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CONSTANT TERMS OF EISENSTEIN SERIES OVER A TOTALLY REAL FIELD

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INTRODUCTION

This manuscript is a summary of my talk "Constant terms of Eisenstein series over a totally real field" at RIMS workshop "Modular Forms and Automorphic Representations," which was held from February 2 to 6, 2015. In his paper [Oh] published in 2003, M. Ohta computed the constant terms of Eisenstein series of weight 2 over the field $\mathbb{Q}$ of rationals, at all equivalence classes of cusps. S. Dasgupta, H. Darmon and R. Pollack calculated the constant terms of Eisenstein series defined over a totally real field at particular (not all) equivalence classes of cusps in 2011 [DDP]. In my talk at the conference, I presented a computation of constant terms of Eisenstein series defined over a general totally real field at *all* equivalence classes of cusps, and explicitly described the constant terms by Hecke L-values.

This investigation is motivated by Ohta's work [Oh] on congruence modules related to Eisenstein series defined over $\mathbb{Q}$. The notion of congruence modules was first introduced by Hida in the 1980s. Congruence modules measure congruences of Fourier coefficients modulo a prime number between newforms (also called primitive forms). Ohta reformulated the notion of congruence modules introduced by Hida in a broader context, and then defined and computed the congruence modules related to Eisenstein series. In his computation, the constant terms of Eisenstein series over $\mathbb{Q}$ at all equivalence classes of cusps are necessary.

It is expected to extend Ohta's work to the case of totally real fields. Ohta himself used his result to give a finer proof of Iwasawa main conjecture over $\mathbb{Q}$, which was first shown by Mazur-Wiles. His theory of congruence modules has been applied to several other important problems in Iwasawa theory.

Such circumstances concerning Ohta's congruence modules motivated me to conduct this computation. It must be mentioned, nevertheless, that Ohta's congruence modules have not even been defined in the case of totally real fields, and this investigation does not benefit formulating congruence modules.

Layout. Section 1 is devoted to a brief explanation of Ohta's congruence modules, by which this investigation was motivated. In Section 2, we review basics of Hilbert modular forms and give a precise definition of the Eisenstein series we are going to treat. In the last section, we investigate the equivalence classes of cusps of certain congruence subgroups, and compute the constant terms of Eisenstein series at all equivalence classes of cusps.

NOTATION 0.1. Throughout this paper we use the following notation:

- $i \in \mathbb{C}$: a fixed square root of -1;
- $\mathfrak{H}$: the upper half plane $\mathfrak{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$;
- $\infty = \lim_{t \to +\infty} it$: the point at infinity;
- $GL_2(\mathbb{R})$: the group of all 2×2 invertible matrices whose entries are real;
- $GL_2^+(\mathbb{R})$: the subgroup of $GL_2(\mathbb{R})$ consisting of $\gamma \in GL_2(\mathbb{R})$ with $\det(\gamma) > 0$.

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1. MOTIVATION: OHTA'S CONGRUENCE MODULES

In this section, we briefly explain the relation of the congruence modules in the sense of Ohta [Oh] to Eisenstein series. Let $k \geq 2$ be an integer, Γ a congruence subgroup of $SL_2(\mathbb{Z})$, and $M_k(\Gamma)$ (resp. $S_k(\Gamma)$) the $\mathbb{C}$-vector space of elliptic modular forms (resp. cusp forms) of weight k and level Γ. We let

$$M_k(\Gamma, \mathbb{Z}) = \left\{ f(z) = \sum_{n=0}^{\infty} a(n, f) \exp(2\pi i n z) \,\middle|\, a(n, f) \in \mathbb{Z}, \forall n \geq 0 \right\}$$

and put $S_k(\Gamma, \mathbb{Z}) = S_k(\Gamma) \cap M_k(\Gamma, \mathbb{Z})$. We shall first see a toy case of Ohta's congruence modules. Let $\Gamma = SL_2(\mathbb{Z})$ and $k \geq 4$ be an even integer. We choose a prime number $p \geq 5$ so that $k \not\equiv 0 \bmod (p-1)$. $\mathbb{Z}_p$ (resp. $\mathbb{Q}_p$) denotes the ring of p-adic integers (resp. the field of p-adic numbers). Hereafter we fix two field embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. We put $M_k(\Gamma, \mathbb{Z}_p) = M_k(\Gamma, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $S_k(\Gamma, \mathbb{Z}_p) = S_k(\Gamma, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_p$. We have the following exact sequence of flat $\mathbb{Z}_p$-modules:

$$0 \longrightarrow S_k(\Gamma, \mathbb{Z}_p) \longrightarrow M_k(\Gamma, \mathbb{Z}_p) \xrightarrow{\lambda} \mathbb{Z}_p \longrightarrow 0. \tag{1.1}$$

Here λ is defined by $\lambda(f) = a(0, f)$. λ is surjective because the constant term of the Eisenstein series

$$E_k(z) = 2^{-1}\zeta(1-k) + \sum_{n=1}^{\infty} \left(\sum_{0<d|n} d^{k-1} \right) \exp(2\pi i n z)$$

is p-integral and non-zero by our assumption on k and p (von Staudt-Clausen's theorem). We give a splitting $s : \mathbb{Q}_p \to M_k(\Gamma, \mathbb{Q}_p) = M_k(\Gamma, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ of (1.1) defined over $\mathbb{Q}_p$ by $s(1) = 2\zeta(1-k)^{-1}E_k$. Then the congruence module attached to the pair (1.1) and s is $\mathbb{Z}_p/\zeta(1-k)\mathbb{Z}_p$.

In order to explain what we need to compute Ohta's congruence modules, let us have a closer look at the case where the weight k is 2. For a congruence subgroup Γ of $SL_2(\mathbb{Z})$, we let $C(\Gamma)$ be the set of representatives of $\mathbb{P}^1(\mathbb{Q})$ modulo Γ. For each $f \in M_2(\Gamma)$, $\omega_f = 2\pi i f(z) dz$ is a well-defined differential form on the compact modular curve $X(\Gamma)$

associated to Γ, and hence it makes sense to consider $\mathrm{Res}_s(\omega_f)$, the residue of ω_f at the cusp $s \in C(\Gamma)$. A residue mapping Res_Γ is defined as follows:

$$\mathrm{Res}_\Gamma : M_2(\Gamma) \to \mathbb{C}[C(\Gamma)];\ f \mapsto \sum_{s \in C(\Gamma)} \mathrm{Res}_s(\omega_f) \cdot s.$$

Here $\mathbb{C}[C(\Gamma)]$ denotes the $\mathbb{C}$-vector space spanned by the set $C(\Gamma)$. The main point is that $\mathrm{Res}_\Gamma(E_2(\eta, \psi))$ for various Eisenstein series $E_2(\eta, \psi) \in M_2(\Gamma)$ are essentially used to compute Ohta's congruence modules (here η, ψ are suitable Dirichlet characters).

