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Finite element methods for nearly incompressible media

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Abstract

We will summarize and analyze some finite element methods for analysis of nearly or completely incompressible media including linearly elastic solids and viscous Newtonian fluids. Numerical analysis of such problems is difficult especially in selecting appropriate finite element models, and the mixed FEM and discontinuous Galerkin FEM (DG FEM) are often effective to obtain reliable numerical solutions.

1 Introduction

We will present some finite element methods for analysis of nearly (or completely) incompressible media including elastic solids and viscous fluids. Numerical analysis of such problems is difficult especially in selecting appropriate finite element models. Especially, the genuine methods based on displacements or velocities only usually suffer from volumetric locking in the nearly incompressible range, so that various attempts have been made. Among them, the mixed and the stabilization methods are known to be effective in this respect. Nowadays, the discontinuous Galerkin methods combined with the mixed methods become to be realized to be more effective to obtain reliable numerical solutions. In this note, we will summarize some known results as well as our own ones.

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2 Nearly incompressible media

We will mainly discuss the solid cases below. Let $x = \{x_1, ..., x_N\}$ $(N = 2, 3)$ denote the Cartesian coordinates, and $\Omega \subset \mathbb{R}^N$ be a bounded domain occupied by the solid. We will use the notation of small displacements of solids as $\boldsymbol{u} = \{u_i\}_{1\leq i\leq N}$, and the associated small or linearized strains as

$$e_{ij}(\boldsymbol{u}) = (\partial_j u_i + \partial_i u_j)/2 \quad (\partial_i = \partial/\partial x_i;\ 1 \leq i, j \leq N), \tag{1}$$

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which can be regarded as a second-order symmetric tensor. The diagonal components e_{ii} are normal stretching strains, while the off-diagonal ones e_{ij} $(i \neq j)$ are shearing strains. Moreover, the volumetric strain is given by $\operatorname{div} \boldsymbol{u} = \sum_{i=1}^{N} e_{ii}(\boldsymbol{u})$, which plays important roles in nearly incompressible cases.

Remark 1 *For $N = 2$, we assume that the functions do not depend on x_3 and also that $u_3 = 0$ (in the 3D notations), so that $e_{33} = e_{i3} = e_{i3} = 0$ $(i = 1, 2)$.*

The stresses s_{ij} $(1 \leq i, j \leq 3)$ are also treated as a second-order symmetric tensor, and we assume the following generalized Hooke's law for isotropic solids:

$$s_{ij}(\boldsymbol{u}) = \lambda(\operatorname{div} \boldsymbol{u})\delta_{ij} + 2\mu e_{ij}(\boldsymbol{u}) \quad (\lambda > 0,\ \mu > 0\ :\ \text{Lamé's parameters}). \tag{2}$$

Notice that s_{33} may not vanish under the above relation. We will assume in addition that the solid is homogeneous, that is, λ and μ do not depend on x.

The static pressure is defined by the minus of mean normal stress, that is,

$$p := -\frac{1}{3}\sum_{i=1}^{3} s_{ii} = -(\lambda + \frac{2\mu}{3}) \operatorname{div} \boldsymbol{u} \quad \Rightarrow \quad s_{ij} = -\frac{\lambda}{\lambda_B} p + 2\mu\, e_{ij}(\boldsymbol{u}) \quad (\lambda_B := \lambda + 2\mu/3), \tag{3}$$

where λ_B is called the bulk modulus, and, as was noted, the term s_{33} is necessary. The above suggests that $\operatorname{div} \boldsymbol{u} \to 0$ as $\lambda \to +\infty$. (Under appropriate settings, we can also show $p \to p_\infty$ for some p_∞.) In some mathematical literatures, p is simply defined by $p = -\lambda \operatorname{div} \boldsymbol{u}$. Such a nonphysical definition may be more convenient for pure mathematical analysis.

The static equilibrium of stresses is expressed by the following Cauchy equations :

$$-\sum_{j=1}^{N} \partial s_{ij}/\partial x_j = f_i \quad (1 \leq i \leq N), \tag{4}$$

where $\boldsymbol{f} = \{f_i\}_{1 \leq i \leq N}$ is the distributed body force vector. Substituting (1) and (2) into (4), we have Navier's equations for isotropic homogeneous solids:

$$-\lambda\, \partial_i \operatorname{div} \boldsymbol{u} - \mu \sum_{j=1}^{N} \partial_j(\partial_j u_i + \partial_i u_j) = f_i \ \ (1 \leq i \leq N), \tag{5}$$

$$(\text{by (3)}) \quad \Rightarrow \quad \frac{\lambda}{\lambda_B}\partial_i p - \mu \sum_{j=1}^{N} \partial_j(\partial_j u_i + \partial_i u_j) = f_i \ \ (1 \leq i \leq N). \tag{6}$$

3 Weak formulations

For simplicity, we will consider the pure homogeneous Dirichlet boundary conditions, and use the usual Hilbertian Sobolev spaces $H^1(\Omega)$ and $H_0^1(\Omega)$. We will also denote the inner products of $L^2(\Omega)$ or $L^2(\Omega)^N$ by $(\cdot,\cdot)_\Omega$ and their associated norms by $\|\cdot\|_\Omega$. Below we will present two fundamental formulations for the present problem.

3.1 Displacement formulation in u only

The most fundamental weak formulation for finite $\lambda > 0$ is the following one using the displacement only.

[DF] *Given $f \in L^2(\Omega)$, find $u \in H_0^1(\Omega)^N$ that satisfies*

$$\lambda(\operatorname{div} u, \operatorname{div} v)_\Omega + 2\mu \sum_{i,j=1}^{N} (e_{ij}(u), e_{ij}(v))_\Omega = (f, v)_\Omega \quad (\forall v \in H_0^1(\Omega)^N). \tag{7}$$

For mathematical analysis of the above formulation, **Korn's inequalities** are essential, and a typical example of them is: There exists a domain constant $C > 0$ such that

$$\sum_{i,j=1}^{N} \|e_{ij}(v)\|_\Omega^2 + \|v\|_\Omega^2 \geq C\|v\|_{H^1(\Omega)^N}^2 \quad (\forall v \in H^1(\Omega)^N). \tag{8}$$

The present result is generalized to the Sobolev space $W^{1,p}(\Omega)$ with $1 < p < \infty$ for $N = 2$ [17]. For $v \in H_0^1(\Omega)^N$, the above becomes $\sum_{i,j=1}^{N} \|e_{ij}(v)\|_\Omega^2 \geq C\|v\|_{H^1(\Omega)^N}^2$ with possible change of $C > 0$, from which we can conclude the existence and uniqueness of the weak solution u of [DF]. Moreover, keeping μ constant, we have $\lambda \|\operatorname{div} u\|_\Omega^2 \leq C\|f\|_\Omega^2$, so that $\operatorname{div} u \to 0$ in $L^2(\Omega)$ as λ tends to $+\infty$.

3.2 Mixed formulation in u and p

To deal with the nearly and completely incompressible cases, it is natural to add p as an independent unknown function.

[MF] *Given $f \in L^2(\Omega)$, find $\{u, p\} \in H_0^1(\Omega)^N \times L^2(\Omega)$ that satisfies*

$$\begin{cases} 2\mu \sum_{i,j=1}^{N} (e_{ij}(u), e_{ij}(v))_\Omega - \frac{\lambda}{\lambda_B}(p, \operatorname{div} v)_\Omega = (f, v)_\Omega & (\forall v \in H_0^1(\Omega)^N), \\ -\frac{\lambda}{\lambda_B}[(\operatorname{div} u, q)_\Omega + \lambda_B^{-1}(p, q)_\Omega] = 0 & (\forall q \in L^2(\Omega)). \end{cases} \tag{9}$$

We can see formally that p is the Lagrange multiplier for the linear constraint $p + \lambda_B \operatorname{div} u = 0$. Also, by the Gauss formula, we find that $p \in L_0^2(\Omega)$ for $L_0^2(\Omega) = \{q \in L^2(\Omega); (q, 1)_\Omega = 0\}$.

To deal with [MF], it is essential to use the following *inf-sup condition*: *There exists a constant $\kappa > 0$ such that*

$$\inf_{v \in H_0^1(\Omega)^N \setminus \{0\}} \sup_{q \in L_0^2(\Omega) \setminus \{0\}} \frac{(\operatorname{div} v, q)_\Omega}{\|v\|_{H^1(\Omega)^N} \cdot \|q\|_\Omega} \geq \kappa. \tag{10}$$

This condition is related to the solvability of $\operatorname{div} v = g \in L_0^2(\Omega)$ for $v \in H_0^1(\Omega)^N$.

