma_2^\pm + \bar{p} \nabla \cdot u_2^\pm = 0, \quad y \in Q^+, \quad t > 0, \\
u_2^\pm(y, 0) = 0, \quad \sigma_2^\pm(y, 0) = 0, \quad y \in Q^+, \quad t > 0.
\end{cases}$$

(1.25)

whereas $u_2^- = 0, \sigma_2^- = 0$. 

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Making the Fourier transform with respect to all the space variables \((y_1, y_2, y_3)\) and the Laplace transform with respect to \(t\),

\[
\mathcal{F}u = \tilde{u} = \int_{0}^{\infty} e^{-st} dt \int_{Q'} u(y, t) e^{-y' \cdot \xi'} - iy_3 \xi_3 dy',
\]

we reduce (1.24) to the algebraic system

\[
\begin{cases}
\mathbf{\rho}^\pm (s + \nu^\pm |\xi|^2) \tilde{u}_1^\pm + (\nu^\pm + \nu_1^\pm(s)) \mathbf{\xi} \cdot \tilde{u}_1^\pm = \mathbf{f}_\ast^\pm, \\
\mathbf{s} \tilde{\sigma}_1^\pm + \mathbf{\rho}^\pm (\mathbf{\xi} \cdot \tilde{u}_1^\pm) = \tilde{h}_\ast^\pm, \quad |\xi|^2 = |\xi'|^2 + \xi_3^2.
\end{cases}
\]  

(1.26)

We always assume that \(\xi_\alpha = k_\alpha \pi / d_0, k_\alpha = \pm 1, \pm 2, ..., \alpha = 1, 2\).

Elimination of \(\tilde{\sigma}_1^\pm\) leads to the system for \(\tilde{u}_1^\pm\):

\[
(s + \nu^\pm |\xi|^2) \tilde{u}_1^\pm + (\nu^\pm + \nu_1^\pm(s)) \mathbf{\xi} \cdot \tilde{u}_1^\pm = \frac{1}{\mathbf{\rho}^\pm} \left( \mathbf{f}_\ast^\pm - \frac{p'(\mathbf{\rho}^\pm) i \mathbf{\xi} \tilde{h}_\ast^\pm}{s} \right) \equiv \tilde{g}_\ast^\pm,
\]

(1.27)

the solution of which is given by

\[
\tilde{u}_1^\pm = \mathcal{H}^\pm \tilde{g}^\pm / (s + \nu^\pm |\xi|^2),
\]

(1.28)

where \(\mathcal{H}^\pm\) is the matrix with the elements

\[
H_{jk}^\pm = \delta_{jk} - \frac{(\nu^\pm + \nu_1^\pm(s)) \xi_j \xi_k}{s + (2\nu^\pm + \nu_1^\pm(s)) |\xi|^2} = \delta_{jk} - \frac{\xi_j \xi_k}{\frac{s}{a_1^\pm s + b^\pm} (s + \nu^\pm |\xi|^2) + |\xi|^2}.
\]

(1.29)

We notice that

\[
\mathcal{F} \nabla \cdot \mathbf{u}_1^\pm = i \mathbf{\xi} \cdot \tilde{u}_1^\pm = \frac{i \mathbf{\xi} \cdot \mathbf{\xi}}{s + (2\nu^\pm + \nu_1^\pm(s)) |\xi|^2} = \frac{s}{a_1^\pm s + b^\pm} \frac{\xi_j \xi_k}{(s + \nu^\pm |\xi|^2) + |\xi|^2}.
\]

(1.30)

To estimate the solution of (1.2) near the rigid wall \(S\), we need to consider one more model problem

\[
\begin{cases}
\mathbf{\rho} \partial_t w - \nabla \cdot T(w) + p'(\mathbf{\rho}) \nabla \vartheta = f, \\
\partial_t \vartheta + \mathbf{\rho} \nabla \cdot w = h, \quad y \in Q^+, \quad t > 0, \\
w(y', t) = \alpha(y', t), \quad y' \in \mathcal{Q}', \quad t > 0, \\
w(y, 0) = 0, \quad \vartheta(y, 0) = 0, \quad y \in Q^+
\end{cases}
\]  

(1.31)

with \(\mathbf{\rho} = \mathbf{\rho}^\ast, T(w) = T^\ast(w)\), under the assumption \(\text{supp} w, \text{supp} \vartheta \subset Q^+\).

Let \(f = 0, h = 0\). The Fourier-Laplace transform (1.4) with respect to \(y_1, y_2, t\) converts (1.31) into

\[
\begin{cases}
\mathbf{\rho} \tilde{w}_\alpha - \left( \sum_{\beta=1,2} i \xi_\beta \tilde{T}_{3\beta} (\mathbf{\tilde{w}}) + \frac{d}{dy_3} \tilde{T}_{3\alpha} (\mathbf{\tilde{w}}) \right) + p'(\mathbf{\rho}) i \xi_\alpha \tilde{\vartheta} = 0, \quad \alpha = 1, 2, \\
\mathbf{\rho} \tilde{w}_3 - \left( \sum_{\beta=1,2} i \xi_\beta \tilde{T}_{33} (\mathbf{\tilde{w}}) + \frac{d}{dy_3} \tilde{T}_{33} (\mathbf{\tilde{w}}) \right) + p'(\mathbf{\rho}) \frac{d}{dy_3} \tilde{\vartheta} = 0, \\
s \tilde{\vartheta} + \mathbf{\rho} \left( \sum_{\beta=1,2} (i \xi_\beta \tilde{w}_3 + \frac{d}{dy_3} \tilde{w}_3) \right) = 0, \quad \alpha = 1, 2, \\
\tilde{\vartheta} = \tilde{\alpha}, \quad y_3 = 0, \quad \tilde{w} \rightarrow 0, \quad y_3 \rightarrow +\infty.
\end{cases}
\]  

(1.32)
We also consider the problem

\[
\begin{aligned}
\bar{p}s\tilde{w}_{\alpha} - \left( \sum_{\beta=1,2} i\xi_{\beta}\bar{T}_{\beta\alpha}(\tilde{w}) + \frac{d}{dy_{3}} \bar{T}_{3\alpha}(\tilde{w}) \right) &= 0, \quad \alpha = 1, 2, \\
\bar{p}s\tilde{w}_{3} - \left( \sum_{\beta=1,2} i\xi_{\beta}\bar{T}_{\beta3}(\tilde{w}) + \frac{d}{dy_{3}} \bar{T}_{33}(\tilde{w}) \right) &= 0, \quad y_{3} > 0, \\
\tilde{w} &= \tilde{a}, \quad y_{3} = 0, \quad \tilde{w} \to 0, \quad y_{3} \to +\infty.
\end{aligned}
\tag{1.33}
\]

