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On a Willmore-Helfrich L^2-flow of open curves in $\mathbb{R}^n$: a different approach

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1 Introduction

In [3] we consider regular open curves in $\mathbb{R}^n$ with fixed boundary points and moving according to the L^2-gradient flow for a generalisation of the Helfrich functional. Natural boundary conditions are imposed along the evolution. A long-time existence result together with sub-convergence to critical points is proven.

The aim of the present work is to propose and sketch *a different proof* of the long-time existence result. This is interesting in its own right but most importantly it gives us the opportunity to discuss from yet another point of view some of the most important ideas that underly the proof given in [3] and the related results presented in [4], [5], [1], and [2]. In order to focus on the ideas and in order not to burden the reader with details and technicalities, it is our choice to omit some steps of the proof. More precisely, being the proof by induction, we concentrate on the first step and provide the interested reader with the formulas needed to perform the induction step. Also, for the sake of brevity, we refrain from giving any history about the problem and motivation for studying it, but simply refer to the above mentioned works for more information.

2 Statement of the problem and notation

We consider a time dependent curve $f : [0,T) \times \bar{I} \rightarrow \mathbb{R}^n$, $f = f(t,x)$, with $n \geq 2$, $I = (0,1)$ and with endpoints fixed in time, that is $f(t,0) = f_-$, $f(t,1) = f_+$ for given vectors $f_-, f_+ \in \mathbb{R}^n$, $f_- \neq f_+$.

We denote by s the arc-length parameter. Then $ds = |f_x|dx$, $\partial_s = \frac{1}{|f_x|}\partial_x$, $\tau = \partial_s f$ is the tangent unit vector and the curvature vector is given by $\vec{\kappa} = \partial_{ss} f$. In the following, vector fields with an arrow on top are normal vector fields. The standard scalar product in $\mathbb{R}^n$ is denoted by $\langle \cdot, \cdot \rangle$, while $\nabla_s \phi$ (resp. $\nabla_t \phi$) is the normal component of $\partial_s \phi$ (resp. $\partial_t \phi$) for a vector field ϕ. That is,

$$\nabla_s \phi = \partial_s \phi - \langle \partial_s \phi, \tau \rangle \tau \qquad (\text{resp.} \quad \nabla_t \phi = \partial_t \phi - \langle \partial_t \phi, \tau \rangle \tau) \,.$$

The Willmore-Helfrich energy for the curve f is given by

$$W_\lambda(f) = \int_I \left(\frac{1}{2} |\vec{\kappa}|^2 - \langle \vec{\kappa}, \zeta \rangle \right) ds + \lambda \int_I ds, \tag{2.1}$$

where ζ is a given vector in $\mathbb{R}^n$ and $\lambda \geq 0$ a second parameter. In this paper we study

$$\partial_t f = -\nabla_s^2 \vec{\kappa} - \frac{1}{2}|\vec{\kappa}|^2 \vec{\kappa} + \lambda \vec{\kappa}, \tag{2.2}$$

for a smooth regular curve f subject to the boundary conditions

$$\begin{aligned} & f(t,0) = f_- , \quad f(t,1) = f_+ , \\ & \vec{\kappa}(t,0) = \zeta - \langle \zeta, \tau(t,0) \rangle \tau(t,0) , \qquad \text{for all } t \in (0,T) \\ & \vec{\kappa}(t,1) = \zeta - \langle \zeta, \tau(t,1) \rangle \tau(t,1) , \end{aligned} \tag{2.3}$$

and for some smooth initial data f_0. In [3] (cf. also Lemma 3.3 below) we showed that equation (2.2) corresponds to the L^2-gradient flow for W_λ and that the boundary conditions considered are natural in the usual sense of calculus of variation.

Moreover we proved that for smooth initial data $f(0,\cdot) = f_0(\cdot)$ the flow exists for all time, more precisely