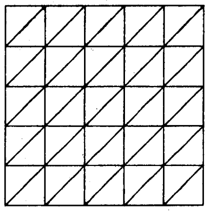
Using the above, we can conclude the existence and uniqueness of $\{u, p\}$ with

$$\|u\|_{H^1(\Omega)^N} + \|p\|_\Omega \leq C\|f\|_\Omega \quad \text{uniformly in } \lambda \geq \lambda_0 = \text{positive constant}. \tag{11}$$

Moreover, as λ tends to ∞, $\{u, p\}$ converges strongly in $H_0^1(\Omega)^N \times L^2(\Omega)$ to a $\{u_\infty, p_\infty\} \in H_0^1(\Omega)^N \times L_0^2(\Omega)$, which satisfies [MF] formally with $\lambda/\lambda_B = 1$ and $\lambda_B^{-1} = 0$:

$$\begin{cases} 2\mu \sum_{i,j=1}^{N} (e_{ij}(u_\infty), e_{ij}(v))_\Omega - (p_\infty, \operatorname{div} v)_\Omega = (f, v)_\Omega \quad (\forall v \in H^1(\Omega)^N), \\ -(\operatorname{div} u_\infty, q)_\Omega = 0 \quad (\forall q \in L^2(\Omega)) \quad \cdots\cdots \quad \textit{incompressibility condition}. \end{cases} \tag{12}$$

Notice that u_∞ is unique but $p_\infty \in L_0^2(\Omega)$ is not so when considered over the whole $L^2(\Omega)$.



For Frierdrichs-Keller meshes, kernel of $\mathrm{div}|_{(V_0^h)^2}$ *for* $k = 1$ *is* $\{0\}$.

Figure 1: Triangular meshes of Frierdrichs-Keller type for a square domain

4 FEM based on [DF]

The most standard FEM are based on the N–simplexes and the piecewise polynomial spaces P^k ($k \in \mathbb{N}$). More specifically, we consider a (regular) family of triangulations $\{\mathcal{T}^h\}_{h>0}$ of Ω by N– simplexes (K's), and introduce the finite element spaces

$$V^h = \{v \in H^1(\Omega);\ v|K \in P^k\ (\forall K \in \mathcal{T}^h)\}, \quad V_0^h = V^h \cap H_0^1(\Omega). \tag{13}$$

Then our discrete problem based on [DF] for each $\mathcal{T}^h$ is to find $\boldsymbol{u}_h \in (V_0^h)^N$ that satisfies

$$[DF]_h \qquad \lambda(\mathrm{div}\,\boldsymbol{u}_h, \mathrm{div}\,\boldsymbol{v}_h)_\Omega + 2\mu \sum_{i,j=1}^{N} (e_{ij}(\boldsymbol{u}_h), e_{ij}(\boldsymbol{v}_h))_\Omega = (\boldsymbol{f}, \boldsymbol{v}_h)_\Omega \quad (\forall \boldsymbol{v}_h \in (V_0^h)^N). \tag{14}$$

Unfortunately, such finite element models usually behave very poorly when λ becomes larger (*Locking*). In fact, for some meshes such as the Friedrichs-Keller ones (Fig. 1), $\boldsymbol{u}_h$ obtained by the piecewise linear (P^1) triangular elements tends to zero to keep the divergence term small, since we have the estimation $\lambda\|\mathrm{div}\,\boldsymbol{u}_h\|_\Omega^2 \le C\|\boldsymbol{f}\|_\Omega^2$ and the condition $\mathrm{div}\,\boldsymbol{u}_h = 0$ implies $\boldsymbol{u}_h = 0$ under the pure homogeneous Dirichlet condition.

5 FEM based on [MF]

This approach uses p besides $\boldsymbol{u}$ as independent unknown functions, so that we must also prepare a finite element element space $W^h \subset L_0^2(\Omega)$ for each $\mathcal{T}^h$.
$[\mathrm{MF}]_h$ *Given* $\boldsymbol{f} \in L^2(\Omega)$, *find* $\{\boldsymbol{u}_h, p_h\} \in (V_0^h)^N \times W^h$ *that satisfies*

$$\begin{cases} 2\mu \sum_{i,j=1}^{N} (e_{ij}(\boldsymbol{u}_h), e_{ij}(\boldsymbol{v}_h))_\Omega - \frac{\lambda}{\lambda_B}(p_h, \mathrm{div}\,\boldsymbol{v}_h)_\Omega = (\boldsymbol{f}, \boldsymbol{v}_h)_\Omega & (\forall \boldsymbol{v}_h \in (V_0^h)^N), \\ -\frac{\lambda}{\lambda_B}[(\mathrm{div}\,\boldsymbol{u}_h, q_h)_\Omega + \lambda_B^{-1}(p_h, q_h)_\Omega] = 0 & (\forall q_h \in W^h). \end{cases} \tag{15}$$

But for this approach to behave nicely, we should take full care of the combination of $\boldsymbol{u}_h$ and p_h to satisfy the discrete inf-sup condition. A typical approach is to use $(V_0^h)^N$ based on P^k for $\boldsymbol{u}_h$ and also W^h based on continuous or discontinuous P^{k_p} for p_h. Usually, we choose k_p to be smaller than k, but the validity of such approximations depends strongly on the arrangement of nodes as well. For example, the $P^1 - P^0$ triangle (continuous P^1 for u_h and discontinuous P^0 for p_h) is not appropriate, while $P^2 - P^0$ triangle works. Moreover, P^2– cont. P^1 works, while P^2– discont. P^1 behaves badly. See Fig.2 for a few typical combinations, and Boffi-Brezzi-Fortin[6] for more examples.

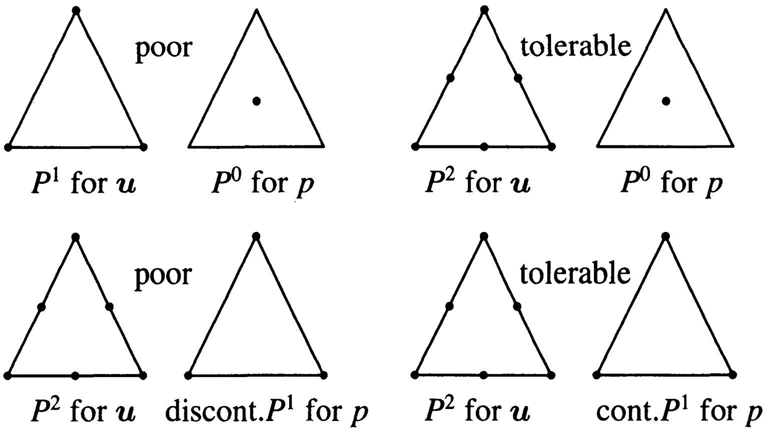


Figure 2: Some combinations of $\boldsymbol{u}$ and p for triangles

To analyze $[\mathrm{MF}]_h$, it is essential to show the following *discrete inf-sup condition*: *There exists a constant* $\kappa > 0$ *such that,* $\forall h > 0$,

$$\inf_{\boldsymbol{v}_h \in (V_0^h)^N \setminus \{0\}} \sup_{q_h \in W_0^h \setminus \{0\}} \frac{(\operatorname{div} \boldsymbol{v}_h, q_h)_\Omega}{\|\boldsymbol{v}_h\|_{H_0^1(\Omega)^N} \cdot \|q_h\|_\Omega} \geq \kappa . \tag{16}$$

A typical approach to show the above is to construct a *Fortin operator* $\Pi_h^F : H_0^1(\Omega)^N \to (V_0^h)^N$; $\forall \boldsymbol{v} \in H_0^1(\Omega)^N$,

$$\|\Pi_h^F \boldsymbol{v}\|_{H^1(\Omega)^N} \leq C \|\boldsymbol{v}\|_{H^1(\Omega)^N}, \quad (\operatorname{div}(\Pi_h^F \boldsymbol{v} - \boldsymbol{v}), q_h)_\Omega = 0; \ \forall q_h \in W^h. \tag{17}$$

A number of trials have been made to find such Π_h^F [6], though it is not so easy a task.

6 Hybrid discontinuous Galerkin FEM

We can also use discontinuous, more flexible approximate displacements based on the hybrid discontinuous Galerkin FEM (HDG FEM). In this case, we use discontinuous element-wise polynomial functions for $\boldsymbol{u}$ (and p) and also the so-called fluxes $\hat{\boldsymbol{u}}$ (inter-element displacements) as independent unknown functions [8, 13]. On the other hand, in the original discontinuous Galerkin (DG) FEM, fluxes are calculated from $\boldsymbol{v}$ [2].

We will consider only the 2-D cases ($N = 2$), and assume that Ω is a bounded polygonal domain with boundary $\partial\Omega$. Moreover, we will also use (possibly fractional) Sobolev spaces $W^{s,p}(\Omega)$ and $W_0^{s,p}(\Omega)$ for $1 \leq p \leq \infty$, $s \geq 0$, whose norm and standard semi-norm are denoted by $\|\cdot\|_{s,p,\Omega}$ and $|\cdot|_{s,p,\Omega}$, respectively. Moreover, s is omitted when $s = 0$. When $p = 2$, they are also written by $H^s(\Omega)$ and $H_0^s(\Omega)$, and the the subscript p in the norms and semi-norms is omitted. We will essentially deal with the Hilbertian cases ($p = 2$), but sometimes consider more general cases. To take the consistency between the case $p = 2$ and others ($p \neq 2$) for (semi-)norms, we for example define $\|\nabla u\|_{p,\Omega}$ by $\|\nabla u\|_{p,\Omega}^p = \sum_{i=1}^2 \|\partial u/\partial x_i\|_{p,\Omega}^p$ for $1 \leq p < \infty$ with the usual modification for $p = \infty$.