2. Hilbert modular forms

In Section 2, we first recall the definitions and basic properties of Hilbert modular forms, and the Eisenstein series constructed by Shimura in [S]. Section 2 is based on [DDP] Section 2, [H1] Chapter 9, and [S].

Notation 2.1. Throughout Sections 2 and 3, we use the following notation:

- F: a totally real number field of degree g;
- O: the ring of integers of F;
- I: the set of the embeddings of F into $\mathbb{R}$;
- F_+: the set of the totally positive elements of F;
- $GL_2(F)$: the group of all 2×2 invertible matrices whose entries are in F;
- $GL_2^+(F)$: the subgroup of $GL_2(F)$ consisting of $\gamma \in GL_2(F)$ with $\det(\gamma) \in F_+$;
- $SL_2(F)$: the subgroup of $GL_2^+(F)$ consisting of $\gamma \in GL_2^+(F)$ with $\det(\gamma) = 1$;
- $\mathfrak{d}$: the different of $F/\mathbb{Q}$;
- $\mathrm{N} = \mathrm{N}_{F/\mathbb{Q}}$: the norm of $F/\mathbb{Q}$;
- For $a \in F$ and $\sigma \in I$, a^σ is the image of a in $\mathbb{R}$ under σ;
- For $a \in F$ and a vector $r = (r_\sigma)_{\sigma \in I} \in (\mathbb{Z}/2\mathbb{Z})^g$, $\mathrm{sgn}(a)^r = \prod_{\sigma \in I} \mathrm{sgn}(a^\sigma)^{r_\sigma}$.

2.1. **Narrow ray class groups and characters.** We begin by recalling the definition of narrow ray class characters of F. Let $\mathfrak{m}$ be a non-zero integral ideal of F. We put

$$I(\mathfrak{m}) = \left\{ \frac{\mathfrak{n}}{\mathfrak{l}} \middle| \mathfrak{n} \text{ and } \mathfrak{l} \text{ are integral ideals and prime to } \mathfrak{m} \right\},$$
$$P_+ = \{aO \mid a \in F_+\}, \text{ and}$$
$$P_+(\mathfrak{m}) = P_+ \cap \{aO \mid a \equiv 1 \bmod {}^\times\mathfrak{m}\},$$

where $a \equiv 1 \bmod {}^\times\mathfrak{m}$ if and only if $aO \in I(\mathfrak{m})$ and there exists an element $b \in F_+$ such that $bO \in I(\mathfrak{m})$, $b \in O$, $ab \in O$ and $ab \equiv b \bmod \mathfrak{m}$. We call the quotient group $\mathrm{Cl}(\mathfrak{m}) = I(\mathfrak{m})/P_+(\mathfrak{m})$ the narrow ray class group modulo $\mathfrak{m}$. When $\mathfrak{m} = O$, we write Cl_F^+ rather than $\mathrm{Cl}(O)$, and we call this group the narrow ideal class group of F. We let $h = \#\mathrm{Cl}_F^+$ denote the narrow class number of F.

Definition 2.2. A narrow ray class character modulo an integral ideal $\mathfrak{m}$ is a group homomorphism $\psi : \mathrm{Cl}(\mathfrak{m}) \to \mathbb{C}^\times$.

We let $\mathrm{cond}(\psi)$ denote the conductor of ψ. It is known that there exists a unique vector $r \in (\mathbb{Z}/2\mathbb{Z})^g$ such that

$$\psi(aO) = \mathrm{sgn}(a)^r \text{ for all } a \in O \text{ with } a \equiv 1 \bmod \mathfrak{m}.$$

We call r the signature of ψ. Then we have a well-defined character $\psi_f : (O/\mathfrak{m})^\times \to \mathbb{C}^\times$ associated to ψ given by $\psi_f(a) = \psi(aO)\mathrm{sgn}(a)^r$. We will always regard the right-hand side as a character on $(O/\mathfrak{m})^\times$, without any notice.

2.2. **Hilbert modular forms.** We now describe the definition of (parallel weight) Hilbert modular forms over F. First we choose a representative fractional ideal $\mathfrak{t}_\lambda$ of λ for each $\lambda \in \mathrm{Cl}_F^+$, and define a subgroup $\Gamma_\lambda(\mathfrak{m})$ of $GL_2^+(F)$ by

$$\Gamma_\lambda(\mathfrak{m}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(F) \middle| a, d \in O, b \in (\mathfrak{d}\mathfrak{t}_\lambda)^{-1}, c \in \mathfrak{m}\mathfrak{d}\mathfrak{t}_\lambda,\ ad - bc \in O^\times \right\}.$$

DEFINITION 2.3 (cf. [S] Sections 1 and 2). Let $k \geq 0$ be an integer, and $\mathfrak{m}, \psi$ as above. The space $M_k(\mathfrak{m}, \psi)$ of Hilbert modular forms of (parallel) weight k, level $\mathfrak{m}$ and character ψ consists of elements f such that

(i) $f = (f_\lambda)_{\lambda \in \mathrm{Cl}_F^+}$ is an h-tuple of holomorphic functions $f_\lambda : \mathfrak{H}^I \to \mathbb{C}$;

(ii) for each $\lambda \in \mathrm{Cl}_F^+$, f_λ satisfies the following modularity property:

$$f_\lambda|_k\gamma = \psi_f(d) f_\lambda \text{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\lambda(\mathfrak{m}). \tag{2.1}$$

Here

$$\det(\gamma) = \prod_{\sigma \in I} \det(\gamma)^\sigma,\ cz + d = \prod_{\sigma \in I} (c^\sigma z_\sigma + d^\sigma),\ \gamma z = \left(\frac{a^\sigma z_\sigma + b^\sigma}{c^\sigma z_\sigma + d^\sigma} \right)_{\sigma \in I}$$

and $f_\lambda|_k\gamma$ is a function on $\mathfrak{H}^I$ defined by

$$(f_\lambda|_k\gamma)(z) = \det(\gamma)^{\frac{k}{2}} (cz + d)^{-k} f_\lambda(\gamma z),$$

Since each f_λ is a function on $\mathfrak{H}^I$, we regard z a g-tuple of variables z_σ. We also note that $\gamma z \in \mathfrak{H}^I$ for any $\gamma \in GL_2^+(F)$. We often omit the subscript k of $f_\lambda|_k\gamma$ when there is no ambiguity concerning weight.

(iii) when $F = \mathbb{Q}$, we also impose the holomorphy condition around each cusp; that is, for any $\gamma \in SL_2(\mathbb{Z})$, we have

$$(f|_k\gamma)(z) = \sum_{n=0}^{\infty} a\left(\frac{n}{M}, f|_k\gamma\right) \exp\left(2\pi i \frac{nz}{M}\right)$$

where M is the positive integer uniquely determined by $M\mathbb{Z} = \mathfrak{m}$.