We assume that \( \xi = (\pi k_{1}/d_{0}, \pi k_{2}/d_{0}) \), \( k = \pm 1, \pm 2, \ldots \). The solution of (1.33) is given by the formula similar to (1.9), i.e.,

\[
\tilde{w}(\xi, s, y_{3}) = \begin{pmatrix}
    h_{1}r + i\xi_{1}h_{3} \\
    h_{2}r + i\xi_{2}h_{3} \\
    H - r_{1}h_{3}
\end{pmatrix}
\begin{pmatrix}
    i\xi_{1} \\
    i\xi_{2} \\
    -r_{1}
\end{pmatrix}
\begin{pmatrix}
    e^{-r_{1}y_{3}} - e^{-ry_{3}} \\
    r_{1} - r
\end{pmatrix}, \quad y_{3} > 0, \tag{1.34}
\]

where \( r = \sqrt{s/\nu - |\xi|^{2}}, \ r_{1} = \sqrt{s/(2\nu + \nu_{1}) + |\xi|^{2}}, \ H = \sum_{\beta=1}^{2} i\xi_{\beta}h_{\beta} \) (see [2]). The boundary conditions imply

\[
H_{\alpha}r + i\xi_{\alpha}h_{3} = a_{\alpha}, \quad \alpha = 1, 2, \\
Hr - \xi^{2}h_{3} = \tilde{A} = \sum_{\beta=1}^{2} i\xi_{\beta}\tilde{a}_{\beta}, \\
H - h_{3}r_{1} = \tilde{a}_{3},
\]

hence

\[
\begin{aligned}
    h_{3} &= \frac{\tilde{A} - r\tilde{a}_{3}}{rr_{1} - |\xi|^{2}}, \\
    H &= \tilde{a}_{3} + r_{1} \frac{\tilde{A} - r\tilde{a}_{3}}{rr_{1} - |\xi|^{2}}, \\
    h_{3}(r_{1} - r) &= \frac{(\tilde{A} - r\tilde{a}_{3})(r_{1} - r)}{rr_{1} - |\xi|^{2}} = \frac{\tilde{A} - r\tilde{a}_{3}(s/2\nu + \nu_{1} - 1/\nu)\nu}{sR^{2}} = -\frac{\nu + \nu_{1}}{2\nu + \nu_{1} + \sqrt{-r}}.
\end{aligned}
\tag{1.35}
\]

so (1.34) is equivalent to

\[
\tilde{w}(\xi, s, y_{3}) = \begin{pmatrix}
    \tilde{a}_{1} \\
    \tilde{a}_{2} \\
    \tilde{a}_{3}
\end{pmatrix}
\begin{pmatrix}
    e^{-r_{1}y_{3}} - e^{-ry_{3}} \\
    r_{1} - r
\end{pmatrix}, \quad y_{3} > 0, \tag{1.36}
\]

with \( h_{3}(r_{1} - r) \) given in (1.35).

The solution of (1.32) is obtained by replacing \( \nu_{1} \) with \( \nu_{1}(s) \) in (1.36), which yields

\[
\tilde{w}(\xi, s, y_{3}) = \begin{pmatrix}
    \tilde{a}_{1} \\
    \tilde{a}_{2} \\
    \tilde{a}_{3}
\end{pmatrix}
\begin{pmatrix}
    e^{-r_{11}y_{3}} - e^{-ry_{3}} \\
    r_{11} - r
\end{pmatrix}, \quad y_{3} > 0, \tag{1.37}
\]

with \( h_{3}(r_{11} - r) \) given in (1.35).
To reduce (1.31) to a similar homogeneous problem, we introduce auxiliary functions \( u_1 \) and \( \sigma_2 \) satisfying the relations analogous to (1.24):
\[
\begin{cases}
\overline{\rho}u_{1t} - \nabla \cdot T(u_{1}^\pm) + p'((\overline{\rho})\nabla\sigma_1 = f^*, \\
\sigma_{1t} + \overline{\rho}\nabla \cdot u_1 = h^*, \\
u_1(y, 0) = 0, \quad \sigma_1(y, 0) = 0, \quad y \in Q.
\end{cases}
\]
(1.38)
with \( \overline{\rho} = \overline{\rho} \) and \( f^* \) and \( h^* \) defined as the extensions of \( f \) and \( h \) into \( Q = Q' \times \mathbb{R} \) with preservation of class. By the Fourier-Laplace transform with respect to \( y_1, y_2, y_3, t \) (we assume that \( |\xi'| \geq \pi/d_0 \)) equations (1.38) reduce to
\[
\begin{cases}
\overline{\rho}(s + \nu|\xi|^2)\tilde{u}_1 + (\nu + \nu_1)|\xi|\tilde{u}_1 + p'(\overline{\rho})i\xi\tilde{\sigma}_1 = \tilde{f}^*, \\
s\tilde{\sigma}_1 + \overline{\rho}|\xi|\tilde{u}_1 = \tilde{h}^*, \\
|\xi|^2 = |\xi'|^2 + \xi_3^2.
\end{cases}
\]
(1.39)
which implies
\[
(s + \nu|\xi|^2)\tilde{u}_1 + (\nu + \nu_1(s))|\xi|\tilde{u}_1 = \frac{1}{\rho}(\tilde{f}^* - \frac{p'(\overline{\rho})i\xi\tilde{h}^*}{s}) \equiv \tilde{g}.
\]
(1.40)
As above, we have
\[
\tilde{u}_1 = \mathcal{H}\tilde{g}/(s + \nu|\xi|^2),
\]
(1.41)
where \( \mathcal{H} = (H_{jk})_{j,k=1,2,3} \)
\[
H_{jk} = \delta_{jk} - \frac{|\xi_j|}{\frac{s}{a_1s+b}(s+v|\xi|^2)+|\xi|^2}.
\]
(1.42)
Problems (1.5), (1.32) with \( |\xi| = 0 \) are treated in Section 2.

2 \( L_p \)-estimates.

In this section, we prove \( L_p \)-estimates for solutions of model problems studied in Section 1.
We obtain the following result.

**Theorem 1.** The functions \( w \) and \( \theta \) with \( \text{supp} w, \text{supp} \theta \subset \overline{\Omega} \) satisfying (1.3) with \( b(y', t) \) and vanishing for \( t = 0 \) are subject to the inequality
\[
\sum_{\pm}(\|e^{\beta t}w\|_{W_{p}^{2,1}(Q_{\infty}^\pm)} + \|e^{\beta t}\theta\|_{W_{p}^{1,0}(Q_{\infty}^\pm)} + \|e^{\beta t}\theta_t\|_{W_{p}^{1,0}(Q_{\infty}^\pm)})
\]
\[
\leq c(\sum_{\pm}(\|e^{\beta t}f\|_{L_{p}(Q_{\infty}^\pm)} + \|e^{\beta t}h\|_{W_{p}^{1,0}(Q_{\infty}^\pm)} + \|e^{\beta t}b\|_{W_{p}^{1-1/p,1/2-1/2p}(\mathfrak{Q}_{\infty}')})),
\]
(2.1)
where \( Q_\infty^\pm = \Omega^\pm \times (0, \infty), Q_\infty' = \Omega' \times (0, \infty), \beta > 0. \)

By \( W_{p}^{2,1}(\Omega \times (0, T)) \), \( \Omega \subset \mathbb{R}^n \), we mean the space \( W_{p}^{2,0}(\Omega \times (0, T)) \cap W_{p}^{1,1}(\Omega \times (0, T)) \), where
\[
W_{p}^{2,0}(\Omega \times (0, T)) = L_{p}(0, T; W_{p}^{2}(\Omega)), \quad W_{p}^{1,1}(\Omega \times (0, T)) = W_{p}^{1}(0, T; L_{p}(\Omega)).
\]
The norm in isotropic space \( W_{p}^{l}(\Omega) \) is given by
\[
\|u\|_{W_{p}^{l}(\Omega)}^{p} = \sum_{|l|\leq t} \int_{\Omega} |D^lu(x)|^{p}dx,
\]
if \( l \) is an integer, and
\[
\|u\|_{W^{l}(\Omega)}^{p} = \sum_{|j|=\lfloor l \rfloor} \int_{\Omega} \int_{\Omega} \frac{|D^{j}u(x) - D^{j}u(y)|^{p}}{|x - y|^{n + p\lambda}} \, dx,
\]
if \( l = \lfloor l \rfloor + \lambda, 0 < \lambda < 1 \). On the manifolds, the above spaces are defined in a standard way, i.e., with the help of the partition of unity and local maps.