6.1 Triangulations by finite elements

We will use $\{\mathcal{T}^h\}_{h>0}$ as a "regular" family of triangulations of Ω. The precise meaning of "regular" is omitted here (see e.g. [3, 12]): roughly speaking, it means that the shapes of finite elements (see below) are not too much distorted and their sizes are comparable. In the present settings, each triangulation $\mathcal{T}^h$ consists of finite number of bounded m-polygonal finite elements K's, where m is an integer such that $3 \le m \le M$ ($M \ge 3$ is a constant) and can differ with K. The boundary of $K \in \mathcal{T}^h$ is a closed simple polygonal line and denoted by ∂K. Notice here that each finite element (or element, in short) K is not necessarily convex and vertexes with the flat angle are allowed. The number h_K stands for the diameter of K, and the mesh parameter h of the triangulation is defined by $h = \max_{K\in\mathcal{T}^h} h_K$. An (open) edge of K is designated by e, and its length by $|e|$. We define $\mathcal{E}^K$ and $\mathcal{E}^h$ as the sets of all edges in $K \in \mathcal{T}^h$ and $\mathcal{T}^h$, respectively. Moreover, $\Gamma^h = \cup_{e\in\mathcal{E}^h}\bar{e}$ is called the skeleton of $\mathcal{T}^h$. Almost everywhere on $\partial\Omega$ and ∂K, we can define the unit outward normal vector $\boldsymbol{n} = \{n_1, n_2\}$.

As duality pairings or inner products related to each element $K \in \mathcal{T}^h$, we will use:

$(\cdot,\cdot)_K$: duality pairing between $L^p(K)^\ell$ and $L^q(K)^\ell$ ($\ell = 1, 2,\ 1/p+1/q = 1$) as the extension of the inner product of $L^2(K)^\ell$, i. e., for example for $\ell = 1$, $(u, v)_K = \int_K u\,v\,dx$ ($u \in L^p(K)$, $v \in L^q(K)$),

$[\cdot,\cdot]_{\partial K}$, ($[\cdot,\cdot]_e$, resp.) : duality pairing between $L^p(\partial K)^\ell$ ($L^p(e)^\ell$ for edge $e \in \mathcal{E}^K$, resp.) and $L^q(\partial K)^\ell$ ($L^q(e)^\ell$, resp.) ($\ell = 1, 2,\ 1/p + 1/q = 1$) as the extension of the inner product of $L^2(\partial K)^\ell$ ($L^2(e)^\ell$, resp.).

Moreover, L^p type norms related to K are denoted by:

$\|\cdot\|_{p,K}$: norm of $L^p(K)^\ell$ ($\ell = 1, 2$),
$|\cdot|_{p,\partial K}$ ($|\cdot|_{p,e}$, resp.) : norm of $L^p(\partial K)^\ell$ ($L^p(e)^\ell$, resp.) ($\ell = 1, 2$).

We will often omit the suffix p of norms and inner products for $p = 2$.

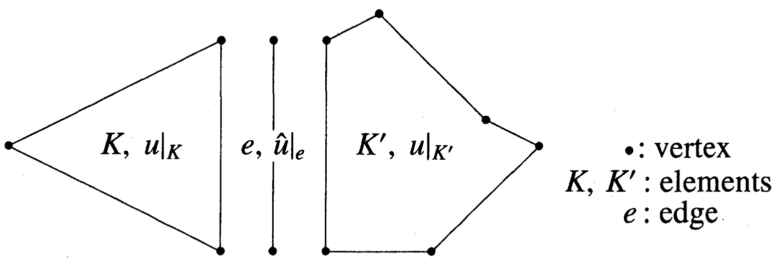


Figure 3: K, K', e, u and $\hat{u}$

6.2 Function spaces dependent on $\mathcal{T}^h$

For our purposes, it is essential to use the broken (or piecewise) Sobolev space $\Pi_{K\in\mathcal{T}^h} W^{s,p}(K)$ ($1 \le p \le \infty,\ s > 0$) on $\mathcal{T}^h$, which is identified with

$$W^{s,p}(\mathcal{T}^h) = \{v \in L^p(\Omega);\ v|_K \in W^{s,p}(K)\ (\forall K \in \mathcal{T}^h)\},\quad H^s(\mathcal{T}^h) = W^{s,2}(\mathcal{T}^h). \tag{18}$$

I is to be noted that, for $v \in W^{1/p+\gamma,p}(\mathcal{T}^h)$ $(\gamma > 0)$, the trace $v|_{\partial K}$ of $v|_K$ $(K \in \mathcal{T}^h)$ to ∂K belongs to $L^p(\partial K)$, and may be double-valued on an inter-element edge e shared by two elements K and K': $(v|_K)|_e$ may not coincide with $(v|_{K'})|_e$.

In HDG FEM, we also use L^p functions on the skeleton Γ^h, which are called *numerical fluxes*. It is important that each $\hat{v} \in L^p(\Gamma^h)$ is single-valued on Γ^h, unlike the traces of $v \in H^{\frac{1}{2}+\gamma}(\mathcal{T}^h)$ $(\gamma > 0)$ to Γ^h. To account for the homogeneous Dirichlet condition, we also introduce the following space on Γ^h:

$$L^p_D(\Gamma^h) = \{\hat{v} \in L^p(\Gamma^h)\,;\ \hat{v} = 0 \text{ on } \partial\Omega\} \tag{19}$$

In the HDG methods, the numerical flux $\hat{v}$ is independent of the function v taken for example from $W^{1,p}(\mathcal{T}^h)$. On the other hand, in the original DG methods, the flux is rather a subsidiary function, and, when necessary, it is derived from v. We will only consider the most typical derivation of $\hat{v}$: if $e \in \mathcal{E}^h$ is shared by two elements K and K', $\hat{v}|e$ is given by

$$\hat{v}|_e = \frac{1}{2}\left((v|_K)|_e + (v|_{K'})|_e\right), \tag{20}$$

while if $e \subset \partial\Omega$, $\hat{v}|_e$ is taken as either 0 or $v|_e$ in accordance with the homogeneous Dirichlet condition is considered or not [2]. The spaces thus induced are also denoted by $\hat{U}^h$ and $\hat{U}^h_D$.

Over $W^{1,p}(\mathcal{T}^h) \times L^p(\Gamma^h)$, define $W^{1,p}$-type semi-norm and norm for $\{v, \hat{v}\} \in W^{1,p}(\mathcal{T}^h) \times L^p(\Gamma^h)$ respectively by

$$|\{v,\hat{v}\}|^p_{1,p,h} = \|\nabla_h v\|^p_{p,\Omega} + \sum_{K\in\mathcal{T}^h}\sum_{e\in\mathcal{E}^K}\frac{1}{|e|^{p-1}}|v-\hat{v}|^p_{p,e}\,, \quad \|\{v,\hat{v}\}\|^p_{1,p,h} = \{v,\hat{v}\}|^p_{1,p,h} + \|v\|^p_{p,\Omega}\,, \tag{21}$$

where $\nabla_h v \in W^{1,p}(\mathcal{T}^h)^2$ is characterized by $(\nabla_h v)|K = \nabla(v|K) \in L^p(K)$ $(\forall K \in \mathcal{T}^h)$. The last term in $|\{v,\hat{v}\}|^p_{1,p,h}$ is used as a measure of discontinuity of v along the inter-element boundaries together with the discrepancy of v from the Dirichlet condition on $\partial\Omega$. A similar term with some coefficients will be used later as the *interior penalty* term.

6.3 Finite element spaces for HDG FEM

For $k \in \mathbb{N}$, let us prepare the following finite dimensional spaces:

$$U^h = \Pi_{K\in\mathcal{T}^h} P^k(K) \subset W^{2,\infty}(\mathcal{T}^h), \tag{22}$$

$$\hat{U}^h = \Pi_{e\in\mathcal{E}^h} P^k(e) \ \ (\text{or } \Pi_{e\in\mathcal{E}^h} P^k(e) \cap C(\Gamma^h)) \subset L^\infty(\Gamma^h), \quad \hat{U}^h_D = \hat{U}^h \cap L^\infty_D(\Gamma^h), \tag{23}$$

where $P^k(K)$ and $P^k(e)$ are polynomial spaces on K and e of degree $\leq k$, respectively. For $\hat{U}^h$, the case $\hat{U}_h \subset C(\Gamma^h)$ (space of continuous numerical fluxes) is in a sense natural as the space of traces from $W^{1,p}(\Omega)$ to Γ^h at least when $p \geq 2$. However, $\hat{U}^h$ whose elements can be discontinuous at vertexes is also used and may be sometimes superior to the continuous one.