REMARK 2.4. The definition of the subgroup $\Gamma_\lambda(\mathfrak{m})$ depends on the choice of a representative fractional ideal $\mathfrak{t}_\lambda$. We take two representative ideals $\mathfrak{t}_{\lambda,i}$ $(i = 1, 2)$ of $\lambda \in \mathrm{Cl}_F^+$ and consider the $\mathbb{C}$-vector space $M_k(\mathfrak{m}, \psi)_i$ consisting of modular forms satisfying the modularity property (2.1) with respect to $\Gamma_{\mathfrak{t}_{\lambda,i}}(\mathfrak{m})$ for each i. By definition we have $\mathfrak{t}_{\lambda,2} = u\mathfrak{t}_{\lambda,1}$

for some $u \in F_+$. Then there is an isomorphism

$$M_k(\mathfrak{m},\psi)_1 \to M_k(\mathfrak{m},\psi)_2;\ (f_\lambda)_{\lambda\in \mathrm{Cl}_F^+} \mapsto \left(f_\lambda|_k \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\right)_{\lambda\in \mathrm{Cl}_F^+}.$$

However we can define Fourier coefficients of f independent of the choice of a representative ideal $\mathfrak{t}_\lambda$ (see Definition 2.6 and Remark 2.7 for details).

2.3. **Fourier expansion of a Hilbert modular forms.** We define Fourier expansion of a Hilbert modular form.

PROPOSITION 2.5. *A Hilbert modular form $f = (f_\lambda)_{\lambda\in \mathrm{Cl}_F^+} \in M_k(\mathfrak{m},\psi)$ has a Fourier expansion (at the cusp $\infty = (\infty,\infty,\dots,\infty)$) of the following form:*

$$f_\lambda(z) = a_\lambda(0) + \sum_{b\in \mathfrak{t}_\lambda \cap F_+} a_\lambda(b) e_F(bz) \text{ for each } \lambda \in \mathrm{Cl}_F^+. \tag{2.2}$$

Here $a_\lambda(0)$, $a_\lambda(b)$ are complex numbers and $e_F(x) = \exp(2\pi i \sum_{\sigma\in I} x_\sigma)$ (we use this notation both for $x \in F$ and for a g-tuple of variables $x = (x_\sigma)_{\sigma\in I}$).

PROOF. The assertion is well known when $F = \mathbb{Q}$. When $F \neq \mathbb{Q}$, ideas of the proof are basically the same as that for $F = \mathbb{Q}$. Namely, the modularity property (2.1) implies that $f_\lambda(z)$ is invariant under the translation by elements of $(\mathfrak{d}\mathfrak{t}_\lambda)^{-1}$, and since f_λ is holomorphic in z we conclude that f is of the form

$$f_\lambda(z) = \sum_{b\in \mathfrak{t}_\lambda} a_\lambda(b) e_F(bz).$$

We need to show that $a_\lambda(b) = 0$ for all $b \in \mathfrak{t}_\lambda$ with $b \notin F_+$ and $b \neq 0$. This is so-called "Koecher's principle" (see [G] Theorem 3.3 of Chapter 2, Section 3). Note that Koecher's principle does not hold when $F = \mathbb{Q}$. □

We call the coefficients $a_\lambda(b)$ the unnormalized Fourier coefficients of f. We also define the normalized one as follows.

DEFINITION 2.6. Let f be as in Proposition 2.5 with the Fourier expansion (2.2). We define the normalized constant term $c_\lambda(0,f)$ of f by

$$c_\lambda(0,f) = a_\lambda(0)\mathrm{N}(\mathfrak{t}_\lambda)^{-\frac{k}{2}}$$

for each $\lambda \in \mathrm{Cl}_F^+$. For each non-zero integral ideal $\mathfrak{n}$ of F, there exists a unique $\lambda \in \mathrm{Cl}_F^+$, and $b \in F_+$ unique up to multiplication by totally positive units, such that $\mathfrak{n} = b\mathfrak{t}_\lambda^{-1}$. Then $b \in \mathfrak{t}_\lambda \cap F_+$ and the normalized Fourier coefficient $c(\mathfrak{n},f)$ associated to $\mathfrak{n}$ is

$$c(\mathfrak{n},f) = a_\lambda(b)\mathrm{N}(\mathfrak{t}_\lambda)^{-\frac{k}{2}}.$$

REMARK 2.7. The following two facts show why $c_\lambda(0,f)$ and $c(\mathfrak{n},f)$ are called "normalized" coefficients. These facts can be deduced from the modularity property (2.1).

(i) $c_\lambda(0,f)$ and $c(\mathfrak{n},f)$ are independent of the choice of a representative ideal $\mathfrak{t}_\lambda$.
(ii) $c(\mathfrak{n},f)$ is independent of the choice of $b \in \mathfrak{t}_\lambda \cap F_+$ such that $\mathfrak{n} = b\mathfrak{t}_\lambda^{-1}$.

2.4. **Eisenstein series.** In this subsection we introduce Eisenstein series, which is one of the most basic example of Hilbert modular forms. Let η (resp. ψ) be a primitive narrow ray class character of conductor $\mathfrak{a}$ (resp. $\mathfrak{b}$) and signature $q \in (\mathbb{Z}/2\mathbb{Z})^g$ (resp. r). Actually we can define Eisenstein series for non-primitive characters, but for simplicity, we content ourselves only with primitive case here. When we consider Eisenstein series, we always impose the following assumption on the weight k:

$$q + r \equiv (k, k, \dots, k) \bmod (2\mathbb{Z})^g. \tag{2.3}$$

PROPOSITION 2.8 ([S] Proposition 3.4). *Under the above condition, there exists a unique Hilbert modular form $E_k(\eta, \psi)$ of weight k, level $\mathfrak{m} = \mathfrak{ab}$ and character $\eta\psi$ with the normalized coefficients*

$$c(\mathfrak{n}, E_k(\eta, \psi)) = \sum_{\mathfrak{n}_1 | \mathfrak{n}} \eta\left(\frac{\mathfrak{n}}{\mathfrak{n}_1}\right) \psi(\mathfrak{n}_1) \mathrm{N}(\mathfrak{n}_1)^{k-1} \tag{2.4}$$

for each non-zero integral ideal $\mathfrak{n}$ and

$$c_\lambda(0, E_k(\eta, \psi)) = \begin{cases} \delta_{\eta,1} 2^{-g} L(\psi, 1-k) & \text{if } k \geq 2, \\ 2^{-g}(\delta_{\eta,1} L(\psi, 0) + \delta_{\psi,1} L(\eta, 0)) & \text{if } k = 1 \end{cases} \tag{2.5}$$

for each $\lambda \in \mathrm{Cl}_F^+$. The sum in (2.4) runs over all integral ideals $\mathfrak{n}_1$ dividing $\mathfrak{n}$. In (2.5), $\delta_{\eta,1} = 1$ if $\eta = 1$ (i.e., $\mathfrak{a} = O$) and 0 otherwise. $L(\eta, s)$ denotes the Hecke L-function attached to the character η (we use the same notation for other characters). We call $E_k(\eta, \psi)$ the Eisenstein series of weight k associated with characters (η, ψ).