The proof of Theorem 1 occupies the major part of the section. We start with estimates of \( u_{1} \) and \( \sigma_{1} \) satisfying (1.26), (1.27). We assume that \( e^{\beta t}f_{*}^{\pm} \leq c e^{\beta t}f^{\pm}, e^{\beta t}h_{*}^{\pm} \leq c e^{\beta t}h^{\pm} \), where \( Q_{\infty} = Q_{\infty}^{+} \cup Q_{\infty}^{-} \). The estimate of \( u_{1} \) is carried out on the basis of the Marcinkiewicz-Mikhlin-Lizorkin theorem:

**Theorem 2** [5,6,7]. Assume that \( \tilde{u} = \mathcal{F}u, \tilde{v} = \mathcal{F}v \) and
\[
\tilde{v} = m(\xi, s)\tilde{u}
\]
where \( \mathcal{F} \) is the Fourier-Laplace transformation \( \mathcal{F}u(\xi, s) = \int_{0}^{\infty} e^{-st} dt \int_{\mathbb{R}^{n}} e^{-i\xi \cdot x} u(x, t) dx \).

If
\[
M_{p}(m) \equiv \sup_{\text{Res}=s_{1}} \sup_{\xi \in \mathbb{R}^{n}} |\mathcal{M}m| \leq c
\]
(2.3)
where
\[
\mathcal{M}m = (|m(\xi, s)| + \sum_{j=1}^{n} \sum_{k_{i} \neq k_{j}} |\xi_{k_{1}} \cdots \xi_{k_{j}}| \left| \frac{\partial^{j}m}{\partial \xi_{k_{1}} \cdots \partial \xi_{k_{j}}} \right| + s \left| \frac{\partial m}{\partial s} \right| + \sum_{j \neq k} |s| |\xi_{j} \xi_{k}| \left| \frac{\partial^{3}m(\xi, s)}{\partial \xi_{1} \partial \xi_{2} \partial \xi_{3}} \right| + \sum_{j=1}^{3} |s| |\xi_{j}| \left| \frac{\partial^{3}m(\xi, s)}{\partial s \partial \xi_{1} \partial \xi_{2}} \right| + \sum_{j \neq k} |s| |\xi_{j}||\xi_{k}| \left| \frac{\partial^{3}m}{\partial s \partial \xi_{1} \partial \xi_{2} \partial \xi_{3}} \right|)
\]
(2.4)
then
\[
\|e^{\beta t}v\|_{L_{p}(\mathbb{R}^{n} \times \mathbb{R}_{+})} \leq c M_{p}(m) \|e^{\beta t}u\|_{L_{p}(\mathbb{R}^{n} \times \mathbb{R}_{+})}, \quad \beta = -\text{Res}.
\]
The function \( m \) with these properties is referred to as \( L_{p} \) Fourier multiplier; the set of the \( L_{p} \)-multipliers is denoted by \( \mathfrak{M} \).

The Marcinkiewicz-Mikhlin-Lizorkin theorem applies also to the Fourier series, if the Fourier coefficients are defined for all real values of \( \xi \) (see [8]). By repeating the arguments in [7] it is easy to show that the condition (2.3) should be replaced in our case with

\[
M_{p}(m) = \sup_{\text{Res}=s_{1}} \sup_{|\xi| \geq \pi/d_{0}} \left( |m(\xi, s)| + \sum_{j=1}^{3} |\xi_{j}| \left| \frac{\partial m}{\partial \xi_{j}} \right| + \sum_{j \neq k} |s| |\xi_{j} \xi_{k}| \left| \frac{\partial^{2}m(\xi, s)}{\partial \xi_{j} \partial \xi_{k}} \right| + \sum_{j=1}^{3} |s| |\xi_{j}| \left| \frac{\partial^{3}m(\xi, s)}{\partial s \partial \xi_{j}} \right| + \sum_{j \neq k} |s| |\xi_{j}||\xi_{k}| \left| \frac{\partial^{3}m}{\partial s \partial \xi_{j} \partial \xi_{k}} \right| \right) < \infty, \quad |\xi|^{2} = |\xi'_{1}|^{2} + \xi_{3}^{2}.
\]
(2.5)
for \( \tilde{m} = \tilde{H}_{jk} \).
Let us consider the expression

$$
\mathbb{P}^\pm(\xi, s) = \frac{s}{a^\pm s + b^\pm} (s + \nu^\pm |\xi|^2) + |\xi|^2 = \frac{sa^\pm |s|^2 + b^\pm s^2}{|a^\pm s + b^\pm|^2} + \frac{s\nu^\pm}{a^\pm s + b^\pm} |\xi|^2 + |\xi|^2.
$$

Since $|\frac{s_1 a^\pm |ss^2}{a^\pm s}|$ is bounded by a constant independent of $s_2$, there holds $|\mathbb{P}| \geq c(|s + |\xi|^2)$, if $d_0$ is small. Moreover,

$$
|M^{-1}P| \leq c|P^{-1}|, \quad M_p(\xi_1\xi_2\mathbb{P}^{-1}) + M_p(s\mathbb{P}^{-1}) \leq c.
$$

We represent $(\tilde{u}_1^\pm, \tilde{\sigma}_1)$ as the sum

$$
\tilde{u}_1^\pm = \tilde{w}_1 + \tilde{w}_2, \quad \tilde{\sigma}_1^\pm = \tilde{\phi}_1 + \tilde{\phi}_2,
$$

where $\tilde{w}_1, \tilde{\phi}_1$ satisfy (1.26) with $\tilde{h}_*^\pm = 0$ and $\tilde{g}_*^\pm = 0$, respectively. Hence $\tilde{w}_1$ is given by the formula (1.28) with $\tilde{h}_*^\pm = 0$ instead of $\tilde{g}$. Applying Theorem 2, we obtain

$$
\|e^{\beta t}w_1\|_{W^{2,1}_p(Q_{\infty})} \leq c\|e^{\beta t}F^{-1}\mathcal{H}\tilde{f}_*^\pm\|_{L_p(Q_{\infty})} \leq c\|e^{\beta t}f_*^\pm\|_{L_p(Q_{\infty}^\pm)},
$$

moreover, from the system of equations (1.26) for $\tilde{w}_1$ and $\tilde{\phi}_1$, we deduce

$$
\|e^{\beta t}\phi_1\|_{W^{1,0}_p(Q_{\infty})} + \|e^{\beta t}\nabla\phi_1\|_{W^{1,0}_p(Q_{\infty})} \leq c(\|e^{\beta t}\mathbb{P}^{-1}\|_{L_p(Q_{\infty})} + \|e^{\beta t}\mathbb{P}^{-1}\|_{L_p(Q_{\infty}^\pm)}).
$$

The functions $(\tilde{w}_2$ and $\tilde{\phi}_2)$ can be viewed as a solution of the transformed Stokes system:

$$
\begin{cases}
\tilde{\rho}^\pm(s + \nu^\pm |\xi|^2)\tilde{w}_2^\pm + p^\pm'(\tilde{\rho})i\xi\tilde{\phi}_2 = -(\nu^\pm + \nu_{1\pm})i\phi, \\
i\xi\cdot \tilde{w}_2^\pm = \tilde{\chi}.
\end{cases}
$$