Now typical examples of finite element spaces used for approximating scalar functions in $W^{1,p}(\Omega)$ and $W^{1,p}_0(\Omega)$ are, respectively,

$$V^h = U^h \times \hat{U}^h \subset W^{2,\infty}(\mathcal{T}^h) \times L^\infty(\Gamma^h), \qquad V^h_D = U^h \times \hat{U}^h_D \subset W^{2,\infty}(\mathcal{T}^h) \times L^\infty_D(\Gamma^h). \tag{24}$$

As was already noted, the approximate flux $\hat{v}$ is derived from $v \in U^h$ in the original DG FEM, so that $\hat{U}^h$ and $\hat{U}^h_D$ are not explicitly defined but determined from U^h implicitly.

6.4 Lifting operators

In DG FEM, various functions on edges and Γ^h appear and play essential roles. To associate them with some functions on K's, we often use the so-called lifting operators, whose typical examples are introduced below.

For $K \in \mathcal{T}^h$ and the former $k \in \mathbb{N}$, let us introduce $Q^K \subset L^\infty(K)$ as

$$Q^K = P^k(K) \text{ or } P^{k-1}(K). \tag{25}$$

Then the *local lifting operators* R_{Ki} $(i = 1, 2) : g \in L^p(\partial K) \mapsto \xi_i \in Q^K$ are defined by

$$(\xi_i, \eta)_K = [g, \eta\, n_i]_{\partial K} \quad (\forall \eta \in Q^K), \tag{26}$$

where $\boldsymbol{n} = \{n_1, n_2\}$ is the outward unit normal on ∂K. Clearly, $\xi_i = R_{Ki} g$ exists uniquely for each i $(= 1, 2)$.

The *global lifting operators* R_{hi} $(i = 1, 2)$ are given, roughly speaking, by assembling the local ones element by element. More precisely, with $Q^h := \Pi_{K \in \mathcal{T}^h} Q^K \subset L^\infty(\Omega)$, R_{hi} for each $i \in \{1, 2\}$ is defined by

$$R_{hi} : \tilde{g} = \{g_{\partial K}\}_{K \in \mathcal{T}^h} \in \Pi_{K \in \mathcal{T}^h} L^p(\partial K) \mapsto \{R_{Ki} g_{\partial K}\}_{K \in \mathcal{T}^h} \in Q^h. \tag{27}$$

The lifting operators of the present form are used to approximate such a boundary integral $[g, n_i \partial u/\partial x_i]_{\partial K}$ $(i \in \{1, 2\})$ by an integral over K. If u is approximated by a P^k function u_h in K, $\partial u_h/\partial x_i|_K$ is a P^{k-1} function, so that $(\xi_i, \eta) = [g, n_i \eta]_{\partial K}$ $(\xi_i = R_{Ki} g,\ \eta \in P^k$ or $P^{k-1})$ gives

$$(R_{Ki} g, \partial u_h/\partial x_i)_K = [g, n_i \partial u_h/\partial x_i]_{\partial K} \quad (i = 1, 2). \tag{28}$$

Moreover, when $k = 1$ and $Q^K = P^0(K)$, any P^1 function v_h in K satisfies $(\partial v_h/\partial x_i, \eta)_K = [v_h, n_i \eta]_{\partial K}$ by the Gauss formula, so that the relation $(R_{Ki} v_h, \eta)_K = [v_h, n_i \eta]_{\partial K}$ recovers $\partial v_h/\partial x_i$, i.e.,

$$R_{Ki}(v_h|_{\partial K}) = \partial v_h/\partial x_i \quad (k = 1,\ Q^K = P^0(K)). \tag{29}$$

It is to be noted that the present flux $\hat{v} \in L^p(\Gamma^h)$ is single-valued on $e \in \mathcal{E}^h$, and is considered an element of $\Pi_{K \in \mathcal{T}^h} L^p(\partial K)$. On the other hand, the trace of $v \in W^{1,p}(\mathcal{T}^h)$ to e can be double-valued. To use R_{hi} to such v, let us introduce the operator:

$$S_h : v \in W^{1,p}(\mathcal{T}^h) \mapsto \{v|_{\partial K}\}_{K \in \mathcal{T}^h} \in \Pi_{K \in \mathcal{T}^h} L^p(\partial K). \tag{30}$$

By using S_h, we can apply R_{hi} $(i = 1, 2)$ to an element such as $S_h v - \hat{v}$, which belongs to $\Pi_{K \in \mathcal{T}^h} L^p(\partial K)$ (also to $\Pi_{K \in \mathcal{T}^h} L^\infty(\partial K)$ in the present choice of V^h in (24) and Q^K in (25)), the domain of definition of R_{hi}.

7 Some theoretical results for DG FEM

In this section, we will present some theoretical results for DG FEM. In the former parts, the functions are essentially scalar ones, while vector functions will be also considered later. We will essentially consider the cases for $1 < p < \infty$, although some results may also hold for $p = 1$ and/or $p = \infty$.

7.1 Regularization of discontinuous functions

7.1.1 Assumptions on the family of triangulations

To derive various theoretical results for HDG FEM, we must impose some assumptions on $\{\mathcal{T}^h\}_{h>0}$. We have already stated that the number of elements in $\mathcal{T}^h$ is finite, and the number of edges of each bounded polygonal finite element $K \in \mathcal{T}^h$ is bounded from above by a positive integer M independently of h and K [3, 12].

We add a geometrical assumption called the chunkiness condition due to Deny-Lions and Brenner-Scott [3]. That is, for any $h > 0$ and $K \in \mathcal{T}^h$, there exists an open disk D_K of radius $\rho_K > 0$ inscribed to K such that K is star-shaped with respect to any points in D_K and

$$\rho_K / h_k \geq \eta, \tag{31}$$

where η is a positive constant common to all the elements in the family $\{\mathcal{T}^h\}_{h>0}$.

For the above D_K of K, let us denote its center by C^K. For each edge $e \in \mathcal{E}^K$, define by θ_e the interior angle at C^K of the triangle composed of C^K and e. Then, we further assume, for a positive constant θ_0,

$$\theta_e \geq \theta_0 \quad (\forall h > 0,\ \forall K \in \mathcal{T}^h,\ \forall e \in \mathcal{E}^K). \tag{32}$$

From this assumption, we can conclude the finiteness of number of edges of K but also that the edge length $|e|$ is bounded from below by a constant times h_K.

7.1.2 Evaluation of lifting operators by interior penalty terms

For analyzing the DG method using lifting operators, we must evaluate them in terms of the interior penalty terms. To do this, we can use the duality operators from $L^p(K)$ to $L^q(K)$ $(1 < p < \infty, q = p/(p-1))$ [5, 7], and then we have the following results.

Lemma 1 *For $h > 0$, $K \in \mathcal{T}^h$ and $i = 1, 2$, it holds for $g \in L^p(\partial K)$ that*

$$\|R_{Ki}g\|_{p,K} \leq C_p \Big[\sum_{e \in \mathcal{E}^K} \frac{1}{|e|^{p-1}} |g|^p_{p,e} \Big]^{1/p}, \tag{33}$$

where $C_p > 0$ is dependent on p but independent of g, h, K and i.

7.1.3 Vertex averaging operator

Let us first consider the vertex averaging operator for the numerical fluxes. Essentially the same idea was introduced by Brenner [4] for the original DG FEM, and has been used in many literatures. In the HDG FEM, such averaging process is unnecessary if $\hat{U}^h \subset C(\Gamma^h)$, while it is effective in general HDG and original DG FEM where numerical fluxes may be discontinuous at vertexes.

Let us consider the numerical flux $\hat{v}_h \in \hat{U}^h$, where $\hat{U}^h$ is either independent of U^h in the present HDG FEM or derived from U^h in the original (non-hybrid) DG FEM. In any cases, they are piecewise polynomials on Γ^h. Then for each vertex $\boldsymbol{p} \in \Gamma^h$, define $(\mathcal{A}_h \hat{v}_h)(\boldsymbol{p})$ by

$$(\mathcal{A}_h \hat{v}_h)(\boldsymbol{p}) = \frac{1}{|\Upsilon_{\boldsymbol{p}}|} \sum_{e \in \Upsilon_{\boldsymbol{p}}} \lim_{q \in e \to \boldsymbol{q}} \hat{v}_h(\boldsymbol{q}), \tag{34}$$

where Υ_p is the set of edges that have p as an endpoint, i.e., $\Upsilon_p = \{e \in \mathcal{E}^h\,;\ \ p \in \bar{e}\}$, and $|\Upsilon_p|$ is the number of edges in Υ_p, which is finite and bounded from above by a positive constant under appropriate regularity conditions on $\{\mathcal{T}^h\}_{h>0}$. For a continuous flux, it clearly holds that $(\mathcal{A}_h\hat{v}_h)(p) = \hat{v}_h(p)$ for any vertex $p \in \Gamma^h$.