PROOF. (Outline of the proof of Proposition 2.8) The Eisenstein series $E_k(\eta, \psi)$ in Proposition 2.8 is explicitly given in [S] Proposition 3.2 and [DDP] Proposition 2.1. We recall the definition. For $s \in \mathbb{C}$, the series

$$E_k(\eta, \psi)_\lambda(z, s) = C\tau(\psi) \frac{\mathrm{N}(\mathfrak{t}_\lambda)^{-\frac{k}{2}}}{\mathrm{N}(\mathfrak{b})} \sum_{\mathfrak{c} \in \mathrm{Cl}_F} \mathrm{N}(\mathfrak{c})^k \times \sum_{\substack{a \in \mathfrak{c}, \\ b \in (\mathfrak{b}\mathfrak{d}\mathfrak{t}_\lambda)^{-1}\mathfrak{c}, \\ (a,b) \bmod U, \\ (a,b) \neq (0,0)}} \frac{\mathrm{sgn}(a)^q \eta(a\mathfrak{c}^{-1}) \mathrm{sgn}(-b)^r \psi^{-1}(-b\mathfrak{b}\mathfrak{d}\mathfrak{t}_\lambda \mathfrak{c}^{-1})}{(az+b)^k |az+b|^{2s}}$$

is convergent on the right half plane $\mathrm{Re}(k + 2s) > 2$. Here Cl_F is the (wide) ideal class group of F,

$$\tau(\psi) = \sum_{x \in (\mathfrak{b}\mathfrak{d})^{-1}/\mathfrak{d}^{-1}} \mathrm{sgn}(x)^r \psi(x\mathfrak{b}\mathfrak{d}) e_F(x)$$

is the Gauss sum of ψ, U is the subgroup of finite index of $O^\times$ defined by

$$U = \{u \in O^\times \mid \mathrm{N}(u)^k = 1, u \equiv 1 \bmod \mathfrak{m}\}$$

which acts on $\{(a, b) \mid a \in \mathfrak{c},\ b \in (\mathfrak{b}\mathfrak{d}\mathfrak{t}_\lambda)^{-1}\mathfrak{c}, (a, b) \neq (0, 0)\}$ by $u \cdot (a, b) = (ua, ub)$, and

$$C = \frac{\sqrt{d_F}\Gamma(k)^g}{[O^\times : U]\mathrm{N}(\mathfrak{d})(-2\pi i)^{kg}}$$

where d_F denotes the discriminant of F. The definition of $E_k(\eta,\psi)_\lambda(z,s)$ here looks slightly different from that in [DDP], but in fact the two definitions are exactly the same. We have already computed some terms of $E_k(\eta,\psi)_\lambda(z,s)$ in [DDP] by using the assumption that ψ is primitive.

It suffices to show that $E_k(\eta,\psi)_\lambda(z,s)$ has a meromorphic continuation in s to the whole complex plane and is holomorphic at $s=0$, and that the h-tuple $(E_k(\eta,\psi)_\lambda(z,0))_{\lambda\in \mathrm{Cl}_F^+}$ is a Hilbert modular form of prescribed weight, level and character, with the desired Fourier coefficients. This is done in a parallel manner with Hecke's technique (later developed by Shimura) for obtaining holomorphic Eisenstein series via meromorphic continuation of real analytic Eisenstein series. One can find the details of this argument in [H2] Sections 9.2 and 9.3. □

3. Equivalence classes of cusps and the main theorem

Section 3 is devoted to a formulation and a proof of our main theorem. We keep using the notation at the beginning of Section 2.

3.1. **Constant terms of Eisenstein series under slash operators.** In this subsection, we present a detailed computation of the normalized constant term of $E_k(\eta,\psi)$ under the slash operators defined below. First we introduce some congruence subgroups of $GL_2^+(F)$ and $SL_2(F)$. The notation in the following definition is basically in accordance with [H2] Chapter 4, Section 1.3.

Definition 3.1. Let $\mathfrak{n}$ be an integral ideal and $\mathfrak{j}$ a fractional ideal of F. $\Gamma(\mathfrak{n};O,\mathfrak{j})$ is a subgroup of $GL_2^+(F)$ defined by

$$\Gamma(\mathfrak{n};O,\mathfrak{j})=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in GL_2^+(F)\middle|a,d\in O,b\in(\mathfrak{j}\mathfrak{d})^{-1},c\in\mathfrak{n}\mathfrak{j}\mathfrak{d},\ ad-bc\in O^\times\right\}.$$

Hereafter we mainly consider the subgroup $\Gamma^1(\mathfrak{n};O,\mathfrak{j})=SL_2(F)\cap\Gamma(\mathfrak{n};O,\mathfrak{j})$ of $SL_2(F)$.

Remark 3.2. (i) When $F=\mathbb{Q}$, $\mathfrak{n}=N\mathbb{Z}$ $(N\in\mathbb{Z}_{>0})$ and $\mathfrak{j}=\mathbb{Z}$, we have

$$\Gamma(\mathfrak{n};O,\mathfrak{j})=\Gamma^1(\mathfrak{n};O,\mathfrak{j})=\Gamma_0(N).$$

(ii) When $\mathfrak{n}=\mathfrak{m}$ and $\mathfrak{j}=\mathfrak{t}_\lambda$ for $\lambda\in\mathrm{Cl}_F^+$, $\Gamma(\mathfrak{m};O,\mathfrak{t}_\lambda)=\Gamma_\lambda(\mathfrak{m})$, which was defined just before Definition 2.3.

From now on, we write $\Gamma_\lambda^*(\mathfrak{n})$ for $\Gamma^*(\mathfrak{n};O,\mathfrak{t}_\lambda)$ ($*=1$ or empty). We define the slash operator on the space of Hilbert modular forms.