It can be easily solved; the solution is given by

$$
\tilde{w}_2^\pm = -\frac{i\xi}{|\xi|^2}\tilde{\chi}, \quad \tilde{\phi}_2 = -\frac{1}{p^\pm'(\tilde{\rho})}((\nu^\pm + \nu_{1\pm})\tilde{\chi} + \frac{\tilde{\rho}^\pm(s + \nu^\pm |\xi|^2)}{|\xi|^2}\tilde{\chi}).
$$

From these formulas and from $s\tilde{\sigma}_2 = \tilde{h} - \tilde{\rho}^\pm \tilde{\chi}$ we deduce

$$
\|e^{\beta t}u_2\|_{W^{2,1}_p(Q_{\infty})} + \|e^{\beta t}\nabla\phi_2\|_{W^{1,0}_p(Q_{\infty})} + \|e^{\beta t}\nabla\phi_2\|_{W^{1,0}_p(Q_{\infty})} \leq c\|e^{\beta t}\mathbb{H}\tilde{f}_*^\pm\|_{L_p(Q_{\infty}^\pm)} \leq c\|e^{\beta t}\mathbb{H}\tilde{f}_*^\pm\|_{L_p(Q_{\infty}^\pm)},
$$
Collecting estimates, we arrive at
\[
\sum_{\pm} (\|e^{\beta t}u_{1}^{\pm}\|_{W^{2,1}_{p}(Q_{\infty}^{\pm})} + \|e^{\beta t}\sigma_{1}^{\pm}\|_{W^{1,0}_{p}(Q_{\infty}^{\pm})} + \|e^{\beta t}\sigma_{1t}^{\pm}\|_{W^{1,0}_{p}(Q_{\infty}^{\pm})})
\leq c \sum_{\pm} (\|f^{\pm}\|_{L_{p}(Q_{\infty}^{\pm})} + \|h^{\pm}\|_{W^{1,0}_{p}(Q_{\infty}^{\pm})}).
\] (2.7)

The problem (1.25) is treated below; we show that
\[
\|e^{\beta t}u_{2}\|_{W^{2,1}_{p}(Q_{\infty}^{+})} + \|e^{\beta t}\sigma_{2}^{\pm}\|_{W^{1,0}_{p}(Q_{\infty}^{+})} + \|e^{\beta t}\sigma_{2t}^{\pm}\|_{W^{1,0}_{p}(Q_{\infty}^{+})})
\leq c \sum_{\pm} \|e^{\beta t}u_{1}^{\pm}\|_{W^{2,1}_{p}(Q_{\infty}^{\pm})}.
\] (2.8)

Hence
\[
\sum_{\pm} (\|e^{\beta t}u^{\pm}\|_{W^{2,1}_{p}(Q_{\infty}^{\pm})} + \|e^{\beta t}\sigma^{\pm}\|_{W^{2,1}_{p}(Q_{\infty}^{\pm})} + \|e^{\beta t}\sigma_{t}^{\pm}\|_{W^{1,0}_{p}(Q_{\infty}^{\pm})})
\leq c \sum_{\pm} (\|e^{\beta t}f^{\pm}\|_{L_{p}(Q_{\infty}^{\pm})} + \|e^{\beta t}h^{\pm}\|_{W^{1,0}_{p}(Q_{\infty}^{\pm})}).
\] (2.9)

The differences \(w=v-u, \theta=\theta-\sigma\) solve the homogeneous problem (1.3) (with \(f=0, h=0\)).

We go back to (1.7). The solution of this problem is expressed by (1.18). We again make use of the Marcinkiewicz-Mikhlin-Lizorkin theorem for functions given in \(y' \in \mathfrak{Q}', t>0\), vanishing for \(t<0\) and periodic with respect to \(y'\). In this case the condition (2.3) should be taken in the form
\[
M_{p}(m) \equiv \sup_{\text{Res}=\pi} \sup_{|\xi|\geq \pi/d_{0}} |\mathcal{M}m| \leq c,
\] (2.10)

Moreover, we make use of the following proposition.

**Proposition 1.** Assume that \(P_{0}(\xi, s)/r^{+} \in \mathfrak{M}, P_{1}(\xi, s)/r^{+2} \in \mathfrak{M}\) (this means that \(m = P_{0}/r^{+}, P_{1}r^{+2}\) are subject to (2.10)). Then the functions \(w_{i}^{+}(x, t) = F^{-1}e^{-r^{+}(x_{3}+y_{3})}d_{i}(\xi, s, y_{3})\), where \(e_{1}^{+} = \frac{e^{-r_{1}^{+}(y_{3}^{+}+\text{e}^{-ry_{3}})}}{r_{1}^{+}}\) and \(d_{i}(x', 0) = 0\) (in the case \(p > 3\)), satisfy
\[
\|e^{\beta t}w_{i}^{+}\|_{L_{p}(Q_{\infty}^{+})} \leq c \|e^{\beta t}d_{i}\|_{W^{1-1/p, 1/(2p)}_{p}(Q_{\infty}^{+})}, \quad i = 0, 1.
\] (2.12)

**Proof.** Following Volevich [9], we represent \(w_{i}\) as follows:
\[
\tilde{w}_{0}^{+} = -\int_{0}^{\infty} P_{0}(\xi, s) \frac{d}{dy_{3}}(e^{-r^{+}(x_{3}+y_{3})}d_{0}^{+}(\xi, s, y_{3}))dy_{3} = 2 \int_{0}^{\infty} P_{0}e^{-r^{+}(x_{3}+y_{3})}r^{+}d_{0}^{+}(\xi, s, y_{3})dy_{3},
\]
\[
\tilde{w}_{1}^{+} = 2 \int_{0}^{\infty} e_{1}^{+}(x_{3}+y_{3})P_{1}r^{+}d_{1}^{+}(\xi, s, y_{3})dy_{3} + \int_{0}^{\infty} P_{1}e^{-r^{+}(x_{3}+y_{3})}d_{1}^{+}(\xi, s, y_{3})dy_{3},
\]
where
\[ \tilde{d}_i^+(\xi, s, y_3) = d_i^+(\xi, s)e^{-ry_3}. \]

Since \( \text{Res} = s_1 < 0 \) and \( d_i^+(t=0) = 0 \), there holds
\[ \|e^{\beta t}F^{-1}\tilde{d}_i^+\|_{L_p(Q_\infty^+)} \leq c\|e^{\beta t}d_i^+\|_{W_p^{1-1/p,1/2-1/2}(Q_\infty^+)}. \]

Now we use the inequalities
\[ |M_p r^+e^{-r^+t}| + |M_p r^+2e_1^+| \leq \frac{c}{t}, \quad t > 0, \]

obtained in [10] (or by direct calculation) and get
\[ \|e^{\beta t}w_0^+\|_{L_p(Q_\infty')} \leq c\int_0^\infty \|e^{\beta t}F^{-1}P_0\tilde{d}_0\|_{L_p(Q_\infty')} \frac{dy_3}{x_3+y_3}, \]
\[ \|e^{\beta t}w_1^+\|_{L_p(Q_\infty')} \leq c\int_0^\infty \|e^{\beta t}F^{-1}P_1r^+\tilde{d}_1^+\|_{L_p(Q_\infty')} \frac{dy_3}{x_3+y_3}. \]

From our assumptions concerning \( P_0 \) and \( P_1 \) and the continuity of the Hilbert transform, it follows
\[ \|e^{\beta t}F^{-1}P_0\tilde{d}_0\|_{L_p(Q_\infty^+)} \leq c\|e^{\beta t}F^{-1}r^+\tilde{d}_0\|_{L_p(Q_\infty^+)} \leq c\|e^{\beta t}d_0^+\|_{W_p^{1-1/p,1/2-1/2}(Q_\infty^+)}, \tag{2.13} \]
\[ \|e^{\beta t}F^{-1}P_1\tilde{d}_1^+\|_{L_p(Q_\infty^+)} \leq c\|e^{\beta t}F^{-1}r^+\tilde{d}_1^+\|_{L_p(Q_\infty^+)} \leq c\|e^{\beta t}d_1^+\|_{W_p^{1-1/p,1/2-1/2}(Q_\infty^+)}, \tag{2.14} \]
q.e.d. The functions \( w_0^-(x, t) = F^{-1}P_0e^{r^-y_3}Fd_0^- \) and \( w_1^-(x, t) = F^{-1}P_1e_1^-(y_3)Fd_1^- \) satisfy similar inequalities.