Theorem 2.1. *Let $\lambda \geq 0$, and let vectors $f_+, f_-, \zeta \in \mathbb{R}^n$ with $f_+ \neq f_-$ be given as well as a smooth regular curve $f_0 : \bar{I} \to \mathbb{R}^n$ satisfying*

$$\begin{aligned} & f_0(0) = f_-, \ f_0(1) = f_+ , \\ & \kappa[f_0](x) + \langle \zeta, \tau[f_0](x) \rangle \tau[f_0](x) = \zeta \ \textit{for } x \in \{0,1\} , \end{aligned}$$

with $\vec{\kappa}[f_0]$ and $\tau[f_0]$ the curvature and tangent vector of f_0 respectively, together with suitable compatibility conditions. Then a smooth solution $f : [0,T) \times [0,1] \to \mathbb{R}^n$ of the initial value problem

$$\begin{cases} \partial_t f = -\nabla_s^2 \vec{\kappa} - \frac{1}{2}|\vec{\kappa}|^2 \vec{\kappa} + \lambda \vec{\kappa} \\ f(0,x) = f_0(x) \ \textit{for } x \in [0,1] \\ f(t,0) = f_-, f(1,t) = f_+ \ \textit{for } t \in [0,T) \\ \vec{\kappa}(t,x) + \langle \zeta, \tau(t,x) \rangle \tau(t,x) = \zeta \ \textit{for } x \in \{0,1\} \ \textit{and for } t \in [0,T), \end{cases} \tag{2.4}$$

exists for all times, that is we may take $T = \infty$. Moreover if $\lambda > 0$, then as $t_i \to \infty$ the curves $f(t_i,.)$ subconverge, when reparametrized by arc-length, to a critical point of the Willmore-Helfrich functional with fixed endpoints, that is to a solution of

$$\begin{cases} -\nabla_s^2 \vec{\kappa} - \frac{1}{2}|\vec{\kappa}|^2 \vec{\kappa} + \lambda \vec{\kappa} = 0 , \\ f(0) = f_- , f(1) = f_+ , \\ \vec{\kappa}(x) + \langle \zeta, \tau(x) \rangle \tau(x) = \zeta \ \textit{for } x \in \{0,1\} . \end{cases} \tag{2.5}$$

In the following we want to sketch a new proof for the long time existence result. For simplicity we restrict to the (from a geometrical point of view most interesting) case where $\lambda > 0$.

First of all we have to recall some important facts.

3 Preliminary results

3.1 Geometrical lemmata

In the following lemma we collect important formulae for the variation of some geometrical quantities of the flow. Note that the velocity in (2.2) has no tangential component.

Lemma 3.1. *Let $f : [0,T) \times \bar{I} \to \mathbb{R}^n$, $f = f(t,x)$, be a smooth solution of $\partial_t f = \vec{V}$ for $t \in (0,T)$, $x \in I$, and with $\vec{V}$ the normal velocity. Given $\vec{\phi}$ any smooth normal field along f, the following formulae hold.*

$$\partial_t(ds) = -\langle \vec{\kappa}, \vec{V} \rangle ds \,, \tag{3.1}$$

$$\partial_t \partial_s - \partial_s \partial_t = \langle \vec{\kappa}, \vec{V} \rangle \partial_s \,, \tag{3.2}$$

$$\partial_t \tau = \nabla_s \vec{V} \,, \tag{3.3}$$

$$\partial_t \vec{\phi} = \nabla_t \vec{\phi} - \langle \nabla_s \vec{V}, \vec{\phi} \rangle \tau \,, \tag{3.4}$$

$$\partial_t \vec{\kappa} = \partial_s \nabla_s \vec{V} + \langle \vec{\kappa}, \vec{V} \rangle \vec{\kappa} \,, \tag{3.5}$$

$$\nabla_t \vec{\kappa} = \nabla_s^2 \vec{V} + \langle \vec{\kappa}, \vec{V} \rangle \vec{\kappa} \,, \tag{3.6}$$

$$(\nabla_t \nabla_s - \nabla_s \nabla_t)\vec{\phi} = \langle \vec{\kappa}, \vec{V} \rangle \nabla_s \vec{\phi} + [\langle \vec{\kappa}, \vec{\phi} \rangle \nabla_s \vec{V} - \langle \nabla_s \vec{V}, \vec{\phi} \rangle \vec{\kappa}]. \tag{3.7}$$

Proof. All statement follow by direct calculation. See [3, Lemma 2.1] and references given in there. □

In the next lemma we highlight the fact that, due to the boundary conditions, some quantities are zero at the boundary.