Definition 3.3. Recall that $h=\#\mathrm{Cl}_F^+$. Let $f=(f_\lambda)_{\lambda\in\mathrm{Cl}_F^+}$ be a Hilbert modular form and $A=(A_\lambda)_{\lambda\in\mathrm{Cl}_F^+}\in SL_2(F)^h$ an h-tuple of matrices. The slash operator is defined by

$$f|A=(f_\lambda|A_\lambda)_{\lambda\in\mathrm{Cl}_F^+}.$$

The main result of this subsection is as follows:

PROPOSITION 3.4. *For an integer $k \geq 2$, let η and ψ be as in Section 2.4 (in particular satisfying (2.3)), and let $A = (A_\lambda)_{\lambda \in \mathrm{Cl}_F^+}$ be a slash operator with*

$$A_\lambda = \begin{pmatrix} \alpha_\lambda & \beta_\lambda \\ \gamma_\lambda & \delta_\lambda \end{pmatrix} \in \Gamma_\lambda^1(O)$$

for each $\lambda \in \mathrm{Cl}_F^+$. Then

$$c_\lambda(0, E_k(\eta, \psi)|A) = 0$$

unless $\gamma_\lambda \in \mathfrak{b}\mathfrak{d}\mathfrak{t}_\lambda$. If this is the case, we have

$$\begin{aligned} &c_\lambda(0, E_k(\eta, \psi)|A) \\ &= \frac{1}{2^g} \frac{\tau(\eta\psi^{-1})}{\tau(\psi^{-1})} \left(\frac{\mathrm{N}(\mathfrak{b})}{\mathrm{N}(\mathfrak{f})}\right)^k \mathrm{sgn}(-\gamma_\lambda)^q \eta(\gamma_\lambda(\mathfrak{b}\mathfrak{d}\mathfrak{t}_\lambda)^{-1}) \mathrm{sgn}(\alpha_\lambda)^r \psi^{-1}(\alpha_\lambda O) \\ &\qquad \times L(\eta^{-1}\psi, 1-k) \prod_{\mathfrak{q}|\mathfrak{m},\ \mathfrak{q}\nmid\mathfrak{f}} (1 - \eta\psi^{-1}(\mathfrak{q})\mathrm{N}(\mathfrak{q})^{-k}), \end{aligned}$$

where $\mathfrak{f} = \mathrm{cond}(\eta^{-1}\psi)$ and the last product runs over all prime ideals $\mathfrak{q}$ dividing $\mathfrak{m} = \mathfrak{a}\mathfrak{b}$ but not dividing $\mathfrak{f}$.

REMARK 3.5. Here we make two remarks on previously known results.

(i) Ohta computed the constant terms of Eisenstein series of weight 2 and level $\Gamma_1(Np^r)$ over $\mathbb{Q}$, at all equivalence classes of cusps (Proposition 2.5.5 of [Oh]). Here $p \geq 5$ is a prime number, N is a positive integer prime to p, and $r \geq 1$ is an integer. Proposition 3.4 is a generalization of his result. Indeed we have $\Gamma_\lambda^1(O) = SL_2(\mathbb{Z})$ when $F = \mathbb{Q}$ and the condition $\gamma_\lambda \in \mathfrak{b}\mathfrak{d}\mathfrak{t}_\lambda$ here corresponds to $u \mid c$ in [Oh].

(ii) This proposition also implies Proposition 2.3 of [DDP], where Dasgupta, Darmon and Pollack computed $c_\lambda(0, E_k(\eta, \psi)|A)$ for particular form of A. It should be noticed that they carried out the computation in order to express a product of two Eisenstein series of weight 1 as a linear combination of Eisenstein series and cusp forms of weight 2, and constant terms for such A will do for that purpose.

PROOF. Hereafter we fix $\lambda \in \mathrm{Cl}_F^+$. We write down $(E_k(\eta, \psi)_\lambda|A_\lambda)(z, s)$ according to the definition:

$$\begin{aligned} &(E_k(\eta, \psi)_\lambda|A_\lambda)(z, s) \\ &= C\tau(\psi) \frac{\mathrm{N}(\mathfrak{t}_\lambda)^{-\frac{k}{2}}}{\mathrm{N}(\mathfrak{b})} \sum_{\mathfrak{c} \in \mathrm{Cl}_F} \mathrm{N}(\mathfrak{c})^k \\ &\times \sum_{\substack{a \in \mathfrak{c}, \\ b \in (\mathfrak{b}\mathfrak{d}\mathfrak{t}_\lambda)^{-1}\mathfrak{c}, \\ (a,b) \bmod U, \\ (a,b) \neq (0,0)}} \frac{\mathrm{sgn}(a)^q \eta(a\mathfrak{c}^{-1}) \mathrm{sgn}(-b)^r \psi^{-1}(-b\mathfrak{b}\mathfrak{d}\mathfrak{t}_\lambda\mathfrak{c}^{-1}) |\gamma_\lambda z + \delta_\lambda|^{2s}}{((a\alpha_\lambda + b\gamma_\lambda)z + (a\beta_\lambda + b\delta_\lambda))^k |(a\alpha_\lambda + b\gamma_\lambda)z + (a\beta_\lambda + b\delta_\lambda)|^{2s}}. \end{aligned}$$

We note that the constant term arises from terms with $a\alpha_\lambda + b\gamma_\lambda = 0$. This condition is equivalent to $b\gamma_\lambda = -a\alpha_\lambda$, which implies $b\gamma_\lambda \in \mathfrak{c}$. On the other hand the condition $\gamma_\lambda \in \mathfrak{d}\mathfrak{t}_\lambda$ implies that there exists an integral ideal $\mathfrak{n}$ with $\gamma_\lambda O = \mathfrak{n}\mathfrak{d}\mathfrak{t}_\lambda$ and hence $b\gamma_\lambda \in \mathfrak{n}\mathfrak{b}^{-1}\mathfrak{c}$. Our

strategy is to focus on the ideal $(\mathfrak{n}\mathfrak{b}^{-1}\mathfrak{c}) \cap \mathfrak{c} = (\mathfrak{n} \cap \mathfrak{b})\mathfrak{b}^{-1}\mathfrak{c}$. We divide the argument into two cases:

Case 1: $\mathfrak{b} \nmid \mathfrak{n}$. There exists a prime factor $\mathfrak{p}$ of $\mathfrak{b}$ which satisfies

$$\mathfrak{n} = \mathfrak{p}^e\mathfrak{n}', \ \mathfrak{b} = \mathfrak{p}^f\mathfrak{b}' \ (e \in \mathbb{Z}_{\geq 0}, \ f \in \mathbb{Z}_{>0}, \ \mathfrak{p} \nmid \mathfrak{n}'\mathfrak{b}') \text{ and } e < f.$$

Then $b\gamma_\lambda \in \mathfrak{n}\mathfrak{b}^{-1}\mathfrak{c} \cap \mathfrak{c} = (\mathfrak{b}')^{-1}(\mathfrak{n}' \cap \mathfrak{b}')\mathfrak{c}$. Thus $b \in \mathfrak{n}^{-1}(\mathfrak{b}')^{-1}(\mathfrak{n}' \cap \mathfrak{b}')\mathfrak{d}^{-1}\mathfrak{t}_\lambda^{-1}\mathfrak{c}$ and $b\mathfrak{b}\mathfrak{d}\mathfrak{t}_\lambda\mathfrak{c}^{-1} \subset \mathfrak{p}^{f-e}(\mathfrak{n}')^{-1}(\mathfrak{n}' \cap \mathfrak{b}') \subset \mathfrak{p}^{f-e}$. Since $f - e > 0$, $b\mathfrak{b}\mathfrak{d}\mathfrak{t}_\lambda\mathfrak{c}^{-1}$ is not prime to $\mathfrak{b}$ and thus $\operatorname{sgn}(-b)^r\psi^{-1}(-b\mathfrak{b}\mathfrak{d}\mathfrak{t}_\lambda\mathfrak{c}^{-1}) = 0$.