Using Proposition 1, formula (1.18), inequality (1.22), the relations
\[ \frac{de_1^+}{dy_3} = -r_1^+e^{-r_1^+} - r_1^+e_1^+(y_3) \]
and
\[ |M r| + |M r_1^-| \leq c|r|, \quad |M r^+| + |M r_1^-| \leq c|r|^{-1}, \]
\[ M_p(R^\mp) \leq c, \quad |MD_0^-| \leq c|r|^{-2}, \tag{2.15} \]
we obtain the desired estimate for the solution of a homogeneous problem (1.3):
\[ \sum_{\pm}\|e^{\beta t}w^\pm\|_{W_p^{2,1}(Q_\infty^\pm)} \leq c\|e^{\beta t}b\|_{W_p^{1-1/p,1/2-1/2}(Q_\infty^+)}. \]

Now, we turn to the problem (1.3), (1.7), (1.23). The solution of (1.23) is also given by (1.18), but with \( \nu_1^+(s) \) and \( r_1^+ \) instead of \( \nu_1^+(s) \) and \( r_1^+ \), respectively. We show that the assumptions of Proposition 1 still hold.

We consider \( r_{11} = \sqrt{\frac{s^2}{as+b} + |\xi|^2} \) and \( R_1 = \frac{r_{11}+\nu_1(s)}{r_{11}+r} \) (omitting indices \( \pm \)), where \( \nu_1(s) = \frac{\nu_2}{as+b} \).

We have
\[ r_{11}^2 = \frac{s_1a|s|^2 + b(s_1^2 - s_2^2)}{|as + b|^2} + is_2 \frac{a|s|^2 + 2bs_1}{|as + b|^2} + |\xi|^2. \]
Since \( \frac{s_{1}a|s|^{2}+b(s_{1}^{2}-s_{2}^{2})}{|as+b|^{2}} \) and \( \frac{a|s|^{2}+2bs_{1}}{|as+b|^{2}} \) are uniformly bounded by constants independent of \( s_{2} \), there hold
\[
 c_{1}(|s| + |\xi|^{2}) \leq |r_{11}| \leq c_{2}(|s| + |\xi|^{2})
\]
and
\[
 c_{3}(|s| + |\xi|^{2})^{1/2} \leq \text{Re} r_{11} \leq |r_{11}| \leq c_{4}(|s| + |\xi|^{2})^{1/2},
\]
if \( d_{0} \) is small. Moreover, easy calculations show that
\[
 |\mathcal{M}r_{11}| \leq c|r| \leq c(|s| + |\xi|^{2})^{1/2},
\]
\[
 |r_{1} - r_{11}| = \frac{|r_{1}^{2} - r_{11}^{2}|}{|r_{1} + r_{11}|} \leq c\frac{cs}{(as+b)(r_{1} + r_{11})} \leq \frac{c|r|}{|as+b|} \leq \delta|r|
\]
if \( |s| \geq h \) (\( \delta \to 0 \), if \( h \) grows without limits; we choose \( h \) in an appropriate way). This implies
\[
 |R_{1} - R| \leq c\delta, \quad c_{5} \leq |R_{1}| \leq c_{6}.
\]
(2.18)

The function \( D_{0} \) goes over into
\[
 D_{1} = (-\mu^{+}(r^{2+} + |\xi|^{2}) + \mu^{-}(r^{2-} + |\xi|^{2}))(\frac{R_{1}^{-}}{\mu^{-}}(1 - 2R_{1}^{+}) + \frac{R_{1}^{+}}{\mu^{+}}(1 - 2R_{1}^{-}))
\]
\[
 + 2(\mu^{+}r^{+} + \mu^{-}r^{-})(\frac{r_{11}^{+}R_{1}^{-}}{\mu^{-}} + \frac{r_{11}^{-}R_{1}^{+}}{\mu^{+}}) + (r^{+} + r^{-})(r_{11}^{+}(1 - 2R_{1}^{-}) + r_{11}^{-}(1 - 2R_{1}^{+}))
\]
(2.19)

Making use of (2.16) - (2.19) we obtain
\[
 |D_{1} - D_{0}| \leq \delta|r|^{2}, \quad c'|r|^{2} \leq |D_{1}| \leq c"|r^{2}|
\]
and
\[
 |\mathcal{M}D_{1}| \leq c|D_{1}|, \quad |\mathcal{M}D^{-1}| \leq c|D_{1}|^{-1}
\]
provided \( |s| \geq h \). If \( |s| \leq h \) and \( d_{0} \) is small, then \( |s| \leq \delta_{1}|\xi|^{2} \) with a small \( \delta_{1} \). In this case (2.20) can be proved by comparing \( D_{1}(\xi, s) \) with \( D_{1}(\xi, 0) \). It is easily seen that \( R_{1}^{\pm}(\xi, 0) = 1/2 \) and \( D_{1}(\xi, 0) = |\xi|^{2}(\mu^{+} + \mu^{-})(\frac{1}{\mu^{+}} + \frac{1}{\mu^{-}}) \), hence \( |D_{1}(\xi, s) - D_{1}(\xi, 0)| \leq c\delta_{1}|\xi|^{2} \), \( |D_{1}(\xi, s)| \geq c|r|^{2} \), if \( \delta_{1} \) is small. This shows that (2.20) hold also for \( |s| \leq h \).

The above arguments prove that the assumptions of proposition 1 are satisfied also for \( D_{y}^{J}D_{t}^{k}w(y, t) \), where \( \tilde{w} \) is a solution of (1.23). Hence \( w \) satisfies (2.15). The estimate of \( \theta \) is deduced from the equations
\[
 \left\{ \begin{array}{l}
 \bar{\rho}_{t}v_{t} - \nabla \cdot T(v) + p'(\bar{\rho})\nabla \theta = 0, \\
 \theta_{t} + \bar{\rho}\nabla \cdot v = 0, \quad y \in \mathbb{R}_{+}^{3} \cup \mathbb{R}_{-}^{3}
\end{array} \right.
\]
and as a result we obtain
\[
 \sum_{\pm}(|e^{\beta t}w||_{W_{p}^{2,1}(Q_{\infty}^{\pm})} + |e^{\beta t}\theta||_{W_{p}^{1,0}(Q_{\infty}^{\pm})} + |e^{\beta t}\bar{\rho}_{t}||_{W_{p}^{1,0}(Q_{\infty}^{\pm})}) \leq c||e^{\beta t}b||_{W_{p}^{1-1/p,1/2-1/2p}(Q_{\infty})},
\]
(2.21)

where \( \Omega_{\infty}^{\pm} = \Omega^{\pm} \times (0, \infty) \), \( \Omega_{\infty}^{k'} = \Omega^{k'} \times (0, \infty) \).
To complete the estimate of the solution of (1.3), we need to analyze the case \( \xi = 0 \) in (1.7), (1.26), i.e., to estimate the zero mode of the solution. If \( \xi = 0 \), then (1.3) is decomposed in two one-dimensional problems

\[
\begin{aligned}
\left\{ \begin{array}{l}
\overline{\rho} w_{\alpha t} - \mu \frac{d^2 w_{\alpha}}{dy_3^2} = f_{1\alpha}(y_3, t), \quad y_3 \in I^\pm, \\
[w_\alpha] = 0, \quad [\mu \frac{d w_\alpha}{dy_3}] = b_\alpha(t), \quad y_3 = 0, \\
w_\alpha|_{y_3=\pm d_0} = 0, \\
w_\alpha|_{t=0} = 0
\end{array} \right.
\end{aligned}
\]  