As Lemma 2.1 of [4], the following lemma holds.

Lemma 2 *There exists a positive constant common to $\{\mathcal{T}^h\}_{h>0}$ such that, for any $\{v_h, \hat{v}_h\} \in U^h \times \hat{U}^h$, any vertex $p \in \Gamma^h$ and any $e \in \Upsilon_p$,*

$$| \lim_{q\in e\to p} \hat{v}_h(q) - (\mathcal{A}_h\hat{v}_h)(p)| \le C \sum_{e\in\Upsilon_p} \sum_{K\in\mathcal{T}^e} | \lim_{q\in K\to p} v_h(q) - \lim_{q^*\in e\to p} \hat{v}_h(q^*)| \,, \tag{35}$$

where $\mathcal{T}^e$ is the set of elements that have $e \in \mathcal{E}^h$ as their edge, i.e., $\mathcal{T}^e = \{K \in \mathcal{T}^h;\ e \subset \partial K\}$.

7.1.4 Discrete inverse trace theorems

We will introduce a 2D discrete inverse trace, or trace lifting, operator, which maps continuous numerical fluxes in $\hat{U}^h \subset C(\partial K)$ to functions in $W^{1,p}(K)$.

Let us state the trivial 1D discrete trace theorem, which is shown by using the linear interpolation functions with two end-points used as the sampling points.

Lemma 3 *Let $I = [0, L]$ for $L > 0$, and a, $b \in \mathbb{R}$. Then there exists a unique linear polynomial function $\hat{v} \in C(I)$ such that $\hat{v}(0) = a$, $\hat{v}(L) = b$, and*

$$|\hat{v}|_{p,I} \le \left(\frac{2^{p-1}L}{p+1}\right)^{1/p} (|a|^p + |b|^p)^{1/p} \ \ (1 \le p < \infty),\ \ |\hat{v}|_{\infty,I} \le \max\{|a|, |b|\}\ \ (p = \infty)\,, \tag{36}$$

where $|\cdot|_{p,I}$ denotes the norm over $L^p(I)$.

The following lemma is a discrete analog of the inverse trace theorem, but in the original inverse trace theorem, functions on ∂K are taken from $W^{1-1/p,p}(\partial K)$, the trace space of $W^{1,p}(K)$. The assumption $\hat{v} \in C(\partial K)$ below is essential for $\hat{v}$ to belong to $W^{1-1/p,p}(\partial K)$ especially when $p \ge 2$.

Lemma 4 *Let $h > 0$, $K \in \mathcal{T}^h$, $k \in \mathbb{N}$, $1 < p < \infty$ and $\hat{v} \in \Pi_{e\in\mathcal{E}^K} P^k(e) \cap C(\partial K)$. Then there exists a function $v \in W^{1,p}(K)$ such that $v|_{\partial K} = \hat{v}$ and*

$$|v|_{1,p,K} + h_K^{-1}\|v\|_{p,K} \le C_p h_K^{1/p-1} |\hat{v}|_{p,\partial K}\,, \tag{37}$$

where h_K is the diameter of K, and C_p is a positive constant depending only on p under the present regularity conditions on $\{\mathcal{T}^h\}_{h>0}$.

7.1.5 Regularization of discontinuous functions in U^h

By Lemmas 2 and 3, we can construct a continuous flux from the original one, and the difference between them is bounded by the interior penalty. Then using Lemma 4 and the above continuous flux, we obtain a $W^{1,p}(\Omega)$ function, whose difference from the original $W^{1,p}(\mathcal{T}^h)$ function is again evaluated by the interior penalty term.

The present process is essentially the same as the use of the reconstruction presented by Brenner [4], and we have the following results.

Lemma 5 *Let us consider* $\{\mathcal{T}^h\}_{h>0}$ *and* $\{v_h, \hat{v}_h\} \in U^h \times \hat{U}^h$ *associated with a* $\mathcal{T}^h$. *Then there exists a function* $v^h \in W^{1,p}(\Omega)$ *such that*

$$\|\nabla v^h - \nabla_h v_h\|_{p,\Omega} + h^{-1}\|v^h - v_h\|_{p,\Omega} \leq C_p \left(\sum_{K\in\mathcal{T}^h} \sum_{e\in\mathcal{E}^K} \frac{1}{|e|^{p-1}} |v_h - \hat{v}_h|^p_{p,e} \right)^{1/p} \quad (h = \max_{K\in\mathcal{T}^h} h_K), \quad (38)$$

where $C_p > 0$ *depends only on* p *under the present regularity conditions on* $\{\mathcal{T}^h\}_{h>0}$.

Using the above results, we can derive the discrete versions of the Poincaré-Friedrichs inequalities, the Rellich-Kondrashov theorem, the Korn inequalities etc., which are all important tools in numerical functional analysis.

7.2 2D discrete Rellich-Kondrashov theorem

Since $W^{1,p}(\mathcal{T}^h)$ is much wider than $W^{1,p}(\Omega)$, we must prove various discrete versions of theorems related to the Sobolev spaces such as the Rellich-Kondrashov compactness theorem. The author derived the Rellich-type theorem ($p = 2$) [12, 14], and here give its extension to more general cases, i.e., discrete Rellich-Kondrashov theorem by using the discrete trace lifting theorem and Lemma 1. It is to be noted that the strong convergence in $L^p(\Omega)$ below is generalized to more general values other than p depending on the value of p, although we omit the detail here.

Theorem 1 *Under the regularity conditions* [3, 12] *of* $\{\mathcal{T}^h\}_{h>0}$, *consider any family* $\{\{u_h, \hat{u}_h\} \in V^h\}_{h>0}$ ($\{\{u_h, \hat{u}_h\} \in V^h_D\}_{h>0}$, *respectively*) *associated to* $\{\mathcal{T}^h\}_{h>0}$ *such that* $|\{u_h, \hat{u}_h\}|^p_{1,p,h} + \|u_h\|^p_{p,\Omega} \leq 1$. *Then there exist* $u_0 \in W^{1,p}(\Omega)$ ($u_0 \in W^{1,p}_0(\Omega)$, *respectively*) *and a sub-family, again denoted by* $\{\{u_h, \hat{u}_h\}\}_{h>0}$, *such that as* $h \downarrow 0$

$$u_h \to u_0 \text{ strongly in } L^p(\Omega), \quad u_h|_{\partial\Omega} \to 0 \text{ strongly in } L^p(\partial\Omega) \text{ if } \{u_h, \hat{u}_h\} \in V^h_D, \quad (39)$$

$$\nabla_h u_h + R_h(\hat{u}_h - S_h u_h) \to \nabla u_0 \text{ weakly in } L^p(\Omega)^2 \text{ with } R_h = \{R_{h1}, R_{h2}\}. \quad (40)$$

7.3 Approximate derivatives in DG FEM

In view of the former theorem, the lifting term in $\nabla_h u_h + R_h(\hat{u}_h - S_h u_h)$ is essential, and is related to the jumps on the inter-element boundaries: even when this term is not used, something more or less alike is necessary as an alternative.

As an approximation of usual derivative $\partial v/\partial x_i$ ($i = 1, 2$) for $v \in H^1(\Omega)$, we will use the following $\partial_{h,i}\{v, \hat{v}\}$ for $\{v, \hat{v}\} \in H^1(\mathcal{T}^h) \times L^2(\Gamma^h)$ based on the lifting operator R_{Ki}:

$$(\partial_{h,i}\{v, \hat{v}\})|_K = \partial(v|_K)/\partial x_i + R_{Ki}(\hat{v}|_{\partial K} - (v|_K)|_{\partial K}) \quad (i = 1, 2).$$

In the present HDG FEM, we will use such approximate derivatives instead of the usual ones in the classical FEM.