Case 2: $\mathfrak{b} \mid \mathfrak{n}$. In this case, we know that $\gamma_\lambda \in \mathfrak{b}\mathfrak{d}\mathfrak{t}_\lambda$. Then the matrix A_λ induces an isomorphism

$$\left\{(a,b) \mid a \in \mathfrak{c}, \ b \in (\mathfrak{b}\mathfrak{d}\mathfrak{t}_\lambda)^{-1}\mathfrak{c}, \ a\alpha_\lambda + b\gamma_\lambda = 0\right\}/U \to (\mathfrak{b}\mathfrak{d}\mathfrak{t}_\lambda)^{-1}\mathfrak{c}/U;$$
$$(a,b) \mapsto a\beta_\lambda + b\delta_\lambda$$

(the inverse map is given by $d \mapsto (-d\gamma_\lambda, d\alpha_\lambda)$). Then

$$\begin{aligned}
&c_\lambda(0, E_k(\eta,\psi)|A) \\
&= C\tau(\psi)\frac{\mathrm{N}(\mathfrak{t}_\lambda)^{-k}}{\mathrm{N}(\mathfrak{b})}\sum_{\mathfrak{c}\in \mathrm{Cl}_F}\mathrm{N}(\mathfrak{c})^k \\
&\times \sum_{\substack{d\in(\mathfrak{b}\mathfrak{d}\mathfrak{t}_\lambda)^{-1}\mathfrak{c},\\ d \bmod U,\\ d\neq 0}} \operatorname{sgn}(-d\gamma_\lambda)^q\eta(-d\gamma_\lambda\mathfrak{c}^{-1})\operatorname{sgn}(-d\alpha_\lambda)^r\psi^{-1}(-d\alpha_\lambda\mathfrak{b}\mathfrak{d}\mathfrak{t}_\lambda\mathfrak{c}^{-1})\mathrm{N}(b)^{-k} \\
&= C\frac{\tau(\psi)\mathrm{N}(\mathfrak{b}\mathfrak{d})^k\operatorname{sgn}(\gamma_\lambda)^q\eta(\gamma_\lambda O)\operatorname{sgn}(\alpha_\lambda)^r\psi^{-1}(\alpha_\lambda O)[O^\times : U]}{(-1)^{kg}\mathrm{N}(\mathfrak{b})\eta(\mathfrak{b}\mathfrak{d}\mathfrak{t}_\lambda)} \\
&\qquad \times L(\eta\psi^{-1}, k)\prod_{\mathfrak{q}|\mathfrak{m}, \ \mathfrak{q}\nmid\mathfrak{f}}(1 - \eta\psi^{-1}(\mathfrak{q})\mathrm{N}(\mathfrak{q})^{-k}). \qquad (3.1)
\end{aligned}$$

We use the functional equation for $L(\eta\psi^{-1}, s)$ (see [M] Chapter 3, Section 3):

$$\frac{(d_F)^{\frac{1}{2}-k}\mathrm{N}(\mathfrak{f})^{1-k}(2\pi i)^{kg}}{2^g\Gamma(k)^g\tau(\eta^{-1}\psi)}L(\eta^{-1}\psi, 1-k) = L(\eta\psi^{-1}, k). \qquad (3.2)$$

We obtain the desired result by combining equalities (3.1) and (3.2).

□

3.2. **The equivalence classes of cusps of congruence subgroups.** The purpose of this subsection is to investigate the equivalence classes of cusps by the action of the subgroup $\Gamma_\lambda^1(O)$. First we describe the set of cusps $\mathbb{P}^1(F)$ of $\mathfrak{H}^I$ in terms of a quotient of $SL_2(F)$. Let $B^+(F)$ denote the subgroup of $GL_2^+(F)$ consisting of all upper triangular matrices in $GL_2^+(F)$, and $B^1(F) = B^+(F) \cap SL_2(F)$ its intersection with $SL_2(F)$. The following bijection is well known.

LEMMA 3.6. *There is a bijection*

$$SL_2(F)/B^1(F) \to \mathbb{P}^1(F); \ \gamma \mapsto \gamma(\infty).$$

Let $\mathfrak{j}$ be a fractional ideal of F. Thanks to Lemma 3.6 we know that the set of equivalence classes of cusps by the action of $\Gamma^1(O;O,\mathfrak{j}^{-1})$ is

$$\Gamma^1(O;O,\mathfrak{j}^{-1})\backslash SL_2(F)/B^1(F).$$

We describe this set explicitly (here we consider $\Gamma^1(O;O,\mathfrak{j}^{-1})$ instead of $\Gamma^1(O;O,\mathfrak{j})$, in order to be consistent with the notation used in [H2] Chapter 4, Section 1). To a matrix $m = \left(\begin{smallmatrix} a & * \\ c & * \end{smallmatrix}\right) \in SL_2(F)$, we associate a fractional ideal $il_{\mathfrak{j}}(m) = c\mathfrak{j}\mathfrak{d}^{-1} + aO$. If $\gamma = \left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right)$ is an element of $\Gamma^1(O;O,\mathfrak{j}^{-1})$, we have $il_{\mathfrak{j}}(\gamma m) = il_{\mathfrak{j}}(m)$. Moreover, for upper triangular $b = \left(\begin{smallmatrix} \widetilde{b} & * \\ 0 & * \end{smallmatrix}\right) \in B^1(F)$ we have $il_{\mathfrak{j}}(gb) = \widetilde{b}\cdot il_{\mathfrak{j}}(g)$. Hence we obtain a map

$$il_{\mathfrak{j}} : \Gamma^1(O;O,\mathfrak{j}^{-1})\backslash SL_2(F)/B^1(F) \to \mathrm{Cl}_F;\ \begin{pmatrix} a & * \\ c & * \end{pmatrix} \mapsto c\mathfrak{j}\mathfrak{d}^{-1} + aO.$$

PROPOSITION 3.7 ([G] Proposition 2.22). *The map $il_{\mathfrak{j}}$ is a bijection.*

One can find a detailed proof of the proposition in [G]. However we need a slightly refined version of the surjectivity of $il_{\mathfrak{j}}$ later on so as to compute the constant terms, so we will review the proof of the surjectivity in Proposition 3.8.