Lemma 3.2. *Under the assumption that f solves $\partial_t f = \vec{V} = -\nabla_s^2 \vec{\kappa} - \frac{1}{2}|\vec{\kappa}|^2 \vec{\kappa} + \lambda \vec{\kappa}$ on $(0,T) \times I$ with boundary conditions (2.3), we have that for $m \in \mathbb{N}_0$*

$$\partial_t f = \nabla_t f = 0, \qquad \nabla_t^{m+1} f = 0 \quad \text{and} \quad \nabla_t^{m+1}(\vec{\kappa} + \langle \zeta, \tau \rangle \tau) = 0 \quad \text{for } x \in \{0,1\}\,.$$

Proof. From the boundary conditions (2.3) we infer that $\partial_t^m f = \partial_t^m(\vec{\kappa} + \langle \zeta, \tau \rangle \tau) = 0$ at the boundary for any $m \in \mathbb{N}$. The statement follows. □

Next we show that the energy decreases during the evolution.

Lemma 3.3. *Let $f : [0,T) \times \bar{I} \to \mathbb{R}^n$ be a sufficiently smooth solution of (2.2) satisfying (2.3) for all t. Then,*

$$\frac{d}{dt} W_\lambda(f) \leq 0\,.$$

Proof. For the sake of readability we report here the proof given in [3, Lemma A.2]. Using (3.6), (3.5), (3.1) we can write

$$\frac{d}{dt} W_\lambda(f) = \int_I (\langle \vec{\kappa}, \nabla_t \vec{\kappa} \rangle - \langle \zeta, \partial_t \vec{\kappa} \rangle)\, ds + \int_I \left(\frac{1}{2}|\vec{\kappa}|^2 - \langle \zeta, \vec{\kappa} \rangle + \lambda \right) \partial_t(ds)$$

$$= \int_I \left(\langle \vec{\kappa}, \nabla_s^2 \vec{V} + \langle \vec{\kappa}, \vec{V} \rangle \vec{\kappa} \rangle - \langle \zeta, \partial_s \nabla_s \vec{V} + \langle \vec{\kappa}, \vec{V} \rangle \vec{\kappa} \rangle \right) ds$$
$$- \int_I \left(\frac{1}{2} |\vec{\kappa}|^2 - \langle \zeta, \vec{\kappa} \rangle + \lambda \right) \langle \vec{\kappa}, \vec{V} \rangle ds$$
$$= \int_I \langle \vec{\kappa}, \nabla_s^2 \vec{V} \rangle ds - \int_I \langle \zeta, \partial_s \nabla_s \vec{V} \rangle ds + \int_I \langle \frac{1}{2} \vec{\kappa} |\vec{\kappa}|^2 - \lambda \vec{\kappa}, \vec{V} \rangle ds.$$

Integration by parts, (2.3), and the fact that $\vec{V}$ is zero at the boundary, give

$$\frac{d}{dt} W_\lambda(f) = [\langle \vec{\kappa} - \zeta, \nabla_s \vec{V} \rangle]_0^1 - \int_I \langle \nabla_s \vec{\kappa}, \nabla_s \vec{V} \rangle ds + \int_I \langle \frac{1}{2} \vec{\kappa} |\vec{\kappa}|^2 - \lambda \vec{\kappa}, \vec{V} \rangle ds$$
$$= -[\langle \nabla_s \vec{\kappa}, \vec{V} \rangle]_0^1 + \int_I \langle \nabla_s^2 \vec{\kappa} + \frac{1}{2} \vec{\kappa} |\vec{\kappa}|^2 - \lambda \vec{\kappa}, \vec{V} \rangle ds = - \int_I |\vec{V}|^2 ds \leq 0.$$

□

The next lemma shows how the L^2-norm of an arbitrary normal vector field ϕ develops in time.

Lemma 3.4. *Suppose* $\partial_t f = \vec{V}$ *on* $(0,T) \times I$. *Let* $\vec{\phi}$ *be a normal vector field along* f *and* $Y = \nabla_t \vec{\phi} + \nabla_s^4 \vec{\phi}$. *Then*

$$\frac{d}{dt} \frac{1}{2} \int_I |\vec{\phi}|^2 ds + \int_I |\nabla_s^2 \vec{\phi}|^2 ds = -[\langle \vec{\phi}, \nabla_s^3 \vec{\phi} \rangle]_0^1 + [\langle \nabla_s \vec{\phi}, \nabla_s^2 \vec{\phi} \rangle]_0^1 + \int_I \langle Y, \vec{\phi} \rangle ds - \frac{1}{2} \int_I |\vec{\phi}|^2 \langle \vec{\kappa}, \vec{V} \rangle ds, \tag{3.8}$$

and if furthermore $\vec{\phi} = 0$ *on* ∂I *then*

$$\frac{d}{dt} \frac{1}{2} \int_I |\vec{\phi}|^2 ds + \int_I |\nabla_s^2 \vec{\phi}|^2 ds = [\langle \nabla_s \vec{\phi}, \nabla_s^2 \vec{\phi} \rangle]_0^1 + \int_I \langle Y, \vec{\phi} \rangle ds - \frac{1}{2} \int_I |\vec{\phi}|^2 \langle \vec{\kappa}, \vec{V} \rangle ds. \tag{3.9}$$