(2.22)

with \( \alpha = 1, 2 \), \( I^\pm = \{ \pm y_3 \in (0, d_0) \} \) and

\[
\left\{ \begin{aligned}
\overline{\rho} w_{3t} - (2\mu + \mu_1) \frac{d^2 w_3}{dy_3^2} + p'(\overline{\rho}) \frac{d \theta}{dy_3} &= f_3(y_3, t), \\
\vartheta_t + \overline{\rho} \frac{dw_3}{dy_3} &= h(y_3, t), \quad y_3 \in I^\pm, \\
[w_3] = 0, \quad [-p'(\overline{\rho}) \theta + (2\mu + \mu_1) \frac{dw_3}{dy_3}] &= b_3(t), \quad y_3 = 0, \\
w_3|_{y_3=\pm d_0} = 0, \\
w_3|_{t=0} = 0, \quad \vartheta|_{t=0} = 0
\end{aligned} \right.
\]  

(2.23)

(we recall that by our initial assumptions, \( w \) and \( \theta \) vanish for \( |y_3| \geq d_0 \).) The parabolic problem (2.22) is easily studied, and we restrict ourselves with the analysis of (2.23). We make the Laplace transform, eliminate \( \theta \) and replace \( \nu_1^\pm \) with \( \nu_1^+(s) \). This leads to

\[
\left\{ \begin{aligned}
s\tilde{w}_3^\pm -(2\nu^\pm + \nu_1^+(s)) \frac{d^2 \tilde{w}_3^\pm}{dy_3^2} &= \frac{1}{\overline{\rho}^\pm} \left( \frac{a_0^\pm s + b_0^\pm}{a_0 s + b_0} \right) \frac{d \bar{h}_3^\pm}{dy_3}, \quad y_3 \in I^\pm, \\
[w_3] = 0, \quad [(2\mu + \mu_1(s)) \frac{d w_3}{dy_3}] &= \bar{b}_3 + \left[ \frac{a_0^\pm s + b_0^\pm}{a_0 s + b_0} \right] \bar{h}_3, \quad y_3 = 0, \\
[\tilde{w}_3^\pm]_{y_3=\pm d} = 0
\end{aligned} \right.
\]  

(2.24)

which is equivalent to

\[
\left\{ \begin{aligned}
R^\pm(s) \tilde{w}_3^\pm - \frac{d^2 \tilde{w}_3^\pm}{dy_3^2} &= \frac{s f_3^\pm}{\overline{\rho}^\pm (a_0^\pm s + b_0^\pm)} - \frac{p'(\overline{\rho})^\pm}{a_0^\pm s + b_0^\pm} \frac{d \bar{h}_3^\pm}{dy_3}, \quad y_3 \in I^\pm, \\
[w_3] = 0, \quad [\lambda \frac{d \tilde{w}_3^\pm}{dy_3}] &= \frac{s}{a_0 s + b_0} \tilde{\bar{b}}_3 - \left[ \frac{a_0^\pm s + b_0^\pm}{a_0 s + b_0} \right] \tilde{\bar{h}}_3 \equiv \tilde{\epsilon}, \quad y_3 = 0, \\
[\tilde{w}_3^\pm]_{y_3=\pm d_0} = 0
\end{aligned} \right.
\]  

(2.24)

where

\[
R^\pm(s) = \frac{\overline{\rho}^\pm s^2}{a_0^\pm s + b_0^\pm}, \quad \lambda^\pm = \overline{\rho}^\pm a_0^\pm s + b_0^\pm \overline{\rho}^\pm a_0 s + b_0
\]

and \( a_0, b_0 \) are some fixed positive numbers, for instance, \( a_0 = \max(a^+, a^-) \), \( b_0 = \max(b^+, b^-) \).
We introduce $\tilde{u}_1^\pm$ as the solution of
\[
R^\pm \tilde{u}_1^\pm - \frac{d^2 \tilde{u}_1^\pm}{dy_3^2} = g^\pm, \quad y_3 \in I^\pm, \quad \tilde{u}_1^\pm = 0, \quad y_3 = 0, \pm d_0.
\]
To estimate $\tilde{u}_1^\pm$, we expand $g^\pm$ in the Fourier series in $\sin k \pi y_3^{d_0}$, $k = 1, 2, \ldots$. Then the Fourier coefficients $\hat{u}_1^\pm$ of $\tilde{u}_1^\pm$ are defined by
\[
\hat{u}_1^\pm(\xi, s) = \frac{\hat{g}^\pm(\xi, s)}{R^\pm(s) + |\xi|^2}, \quad \xi = k\pi/d_0, \quad k = 1, 2, \ldots.
\]
If $d_0$ is small, then, by the Marcinkiewicz theorem,
\[
\|e^{\beta t}\tilde{u}_1^\pm\|_{L^p(\mathbb{R}^2)} \leq c\|e^{\beta t}g^\pm\|_{L^p(\mathbb{R}^2)}, \quad I_{\infty}^\pm = I^\pm \times \mathbb{R}_+.
\]
The difference $\tilde{u}_2 = \tilde{w}_3 - \tilde{u}_1$ satisfies
\[
\begin{cases}
R^\pm \tilde{u}_2^\pm - \frac{d^2 \tilde{u}_2^\pm}{dy_3^2} = 0, & y_3 \in I^\pm, \\
[\tilde{u}_2] = 0, & \lambda \frac{d[\tilde{u}_2]}{dy_3} = -[\lambda \frac{d\tilde{u}_1}{dy_3}] = \tilde{A}(s), & y_3 = 0, \\
\tilde{u}_2^\pm = 0, & y = \pm d.
\end{cases}
\]
We need to compute the solution of this problem. It is given by
\[
u^+_2(y_3) = C^+(e^{r^+(d_0-y_3)} - e^{-r^+(d_0-y_3)}), \quad y_3 \in I^+,
\]
\[
u^-_2(y_3) = C^-(e^{r^-(d_0+y_3)} - e^{-r^-(d_0+y_3)}), \quad y_3 \in I^-,
\]
where
\[
r^\pm = \frac{s}{\sqrt{a^\pm s + b^\pm}}.
\]
From the jump conditions on the interface $y_3 = 0$ it follows that
\[
\mathcal{D}(C^+, C^-)^T = (\tilde{A}, 0)^T,
\]
where
\[
\mathcal{D} = \begin{pmatrix}
-\lambda^+ r^+(e^{r^+d_0} + e^{-r^+d_0}) & -\lambda^- r^+(e^{r^-d_0} + e^{-r^-d_0}) \\
e^{dor^+} & -e^{dor^-}
\end{pmatrix},
\]
so that
\[
D = \det \mathcal{D} = \lambda^+ r^+(e^{dor^+} + e^{-dor^+})(e^{dor^-} - e^{-dor^-}) + \lambda^- r^-(e^{dor^-} + e^{-dor^-})(e^{dor^+} - e^{-dor^+}).
\]
Hence
\[
C^+ = -\frac{\tilde{A}}{D}(e^{dor^-} - e^{-dor^-}), \quad C^- = -\frac{\tilde{A}}{D}(e^{dor^+} - e^{-dor^+}).
\]
We also compute
\[
\tilde{U}^\pm = \frac{d\tilde{u}_2^\pm}{dy_3}|_{y_3=0} = \mp r^\pm C^\pm(e^{r^\pm d_0} + e^{-r^\pm d_0}).
\]
These functions are connected with $\tilde{A}$ by
\[
\tilde{U}^+(s) = \frac{r^+ \tilde{A}}{D}(e^{d_0 r^+} + e^{-d_0 r^+})(e^{d_0 r^-} - e^{-d_0 r^-}) \equiv \mathcal{W}^+(\tilde{A}(s)),
\]
\[
\tilde{U}^-(s) = -\frac{r^- \tilde{A}(1+e^{-2d_0 r^-})(1-e^{-2d_0 r^+})}{\lambda+r^+(1+e^{-2d_0 r^+})(1-e^{-2d_0 r^-})+\lambda^- r^-(1+e^{-2d_0 r^-})(1-e^{-2d_0 r^+})} \equiv \mathcal{W}^-(\tilde{A}(s)).
\]
(2.28)