Now we apply Proposition 3.7 for $\mathfrak{j} = \mathfrak{t}_\lambda^{-1}$. Note that $\Gamma^1_\lambda(O) = \Gamma^1(O;O,\mathfrak{t}_\lambda)$ by definition. In the light of Proposition 3.7, what we have computed in the previous subsection is a constant term of $E_k(\eta,\psi)$ at *one* equivalence class of cusps of $\Gamma^1_\lambda(O)$, that is, the equivalence class of ∞. We will compute the constant terms at *all* equivalence classes of cusps of $\Gamma^1_\lambda(O)$ in the next subsection.

3.3. **Constant terms of Eisenstein series under slash operators II.** Hereafter we fix $\lambda \in \mathrm{Cl}_F^+$. As declared at the end of the previous subsection, we compute the constant terms of $E_k(\eta,\psi)$ at all equivalence classes of cusps of $\Gamma^1_\lambda(O)$. We choose an element in Cl_F and fix its representative integral ideal $\mathfrak{c}_0$. We may assume that $\mathfrak{c}_0$ is prime to $\mathfrak{m} = \mathfrak{a}\mathfrak{b}$. We shall prove a slightly refined version of the surjectivity of the map $il_{\mathfrak{j}}$, with $\mathfrak{j} = \mathfrak{t}_\lambda^{-1}$.

PROPOSITION 3.8. *We can choose a matrix*

$$A_\lambda = \begin{pmatrix} \alpha_\lambda & \beta_\lambda \\ \gamma_\lambda & \delta_\lambda \end{pmatrix} \in SL_2(F)$$

with $il_{\mathfrak{t}_\lambda^{-1}}(A_\lambda) = \mathfrak{c}_0$ so that

$$\alpha_\lambda O = \mathfrak{n}_2\mathfrak{c}_0,\ \beta_\lambda \in (\mathfrak{d}\mathfrak{t}_\lambda\mathfrak{c}_0)^{-1},\ \gamma_\lambda O = \mathfrak{n}_1\mathfrak{d}\mathfrak{t}_\lambda\mathfrak{c}_0 \text{ and } \delta_\lambda \in \mathfrak{c}_0^{-1}.$$

Here $\mathfrak{n}_i$ $(i = 1,2)$ are mutually prime integral ideals. Furthermore, the ideal $\mathfrak{n}_1$ can be chosen so that $\mathfrak{n}_1$ is prime to $\mathfrak{b} = \mathrm{cond}(\psi)$.

PROOF. Let $\mathfrak{c}_0$ be as above and $\mathfrak{b} = \prod_{i=1}^w \mathfrak{p}_i^{e_i}$ the prime ideal factorization of $\mathfrak{b}$. We can take a non-zero element $\gamma_\lambda \in \mathfrak{d}\mathfrak{t}_\lambda\mathfrak{c}_0$ so that $\gamma_\lambda \notin \mathfrak{p}_i\mathfrak{d}\mathfrak{t}_\lambda\mathfrak{c}_0$ for all $i = 1,2,\dots,w$. This can be proved as follows: we let $\mathfrak{l} = \mathfrak{p}_1\mathfrak{p}_2\cdots\mathfrak{p}_w\mathfrak{d}\mathfrak{t}_\lambda\mathfrak{c}_0$ and $\mathfrak{l}_i = \mathfrak{l}\mathfrak{p}_i^{-1}$ for each $i = 1,2,\dots,w$. Since $\mathfrak{l} \subsetneq \mathfrak{l}_i$ there exists $c_i \in \mathfrak{l}_i \setminus \mathfrak{l}$ for each i. Then $\gamma_\lambda = c_1 + c_2 + \cdots + c_w$ does the job. We write $\gamma_\lambda O = \mathfrak{n}_1\mathfrak{d}\mathfrak{t}_\lambda\mathfrak{c}_0$ with $\mathfrak{n}_1$ integral and prime to $\mathfrak{b}$. In a similar manner we see that there exists an element $\alpha_\lambda \in \mathfrak{c}_0$ such that $\alpha_\lambda O = \mathfrak{n}_2\mathfrak{c}_0$ with $\mathfrak{n}_2$ integral and prime to $\mathfrak{n}_1$. Then we have $\gamma_\lambda(\mathfrak{d}\mathfrak{t}_\lambda)^{-1} + \alpha_\lambda O = \mathfrak{n}_1\mathfrak{c}_0 + \mathfrak{n}_2\mathfrak{c}_0 = \mathfrak{c}_0$. Since this condition is

equivalent to $\gamma_\lambda(\mathfrak{d}\mathfrak{t}_\lambda\mathfrak{c}_0)^{-1}+\alpha_\lambda\mathfrak{c}_0^{-1}=O$, there exist $\beta_\lambda\in(\mathfrak{d}\mathfrak{t}_\lambda\mathfrak{c}_0)^{-1}$ and $\delta_\lambda\in\mathfrak{c}_0^{-1}$ such that $\alpha_\lambda\delta_\lambda-\beta_\lambda\gamma_\lambda=1$. This proves

$$A_\lambda=\begin{pmatrix}\alpha_\lambda & \beta_\lambda\\ \gamma_\lambda & \delta_\lambda\end{pmatrix}\in SL_2(F) \text{ and } il_{\mathfrak{t}_\lambda^{-1}}(A_\lambda)=\mathfrak{c}_0.$$

□

In consideration of Proposition 3.7, it is sufficient to compute the constant term of $E_k(\eta,\psi)_\lambda|A_\lambda$ for A_λ as in Proposition 3.8. We recall the definition of $(E_k(\eta,\psi)_\lambda|A_\lambda)(z,s)$:

$$\begin{aligned}&(E_k(\eta,\psi)_\lambda|A_\lambda)(z,s)\\ &=C\tau(\psi)\frac{\mathrm{N}(\mathfrak{t}_\lambda)^{-\frac{k}{2}}}{\mathrm{N}(\mathfrak{b})}\sum_{\mathfrak{c}\in\mathrm{Cl}_F}\mathrm{N}(\mathfrak{c})^k\\ &\times\sum_{\substack{a\in\mathfrak{c},\\ b\in(\mathfrak{b}\mathfrak{d}\mathfrak{t}_\lambda)^{-1}\mathfrak{c},\\ (a,b)\bmod U,\\ (a,b)\neq(0,0)}}\frac{\mathrm{sgn}(a)^q\eta(a\mathfrak{c}^{-1})\mathrm{sgn}(-b)^r\psi^{-1}(-b\mathfrak{b}\mathfrak{d}\mathfrak{t}_\lambda\mathfrak{c}^{-1})|\gamma_\lambda z+\delta_\lambda|^{2s}}{((a\alpha_\lambda+b\gamma_\lambda)z+(a\beta_\lambda+b\delta_\lambda))^k|(a\alpha_\lambda+b\gamma_\lambda)z+(a\beta_\lambda+b\delta_\lambda)|^{2s}}.\end{aligned}$$