8 2D plain strain problem

8.1 Preparations for HDG FEM

As the 2D displacement in our approach, we will use $\tilde{\boldsymbol{u}} = \{\boldsymbol{u}, \hat{\boldsymbol{u}}\} \in H^1(\mathcal{T}^h)^2 \times L^2(\Gamma^h)^2$ with $\boldsymbol{u} = \{u_1, u_2\}$ and $\hat{\boldsymbol{u}} = \{\hat{u}_1, \hat{u}_2\}$. Then, as approximate derivatives to be used for approximate strains, we adopt $\partial_{h,i}\{\boldsymbol{v}, \hat{\boldsymbol{v}}\}$ $(i = 1, 2)$ for $\{\boldsymbol{v}, \hat{\boldsymbol{v}}\} \in H^1(\mathcal{T}^h)^2 \times L^2(\Gamma^h)^2$ as proposed in the preceding subsection. Thus the approximate strains $e_{h,ij}(\tilde{\boldsymbol{u}})$ $(1 \le i, j \le 2)$ and divergence are expressed by

$$e_{h,ij}(\tilde{\boldsymbol{u}})|_K = \tfrac{1}{2}(\partial_{h,j}\{u_i, \hat{u}_i\} + \partial_{h,i}\{u_j, \hat{u}_j\}), \quad \mathrm{div}_h\, \tilde{\boldsymbol{u}} = \sum_{i=1}^{2} e_{h,ii}(\tilde{\boldsymbol{u}}), \tag{41}$$

while the stresses are derived from the strains by the generalized Hooke law:

$$s_{h,ij}(\tilde{\boldsymbol{u}}) = \lambda\delta_{ij}\, \mathrm{div}_h \tilde{\boldsymbol{u}} + 2\mu\, e_{h,ij}(\tilde{\boldsymbol{u}}) \quad (1 \le i, j \le 2) \text{ with } p_h(\tilde{\boldsymbol{u}}) = -\lambda_B \mathrm{div}_h\, \tilde{\boldsymbol{u}}. \tag{42}$$

We will essentially use the following bilinear form for $\tilde{\boldsymbol{u}}$, $\tilde{\boldsymbol{v}} \in H^{\frac{3}{2}+\gamma}(\mathcal{T}^h)^2 \times L^2(\Gamma^h)^2$ with $\gamma > 0$:

$$\begin{aligned} a_{h,\sigma}(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{v}}) = &\sum_{i,j=1}^{2} \Big\{ \Big[\sum_{K\in\mathcal{T}^h} (e_{ij}(\boldsymbol{u}), e_{ij}(\boldsymbol{v}))_K + [\hat{u}_i - u_i, e_{ij}(\boldsymbol{v})n_j]_{\partial K} + [\hat{v}_i - v_i, e_{ij}(\boldsymbol{u})n_j]_{\partial K} \Big] \\ &+ \frac{1}{4}(R_{hj}(\hat{u}_i - S_h u_i) + R_{hi}(\hat{u}_j - S_h u_j), R_{hj}(\hat{v}_i - S_h v_i) + R_{hi}(\hat{v}_j - S_h v_j))_\Omega \Big\} \\ &+ \sigma \sum_{K\in\mathcal{T}^h} \sum_{e\in\mathcal{E}^K} \sum_{i=1}^{2} \frac{1}{|e|}[u_i - \hat{u}_i, v_i - \hat{v}_i]_e, \end{aligned} \tag{43}$$

$$\begin{aligned} d_h(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{v}}) = &\sum_{K\in\mathcal{T}^h} \Big[(\mathrm{div}\, \boldsymbol{u}, \mathrm{div}\, \boldsymbol{v})_K + \sum_{i=1}^{2}([\hat{u}_i - u_i, (\mathrm{div}\, \boldsymbol{v})n_i]_{\partial K} + [\hat{v}_i - v_i, (\mathrm{div}\, \boldsymbol{u})n_i]_{\partial K}) \Big] \\ &+ \sum_{i,j=1}^{2} (R_{hi}(\hat{u}_i - S_h u_i), R_{hi}(\hat{v}_i - S_h v_i))_\Omega, \end{aligned} \tag{44}$$

where the assumption $\gamma > 0$ is required to assure the existence of some traces, the last term in $a_{h,\sigma}(\cdot,\cdot)$ is the interior penalty one, and $\sigma \ge 0$ is the penalty parameter which is usually chosen to be $O(1)$. Moreover, the terms involving R_{hi}'s are omitted in some primitive DG and HDG FEM [13], but then σ must be chosen sufficiently large to assure the positivity of the above bilinear forms.

8.2 Basic HDG FEM in $\tilde{\boldsymbol{u}}$ only

We will use the P^k finite element spaces (22) and (23) for $\tilde{\boldsymbol{u}} = \{\boldsymbol{u}, \hat{\boldsymbol{u}}\}$:

$$\tilde{V}^h = (U^h)^2 \times (\hat{U}^h)^2, \quad \tilde{V}^h_D = (U^h)^2 \times (\hat{U}^h_D)^2. \tag{45}$$

Then a typical FE formulation is, for a given $\boldsymbol{f} \in L^2(\Omega)^2$, to find $\tilde{\boldsymbol{u}}_h = \{\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h\} \in \tilde{V}_D^h$ s. t.

$$2\mu\, a_{h,\sigma}(\tilde{\boldsymbol{u}}_h, \tilde{\boldsymbol{v}}_h) + \lambda d_h(\tilde{\boldsymbol{u}}_h, \tilde{\boldsymbol{v}}_h)_\Omega = (\boldsymbol{f}, \boldsymbol{v}_h)_\Omega \quad (\forall \tilde{\boldsymbol{v}}_h = \{\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h\} \in \tilde{V}_D^h), \tag{46}$$

where we can rewrite the approximate bilinear forms by

$$a_{h,\sigma}(\tilde{\boldsymbol{u}}_h, \tilde{\boldsymbol{v}}_h) = \sum_{i,j=1}^{2} (e_{h,ij}(\tilde{\boldsymbol{u}}_h), e_{h,ij}(\tilde{\boldsymbol{v}}_h))_\Omega + \sigma \sum_{K \in \mathcal{T}^h} \sum_{e \mathcal{E}^K} \sum_{i=1}^{2} \frac{1}{|e|} [u_i - \hat{u}_i, v_i - \hat{v}_i]_e, \tag{47}$$

$$d_h(\tilde{\boldsymbol{u}}_h, \tilde{\boldsymbol{v}}_h) = (\mathrm{div}_h \tilde{\boldsymbol{u}}_h, \mathrm{div}_h \tilde{\boldsymbol{v}}_h)_\Omega . \tag{48}$$

The unique solvability of the above problem is straightforward provided that the Korn-type inequalities (see the subsequent subsection) are established together with some standard requirements in DG FEM[2]. However, its behaviors as $\lambda \to \infty$ is not clear at present. Moreover, under appropriate regularity conditions on the exact solution, we can derive various error estimates as routine works.

8.3 A discrete 2D Korn's inequality

For the validity of the present approximation, it is again essential to show discrete versions of Korn's inequalities. We have shown, for P^k/P^k approximation ($P^k(K)$ for u_i, $P^k(e)$ for $\hat{u}_i$) [15] for $p = 2$, which can be generalized for $1 < p < \infty$:

Lemma 6 *There exists a constant $C_{p,\Omega} > 0$ (dependent on p ($1 < p < \infty$) and Ω), s. t., for any small h ($0 < h \le h_0$) and any $\tilde{\boldsymbol{v}}_h = \{\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h\} = \{\{v_{h1}, v_{h2}\}, \{\hat{v}_{h1}, \hat{v}_{h2}\}\} \in \tilde{V}^h$,*

$$\sum_{K \in \mathcal{T}^h} \sum_{i,j=1}^{2} \left\| \frac{1}{2}\left(\partial_j v_{hi} + \partial_i v_{hj}\right) \right\|_{p,K}^p + \sum_{i=1}^{2} \sum_{K \in \mathcal{T}^h} \sum_{e \in \mathcal{E}^K} \frac{1}{|e|^{p-1}} |v_{hi} - \hat{v}_{hi}|_{p,e}^p + \|\boldsymbol{v}_h\|_{p,\Omega}^p \ge C_p \|\tilde{\boldsymbol{v}}_h\|_{1,p,h}^p \tag{49}$$

where the term $\|\boldsymbol{v}\|_{p,\Omega}^p$ can be omitted for $\tilde{\boldsymbol{v}} \in \tilde{V}_D^h$.

Remark 2 *For the non-conforming P^1 triangle of Crouzeix-Raviart, Korn-type inequalities do not hold. If the mesh contains the patch in Fig. 4 and $\boldsymbol{u}_h = \{u_{h1}, u_{h2}\}$ is zero outside it, this non-conforming displacement cannot satisfy Korn-type inequalities* [11], *although it certainly does for the completely incompressible case where the divergence-free condition is imposed element-wise, see* [1]. *Notice that this triangular finite element method is the most elementary DG FEM.*

8.4 Mixed type HDG FEM in $\tilde{u}$ and p

It is also possible to add p as an independent unknown function, and such an approach is an mixed type HDG FEM. We introduce a space W^h for p such as $\Pi_{K \in \mathcal{T}^h} P^{k_p}(K)$ with $0 \le k_p \le k$, and also use $W_0^h = W^h \cap L_0^2(\Omega)$. Then a mixed HDG FEM is to find $\{\tilde{\boldsymbol{u}}_h, p_h\} \in \tilde{V}_D^h \times W^h$ s. t.