It is easily verified that the expressions $\mathcal{W}^{\pm}(s)$ are $L_p$-multipliers, hence
\[
\|e^{\beta t}U^\pm\|_{W_p^{1,2-1/(2p)}(\mathbb{R}_+)} \leq c\|e^{\beta t}A\|_{W_p^{1/2-1/2p}(\mathbb{R}_+)}. 
\]
(2.29)

Now we consider (2.27) as the union of two problems
\[
\begin{cases}
R^\pm \tilde{u}_2^\pm - \frac{d^2 \tilde{u}_2^\pm}{dy_3^2} = 0, & y_3 \in I^\pm, \\
\frac{d \tilde{u}_2^\pm}{dy_3} \bigg|_{y_3=0} = \tilde{U}^\pm, & \tilde{u}_2^\pm = 0, & y = \pm d. 
\end{cases}
\]
(2.30)

The solution of (2.30) can be estimated in the same way as $\tilde{u}_1$ above. We have $\tilde{u}_2^\pm = \tilde{V}^\pm + \overline{W}^\pm$, where
\[
\frac{d \tilde{V}^\pm}{dy_3} \bigg|_{y_3=0} = \tilde{U}^\pm, \\
\tilde{V}^\pm \bigg|_{y_3=0} = \tilde{V}^\pm \bigg|_{y_3=\pm d_0} = 0, \\
\|e^{\beta t}V^\pm\|_{W_p^{2,1}(I_\infty^\pm)} \leq c\|e^{\beta t}A\|_{W_p^{1/2-1/(2p)}(\mathbb{R}_+)}, \quad I_{\infty}^\pm = I^\pm \times (0, \infty),
\]
(2.31)
\[
\begin{cases}
R^\pm(s) \overline{W}^\pm - \frac{d^2 \overline{W}^\pm}{dy_3^2} = -R^\pm(s)\tilde{V}^\pm + \frac{d^2 \tilde{V}^\pm}{dy_3^2} = \overline{F}^\pm, & y_3 \in I^\pm, \\
\frac{d \overline{W}^\pm}{dy_3} \bigg|_{y_3=0} = 0, & \overline{W}^\pm \bigg|_{y_3=d_0} = 0.
\end{cases}
\]

We extend $\overline{F}^\pm$ as an even function of $y_3$ into a symmetric interval $I^\mp$ and apply the Laplace transform. It is clear that $\overline{W}^\pm$ is expressed by the formula similar to (2.25):
\[
\overline{W}^\pm(\xi, s) = \frac{\overline{F}(\xi,s)}{R^\pm(s) + |\xi|^2}.
\]
(2.32)

Applying the Marcinkiewicz theorem and making use of (2.31), we obtain
\[
\sum_{\pm} \|e^{\beta t}\tilde{u}_2^\pm\|_{W_p^{2,1}(I^\pm_{\infty})} \leq c \sum_{\pm} \|e^{\beta t}V^\pm\|_{W_p^{2,1}(I^\pm_{\infty})} \leq c\|e^{\beta t}A\|_{W_p^{1/2-1/(2p)}(\mathbb{R}_+)},
\]
which completes the proof of
\[
\sum_{\pm} \left(\|e^{\beta t}w_3\|_{W_p^{2,1}(I^\pm_{\infty})} + \|e^{\beta t}\theta\|_{W_p^{1,0}(I^\pm_{\infty})} + \|e^{\beta t}\theta_t\|_{W_p^{0,1}(I^\pm_{\infty})}\right) \\
\leq c \left(\sum_{\pm} \left(\|e^{\beta t}f_{13}\|_{L_p(I^\pm_{\infty})} + \|e^{\beta t}h\|_{W_p^{0,1}(I^\pm_{\infty})} + \|e^{\beta t}b\|_{W_p^{1/2-1/(2p)}(\mathbb{R}_+)}\right)\right).
\]
(2.33)
(the estimates of \( \vartheta \) are obtained from the equations (2.23)).

The solution of (1.24) satisfies similar inequalities.

Combining (2.33) with (2.9) and (2.21) we obtain inequality (2.1).

Next, we turn to the problem (1.31) with \( f = 0, h = 0 \) and with \( a \) vanishing for \( t = 0 \) (if \( p \geq 3/2 \)). Assuming as above that \( |\xi| \geq \pi/d_{0} \), we write (1.37) as

\[
\tilde{w}(\xi, s, y_{3}) = \left( \begin{array}{c} r\tilde{a}_{1} \\ r\tilde{a}_{2} \\ r\tilde{a}_{3} \end{array} \right) r^{-1} e^{-ry_{3}} + h_{3}(r_{11} - r) \left( \begin{array}{c} i\xi_{1} \\ i\xi_{2} \\ -r_{11} \end{array} \right) \frac{e^{-r_{11}ys} - e^{-ry_{3}}}{r_{11} - r} y_{3} > 0,
\]

(2.34)

\[
h_{3}(r_{11} - r) = \frac{a_{1}s + b}{as + b} \frac{\tilde{A} - r\tilde{a}_{3}}{r_{11} + \kappa(s)r}, \quad \kappa(s) = \frac{\nu s}{as + b}.
\]

It is easily seen that the assumptions of Proposition 1 are satisfied in the formula for \( \mathcal{F}D_{y}^{j}D_{t}^{k}w \), \( 2|j| + k \leq 2 \), the role of \( \tilde{d}_{i} \) being played by \( r\tilde{a}_{i} \). Hence there holds

\[
\|e^{\beta t}w\|_{W_{p}^{2,1}(Q_{\infty}^{+})} \leq c\|e^{\beta t}F^{-1}r\tilde{a}\|_{W_{p}^{1-1/p,1/2-1/2p}(Q_{\infty})} \leq c\|e^{\beta t}a\|_{W_{p}^{2-1/p,1/-2p}(Q_{\infty})} \leq c\|e^{\beta t}F^{-1}r\tilde{a}\|_{W_{p}^{1-1/p,1/2-1/2p}(Q_{\infty})}.
\]

(2.35)

We notice that this inequality justifies (2.7).