As in the proof of Proposition 3.4, we need to consider terms with $a\alpha_\lambda+b\gamma_\lambda=0$. For each $\mathfrak{c}\in\mathrm{Cl}_F$, we have $a\alpha_\lambda\in\mathfrak{n}_2\mathfrak{c}_0\mathfrak{c}$ and $b\gamma_\lambda\in\mathfrak{b}^{-1}\mathfrak{n}_1\mathfrak{c}_0\mathfrak{c}$. Noting that $\mathfrak{n}_1$ is prime to $\mathfrak{b}$, we see that $b\gamma_\lambda=-a\alpha_\lambda\in(\mathfrak{n}_2\mathfrak{c}_0\mathfrak{c})\cap(\mathfrak{b}^{-1}\mathfrak{n}_1\mathfrak{c}_0\mathfrak{c})=\mathfrak{n}_1\mathfrak{n}_2\mathfrak{c}_0\mathfrak{c}$ and hence $b\in\mathfrak{n}_2(\mathfrak{d}\mathfrak{t}_\lambda)^{-1}\mathfrak{c}$. Consequently we have $\psi^{-1}(-b\mathfrak{b}\mathfrak{d}\mathfrak{t}_\lambda\mathfrak{c}^{-1})=0$ unless $\mathfrak{b}=O$. If this is the case, we use an isomorphism

$$\begin{aligned}\left\{(a,b)\mid a\in\mathfrak{c},\ b\in(\mathfrak{d}\mathfrak{t}_\lambda)^{-1}\mathfrak{c},\ a\alpha_\lambda+b\gamma_\lambda=0\right\}/U&\to(\mathfrak{d}\mathfrak{t}_\lambda\mathfrak{c}_0)^{-1}\mathfrak{c}/U;\\ (a,b)&\mapsto a\beta_\lambda+b\delta_\lambda\end{aligned}$$

to compute (the inverse map is given by $d\mapsto(-d\gamma_\lambda,d\alpha_\lambda)$). The normalized constant term of $E_k(\eta,\psi)_\lambda|A_\lambda$ is equal to

$$\begin{aligned}&C\mathrm{N}(\mathfrak{t}_\lambda)^{-k}\sum_{\mathfrak{c}\in\mathrm{Cl}_F}\mathrm{N}(\mathfrak{c})^k\sum_{\substack{d\in(\mathfrak{d}\mathfrak{t}_\lambda\mathfrak{c}_0)^{-1}\mathfrak{c},\\ d\bmod U,\\ d\neq0}}\mathrm{sgn}(-d\gamma_\lambda)^q\eta(-d\gamma_\lambda\mathfrak{c}^{-1})\mathrm{N}(d)^{-k}\\ &=C\mathrm{N}(\mathfrak{d}\mathfrak{c}_0)^k\mathrm{sgn}(-\gamma_\lambda)^q\eta(\gamma_\lambda(\mathfrak{d}\mathfrak{t}_\lambda\mathfrak{c}_0)^{-1})\mathrm{N}(\mathfrak{d}\mathfrak{t}_\lambda\mathfrak{c}_0)^k\\ &\qquad\times\sum_{\mathfrak{c}\in\mathrm{Cl}_F}\sum_{\substack{d\in(\mathfrak{d}\mathfrak{t}_\lambda\mathfrak{c}_0)^{-1}\mathfrak{c},\\ d\bmod U,\\ d\neq0}}\eta(-d\mathfrak{d}\mathfrak{t}_\lambda\mathfrak{c}_0\mathfrak{c}^{-1})\mathrm{N}(d\mathfrak{d}\mathfrak{t}_\lambda\mathfrak{c}_0\mathfrak{c}^{-1})^{-k}.\end{aligned}$$

Combining this and the functional equation

$$\frac{(d_F)^{\frac{1}{2}-k}\mathrm{N}(\mathfrak{a})^{1-k}(2\pi i)^{kg}}{2^g\Gamma(k)^g\tau(\eta^{-1})}L(\eta^{-1},1-k)=L(\eta,k)$$

for $L(\eta,s)$ (see [M], Chapter 3, Section 3), we see that the constant term is equal to

$$\frac{1}{2^g}\tau(\eta)\left(\frac{\mathrm{N}(\mathfrak{c}_0)}{\mathrm{N}(\mathfrak{a})}\right)^k\mathrm{sgn}(-\gamma_\lambda)^q\eta(\mathfrak{n}_1)L(\eta^{-1},1-k).$$

We give a summary of our computation as a theorem.

THEOREM 3.9. (i) *For a matrix*

$$A_\lambda = \begin{pmatrix} \alpha_\lambda & \beta_\lambda \\ \gamma_\lambda & \delta_\lambda \end{pmatrix} \in \Gamma^1_\lambda(O),$$

we write $\gamma_\lambda O = \mathfrak{n}_1 \mathfrak{d} \mathfrak{t}_\lambda$. *Then the constant term of* $\mathrm{N}(\mathfrak{t}_\lambda)^{-\frac{k}{2}} E_k(\eta, \psi)_\lambda | A_\lambda$ *is equal to* 0 *unless* $\mathfrak{b} \mid \mathfrak{n}_1$. *If this is the case, the constant term is equal to*

$$\frac{1}{2^g} \frac{\tau(\eta\psi^{-1})}{\tau(\psi^{-1})} \left(\frac{\mathrm{N}(\mathfrak{b})}{\mathrm{N}(\mathfrak{f})}\right)^k \mathrm{sgn}(-\gamma_\lambda)^q \eta(\gamma_\lambda(\mathfrak{b}\mathfrak{d}\mathfrak{t}_\lambda)^{-1}) \mathrm{sgn}(\alpha_\lambda)^r \psi^{-1}(\alpha_\lambda O)$$
$$\times L(\eta^{-1}\psi, 1-k) \prod_{\mathfrak{q} \mid \mathfrak{m},\ \mathfrak{q} \nmid \mathfrak{f}} (1 - \eta\psi^{-1}(\mathfrak{q})\mathrm{N}(\mathfrak{q})^{-k}).$$

(ii) *Let*

$$\mathfrak{c}_0,\ A_\lambda = \begin{pmatrix} \alpha_\lambda & \beta_\lambda \\ \gamma_\lambda & \delta_\lambda \end{pmatrix} \in SL_2(F),\ \mathfrak{n}_i\ (i = 1, 2)$$

be as in Proposition 3.8. *Then the constant term of* $\mathrm{N}(\mathfrak{t}_\lambda)^{-\frac{k}{2}} E_k(\eta, \psi)_\lambda | A_\lambda$ *is*

$$\delta_{\psi,1} \frac{1}{2^g} \tau(\eta) \left(\frac{\mathrm{N}(\mathfrak{c}_0)}{\mathrm{N}(\mathfrak{a})}\right)^k \mathrm{sgn}(-\gamma_\lambda)^q \eta(\mathfrak{n}_1) L(\eta^{-1}, 1-k).$$

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