$$\begin{cases} 2\mu\, a_{\sigma,h}(\tilde{\boldsymbol{u}}_h, \tilde{\boldsymbol{v}}_h) + \frac{\lambda}{\lambda_B}(p_h, \mathrm{div}_h\, \tilde{\boldsymbol{v}}_h)_\Omega = (\boldsymbol{f}, \boldsymbol{v}_h)_\Omega & ;\ \forall \tilde{\boldsymbol{v}}_h \in \tilde{V}_D^h, \\ \frac{\lambda}{\lambda_B}[(\mathrm{div}_h\, \tilde{\boldsymbol{u}}_h, q_h)_\Omega + \lambda_B^{-1}(p_h, q_h)_\Omega] = 0 & ;\ \forall q \in W^h . \end{cases} \tag{50}$$

element	u_{h1}	u_{h2}
[1]	$1-2y/h$	$-1+2x/h$
[2]	$-1+2y/h$	$-1-2x/h$
[3]	$-1-2y/h$	$1+2x/h$
[4]	$1+2y/h$	$1-2x/h$

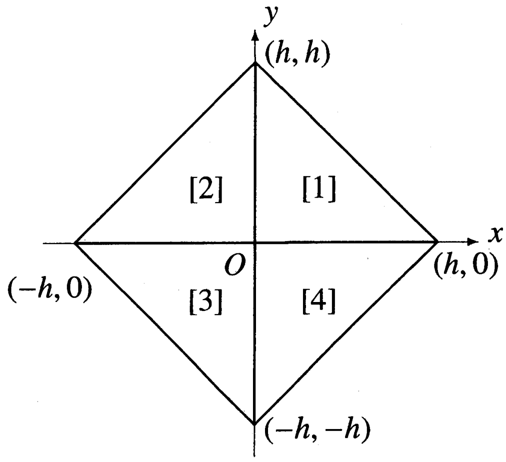


Figure 4: Example of patch of triangular elements

Remark 3 *Since* $\hat{\boldsymbol{u}}_h \in \hat{U}_D^h$, *we can show that* $(\mathrm{div}_h\, \tilde{\boldsymbol{u}}_h, 1)_\Omega = 0$ *by the Green formula. Thus the present* p_h *necessarily belongs to* W_0^h. *If* $\mathrm{div}_h\, \tilde{V}_D^h \subset W^h$ *and* λ *is finite, we have* $p_h = -\lambda_B\, \mathrm{div}_h\, \tilde{\boldsymbol{u}}_h$, *so that the present formulation coincides with the one in* $\boldsymbol{u}$ *only. Such a situation is realized if we choose* $k_p = k-1$ $(k \geq 1)$.

8.5 An inf-sup condition for mixed HDG FEM

For the validity of the present mixed HDG FEM, it is again essential to show some inf-sup conditions such as: *There exists a constant* $\kappa > 0$ *s. t., for all* h $(0 < h \leq h_0)$,

$$\inf_{\tilde{\boldsymbol{v}} \in \tilde{V}_D^h \setminus \{\tilde{0}\}} \sup_{q \in W_0^h \setminus \{0\}} \frac{(\mathrm{div}_h\, \tilde{\boldsymbol{v}}, q)_\Omega}{\|\tilde{\boldsymbol{v}}\|_{1,h} \cdot \|q\|_\Omega} \geq \kappa\,. \tag{51}$$

To show the above, it is again effective to construct a **Fortin operator** $\tilde{\Pi}_h^F : H_0^1(\Omega)^2 \to \tilde{V}_D^h$, which is characterized as follows: *There exists a positive constant* C *such that, for all* $\boldsymbol{v} \in H_0^1(\Omega)^2$,

$$\|\tilde{\Pi}_h^F \boldsymbol{v}\|_{1,h} \leq C \|\boldsymbol{v}\|_{H^1(\Omega)^2}\,, \quad ((\mathrm{div}_h\, \tilde{\Pi}_h^F - \mathrm{div})\, \boldsymbol{v}, q_h)_\Omega = 0\,; \forall q_h \in W^h\,. \tag{52}$$

Although it is in general difficult to find a nice Fortin operator, it is now known that its existence is assured in the arbitrary choice of $0 \leq k_p \leq k$ for the polygonal 2D elements with discontinuous numerical fluxes ($\hat{U}^h = \Pi_{e \in \mathcal{E}^h} P^{k_p}(e)$) [9]. This is in a sense an amazing result compared with the classical mixed FEM using $\boldsymbol{u}$ and p.

For some finite element spaces with (discontinuous) P^{k-1} approximation for W^h, the present mixed HDG FEM give identical results to the displacement HDG FEM for finite λ by eliminating p using the relation $p = -\lambda_B\, \mathrm{div}_h\, \tilde{\boldsymbol{u}}$, cf. Remark 3.

On the other hand, if we add the continuity condition on the numerical fluxes at vertexes ($\hat{U}^h \subset C(\Gamma^h)$), we can at present conclude the existence of the Fortin operators for $0 \leq k_p \leq k-2$ $(k \geq 2)$.

8.6 Reduced-order numerical fluxes

In [16], Oikawa proposed the use of P^{k-1} discontinuous numerical fluxes together with the P^k interior functions. As for the interior penalty term, the trace $(v|_K)|_e$ of $v \in P^k(K)$ ($K \in \mathcal{T}^h$, $e \in \mathcal{E}^K$) is replaced with its $L^2(e)$ projection to $P^{k-1}(e)$. He also pointed out its relation with the Crouzeix-Raviart P^1 nonconforming triangular element when $k = 1$ and K's are triangles. Except for $k = 1$ where Korn type inequalities may not hold, such approximations work well for the finite element analysis of nearly incompressible media.

8.7 Initial stress problems

We have considered the case where the linear elastic media are free from the initial stresses or strains. However, if the media are pre-stressed before further forces are applied, there may remain initial stresses. In modeling such phenomena, we often add some linear forms to the weak form (7). An example of such modified weak forms is given as follows:

$[\mathrm{DF}]_{IS}$ *Given* $\boldsymbol{f} \in L^2(\Omega)$ *and* $\{s^0_{ij}\} \in L^2(\Omega)^4$ $(1 \le i,\ j \le 2;\ s^0_{12} = s^0_{21})$, *find* $\boldsymbol{u} \in H^1_0(\Omega)^2$ *s. t.*

$$\lambda(\operatorname{div}\boldsymbol{u}, \operatorname{div}\boldsymbol{v})_\Omega + 2\mu\sum_{i,j=1}^{N}(e_{ij}(\boldsymbol{u}), e_{ij}(\boldsymbol{v}))_\Omega = (\boldsymbol{f}, \boldsymbol{v})_\Omega - \sum_{i,j=1}^{2}(s^0_{ij}, e_{ij}(\boldsymbol{v}))_\Omega \quad (\forall \boldsymbol{v} \in H^1_0(\Omega)^N). \tag{53}$$

It is clear that the solution $\boldsymbol{u}$ exists uniquely in $H^1_0(\Omega)^2$, but it may not be sufficiently smooth, so that the usual error estimation in terms of h may be difficult to derive.

In DG FEM for the present problem, it is a serious problem to express the right-hand side that is valid for $\tilde{V}^h$ or $\tilde{V}^h_D$. One possible remedy is to replace the strain terms e_{ij}'s in the right-hand side with their approximations $e_{h,ij}$'s. Then (46) should modified as, for all $\tilde{\boldsymbol{v}}_h = \{\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h\} \in \tilde{V}^h_D$,

$$2\mu\, a_{\sigma,h}(\tilde{\boldsymbol{u}}_h, \tilde{\boldsymbol{v}}_h) + \lambda(\operatorname{div}_h\tilde{\boldsymbol{u}}_h, \operatorname{div}_h\tilde{\boldsymbol{v}}_h)_\Omega = (\boldsymbol{f}, \boldsymbol{v}_h)_\Omega - \sum_{i,j=1}^{2}(s^0_{ij}, e_{h,ij}(\tilde{\boldsymbol{v}}_h))_\Omega \quad . \tag{54}$$

We can show the strong L^2 convergence of $\boldsymbol{u}_h$ and $\partial_{hi}\tilde{\boldsymbol{u}}_h$ respectively to the exact $\boldsymbol{u}$ and $\partial_i\boldsymbol{u}$ by using the discrete Rellich theorem, the discrete Korn-type inequalities as well as the lower semi-continuity of the L^2 norm, cf. [12].

9 Numerical Examples

We show some numerical results for a test problem, where the domain Ω is a unit square defined by $\{x = \{x_1, x_2\}; 0 < x_1 < 1,\ 0 < x_2 < 1\}$ and the exact solution is given by

$$\phi(s) := s^2(s-1)^2, \quad \Phi(x) = -\frac{1}{2}\phi(x_1)\,\phi(x_2), \quad \mathbf{u} = \operatorname{rot}\Phi + t\,\lambda^{-1}\operatorname{grad}\Phi, \tag{55}$$

where $t \ge 0$ is a parameter. In the present calculations, we take λ, μ and t as 5000, 1 and 1, respectively. Then the boundary condition associated with the above is the homogeneous

Dirichlet one, and f is specified by applying the operator of the Navier equations to the above $\boldsymbol{u}$.

In the numerical calculations, we choose σ unity and consider four triangulations numbered as $j = 1, 2, 3, 4$, which are obtained by using Gmsh [10] and by dividing each side of the square domain into $10 \times 2^{j-1}$ segments with equal length.