As for the non-homogeneous problem (1.31), it reduces to a homogeneous one exactly as (1.3); detail are omitted. For the solution of (1.31) we obtain the estimate

\[
\|e^{\beta t}w\|_{W_{p}^{2,1}(Q_{\infty}^{+})} + \|e^{\beta t}\sigma_{t}\|_{W_{p}^{1,0}(Q_{\infty}^{+})} + \|e^{\beta t}\sigma\|_{W_{p}^{1,0}(Q_{\infty}^{+})} \leq c(\|e^{\beta t}a\|_{W_{p}^{2-1/p,1/-2p}(Q_{\infty})} + \|e^{\beta t}f\|_{L_{p}(Q_{\infty}^{+})} + \|e^{\beta t}h\|_{W_{p}^{1,0}(Q_{\infty})}).
\]

(2.36)

If \( \xi = 0 \), then (1.32) reduces to two one-dimensional problems

\[
\begin{align*}
\bar{p}\partial_{t}\tilde{w}_{\alpha} - \frac{\mu}{\bar{\rho}} \frac{d^{2}\tilde{w}_{\alpha}}{dy_{3}^{2}} &= \tilde{f}_{\alpha}(y_{3}, t), \quad y_{3} \in I^{+}, \\
\tilde{w}_{\alpha}|_{y_{3}=0} &= \tilde{a}_{\alpha}, \quad \tilde{w}_{\alpha}|_{y_{3}=d_{0}} = 0,
\end{align*}
\]

(2.37)

\( \alpha = 1, 2, \) and

\[
\begin{align*}
\bar{p}\partial_{t}\tilde{w}_{3} - (2\mu + \mu_{1}) \frac{d^{2}\tilde{w}_{3}}{dy_{3}^{2}} + p'(\bar{\rho}) \frac{\partial}{\partial y_{3}} &= \tilde{f}_{3}(y_{3}, t), \\
\partial_{t} \tilde{w}_{3} + \bar{p} \frac{d\tilde{w}_{3}}{dy_{3}} &= h, \quad y_{3} \in I^{+}, \\
\tilde{w}_{3}|_{y_{3}=0} &= \tilde{a}_{3}, \quad \tilde{w}_{3}|_{y_{3}=d_{0}} = 0.
\end{align*}
\]

(2.38)

They are treated as (2.24), even much easier.

Finally, we mention the interior estimates of the solution of (1.3). The corresponding model problem is completely analogous to (1.24):

\[
\begin{align*}
\bar{p}w_{t} - \nabla \cdot T(w) + p'(\bar{\rho})\nabla \vartheta &= f, \\
\vartheta_{t} + \bar{p} \nabla \cdot w &= h, \quad (y, t) \in Q_{\infty}, \\
w(y, 0) &= 0, \quad \vartheta(y, 0) = 0, \quad y \in Q.
\end{align*}
\]

(2.39)
Repeating the above arguments, we obtain
\[
\|e^{\beta t}u\|_{W^{2,1}_{p}(Q_{\infty})} + \|e^{\beta t}\partial_{t}\|_{W^{1,0}_{p}(Q_{\infty})} + \|e^{\beta t}\Theta_{t}\|_{W^{1,0}_{p}(Q_{\infty})}
\leq c(\|e^{\beta t}f\|_{L_{p}(Q_{\infty})} + \|e^{\beta t}h\|_{W^{1,0}_{p}(Q_{\infty})}).
\tag{2.40}
\]

We have considered model problems only with zero initial data. This does not restrict generality, because problem (1.1) can be reduced to the problem with homogeneous initial data by constructing, after passage to the Lagrangian coordinates, auxiliary functions \(U^\pm(y, t), \Theta^\pm(y, t)\) such that
\[
U^\pm(y, 0) = v^\pm_0(y), \quad \Theta^\pm(y, 0) = \theta^\pm_0(y), \quad [U]_{y \in \Gamma_0} = 0,
\]
\[
U(y, t) = 0, \quad \Theta(y, t) = 0 \quad \text{for} \quad t > 2,
\]
\[
\sum_{\pm} \|U^\pm\|_{W^{2,1}_{p}(\Omega_{0}^\pm \times (0, 2))} \leq c \sum_{\pm} \|u^\pm_0\|_{W^{2-2/p}_{p}(\Omega_{0}^\pm)}
\leq c(\|e^{\beta t}f\|_{L_{p}(Q_{\infty})} + \|e^{\beta t}h\|_{W^{1,0}_{p}(Q_{\infty})})
\leq c(\|e^{\beta t}w\|_{W^{2,1}_{p}(Q_{\infty})} + \|e^{\beta t}\theta\|_{W^{1,0}_{p}(Q_{\infty})} + \|e^{\beta t}\theta_{t}\|_{W^{1,0}_{p}(Q_{\infty})})
\leq c(\|e^{\beta t}v\|_{L_{p}(\Omega_{0}\times(0, \infty))} + \|e^{\beta t}\theta\|_{L_{p}(\Omega_{0}\times(0, \infty))})
\tag{2.41}
\]

The existence of \(U^\pm\) with these properties follows from the trace theorem for the anisotropic Sobolev spaces, and \(\Theta^\pm\) can be taken in the form \(\Theta^\pm(y, t) = \theta^\pm_0(y) \zeta(t)\), where \(\zeta(t)\) is a smooth function vanishing for \(t > 2\) and equal to one for small \(t\).

Inequalities (2.2), (2.37), (2.39) in combination with localization procedure yield the following estimate for the linear problem (1.2):
\[
\sum_{\pm} \|e^{\beta t}v^\pm\|_{W^{2,1}_{p}(\Omega_{0}^\pm \times (0, \infty))} + \|e^{\beta t}\theta^\pm\|_{W^{1,0}_{p}(\Omega_{0}^\pm \times (0, \infty))} + \|e^{\beta t}\theta^\pm_{t}\|_{W^{1,0}_{p}(\Omega_{0}^\pm \times (0, \infty))}
\leq c(\sum_{\pm} (\|e^{\beta t}f^\pm\|_{L_{p}(\Omega_{0}^\pm \times (0, \infty))} + \|e^{\beta t}h^\pm\|_{W^{0,1}_{p}(\Omega_{0}^\pm \times (0, \infty))} + \|\theta^\pm_0\|_{W^{1,0}_{p}(\Omega_{0}^\pm)})
\leq c(\|e^{\beta t}v\|_{L_{p}(\Omega_{0}\times(0, \infty))} + \|e^{\beta t}\theta\|_{L_{p}(\Omega_{0}\times(0, \infty))})
\tag{2.42}
\]

The estimate of weighted \(L_{p}\)-norms of \(v\) and \(\theta\) can be obtained on the basis of the resolvent estimate of the operator corresponding to the problem (1.2) and of the standard results of the semi-group theory; this goes beyond the frames of the present paper.

**Remark.** We have studied model problems (1.3) and (1.31) in the class of functions decaying exponentially as \(t \to \infty\). Most often, such problems are treated by applying the integral Fourier-Laplace transformation. It is easily seen that the above arguments are applicable also in this case, but the exponential weight will be with \(\beta \leq 0\), which suffices for estimating solutions of (1.2) in a finite time interval.

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**References**

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