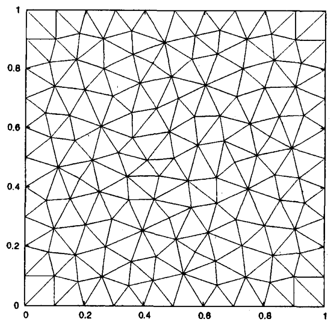
As finite element methods, we considered three types of P^1-based ones:

1. CG: Classical conforming P^1 triangular element.

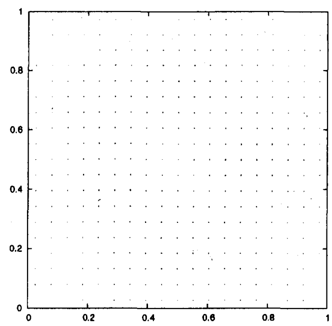
2. DG-D: Hybrid DG triangular element based on discontinuous P^1-$\boldsymbol{u}$ and discontinuous P^1-$\hat{\boldsymbol{u}}$, where the pressure can be calculated element-wise by the relation $p = -\lambda_B \mathrm{div}\, \boldsymbol{u}$.

3. DG-C: Hybrid DG triangular element based on discontinuous P^1-$\boldsymbol{u}$ and continuous P^1-$\hat{\boldsymbol{u}}$.

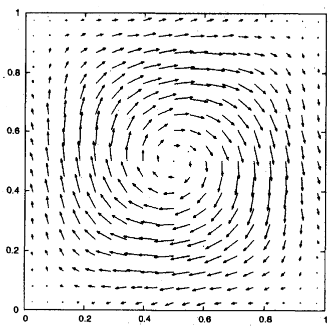
The results are shown in Fig. 5 through 8, and we can observe that the results may strongly depend on element types. That is, when triangulations are coarser, the results based on CG and DG-C are very poor: the displacements are much smaller than the exact one, which is very close to the ones based on DG-D in graphical level. The results are improved as the triangulations become finer, but we need much more computational costs than using the DG-D method. On the contrary, the results based on DG-D are robust to the fineness of triangulations as is expected theoretically. It is to be noted that, for smaller λ (not shown here), the difference is not so severe and sometimes *CG* may give better results.



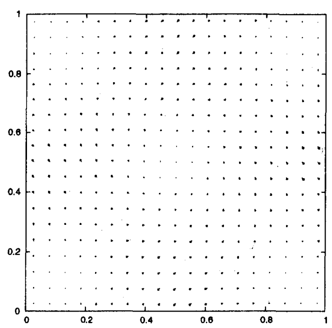
Triangulation 1



Displacement by CG

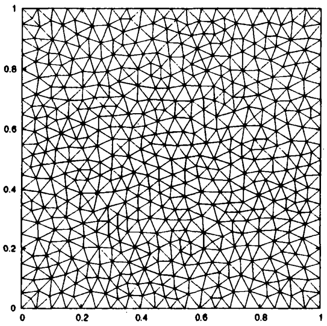


Displacement by DG-D

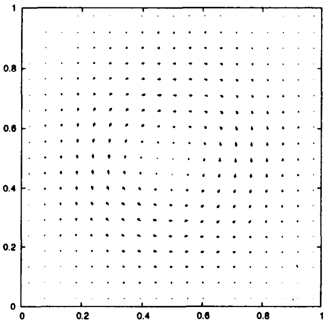


Displacement by DG-C

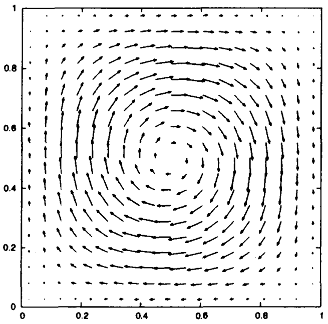
Figure 5: Triangulation 1 and computed displacements



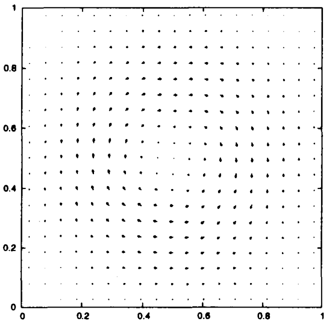
Triangulation 2



Displacement by CG

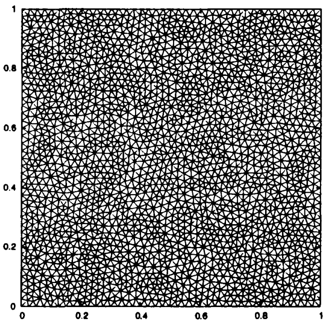


Displacement by DG-D

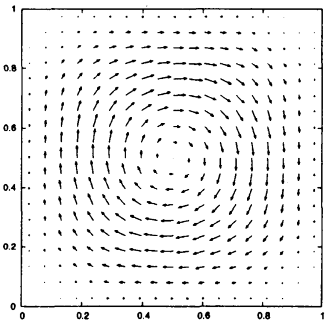


Displacement by DG-C

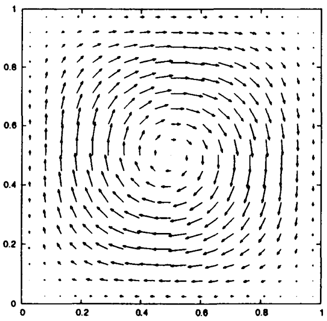
Figure 6: Triangulation 2 and computed displacements



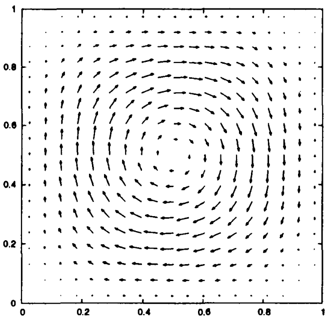
Triangulation 3



Displacement by CG

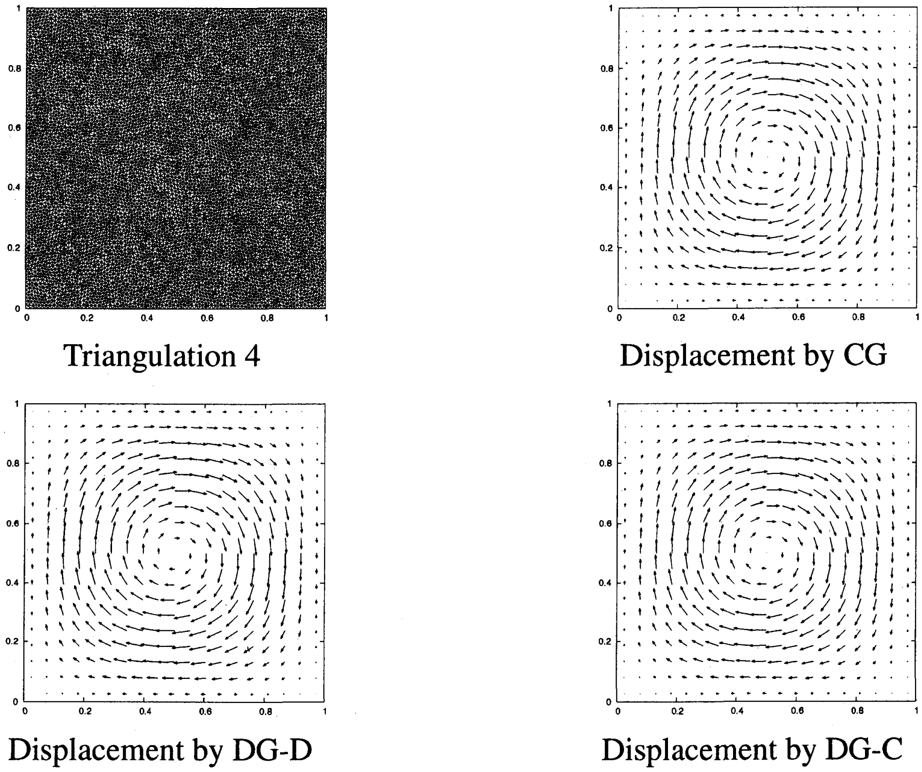


Displacement by DG-D



Displacement by DG-C

Figure 7: Triangulation 3 and computed displacements



Triangulation 4 — Displacement by CG

Displacement by DG-D — Displacement by DG-C

Figure 8: Triangulation 4 and computed displacements

10 Concluding remarks

We have surveyed fundamental finite element methods for nearly incompressible media. As a whole, the mixed methods using the pressure p as well are superior to the methods in displacements (or velocities) only, but it is not so easy to find nice finite element models.

As alternatives, the hybrid DG FEM and their mixed variants may be promising to derive more flexible finite element models that behave nicely in nearly incompressible cases, although their practical feasibility (cost, size of discretized problems, etc.) must be carefully tested. Finally, we have essentially omit the proofs of theoretical results, which must be completed as soon as possible.

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