Proof. See [3, Lemma 2.3], and [4, Lemma 2.2], [5, Lemma 3] for similar statements. The claim follows using (3.1) and integration by parts. □

Typically the previous lemma is used to get upper bounds for the L^2-norm of $\vec{\phi}$ squared using Gronwall's Lemma and suitable interpolation estimates.

3.2 Some technical lemmas

Since the number of terms in the equation explodes every time we interchange spatial and time derivatives (see for instance (3.7), (3.2)), it is important to use a concise notation that captures all relevant information. Here we recall briefly how this is done and list some important technical results.

For normal vector fields $\vec{\phi}_1, \ldots, \vec{\phi}_k$, the product $\vec{\phi}_1 * \cdots * \vec{\phi}_k$ defines for even k a function given by $\langle \vec{\phi}_1, \vec{\phi}_2 \rangle \ldots \langle \vec{\phi}_{k-1}, \vec{\phi}_k \rangle$, while for k odd it defines a normal vector field given by $\langle \vec{\phi}_1, \vec{\phi}_2 \rangle \ldots \langle \vec{\phi}_{k-2}, \vec{\phi}_{k-1} \rangle \vec{\phi}_k$.

For $\vec{\phi}$ a normal vector field, $P_b^{a,c}(\vec{\phi})$ denotes any linear combination of terms of type

$$\nabla_s^{i_1}\vec{\phi} * \cdots * \nabla_s^{i_b}\vec{\phi} \text{ with } i_1 + \cdots + i_b = a \text{ and } \max i_j \leq c\,,$$

with coefficients bounded by some universal constant. Notice that a gives the total number of derivatives, b gives the number of factors and c gives the highest number of derivatives falling on one factor. A simple computation give that $\nabla_s P_b^{a,c}(\vec{\phi}) = P_b^{a+1,c+1}(\vec{\phi})$ when b is an odd natural number.

For sums over a, b and c we set

$$\sum_{\substack{[[a,b]]\leq[[A,B]]\\ c\leq C}} P_b^{a,c}(\vec{\phi}) := \sum_{a=0}^{A}\sum_{b=1}^{2A+B-2a}\sum_{c=0}^{C} P_b^{a,c}(\vec{\phi})\,. \tag{3.10}$$

The range or nature (even/odd) of the b's will also be often specified at the bottom of the summation symbol. Similarly we set $\sum_{\substack{[[a,b]]\leq[[A,B]]\\ c\leq C}} |P_b^{a,c}(\vec{\phi})| := \sum_{a=0}^{A}\sum_{b=1}^{2A+B-2a}\sum_{c=0}^{C} |P_b^{a,c}(\vec{\phi})|\,.$ In [3] we explained that it is important to understand the relation between a and b in the sum: the more derivatives we take the less factors are present. This relation has its origin in the equation that f satisfies and is maintained in the equations obtained by differentiation. Moreover notice that for the application of interpolation inequalities it is important to observe that for all terms in the sum (3.10)

$$a + \frac{1}{2}b \leq a + \frac{1}{2}(2A + B - 2a) = A + \frac{1}{2}B. \tag{3.11}$$

Last but not least we mention that simple computations gives

$$\nabla_t(h\tau) = h\nabla_t\tau, \qquad \nabla_s(h\vec{\phi}) = \partial_s h\,\vec{\phi} + h\nabla_s\vec{\phi}, \qquad \nabla_t(h\vec{\phi}) = \partial_t h\,\vec{\phi} + h\nabla_t\vec{\phi}, \tag{3.12}$$

for a scalar function $h : [0,T) \times I \to \mathbb{R}$ and a normal vector field $\vec{\phi} : [0,T) \times I \to \mathbb{R}^n$.

In the following lemma we collect the formulae needed.

Lemma 3.5. *Suppose $f : [0,T) \times \bar{I} \to \mathbb{R}^n$ is a smooth regular solution to (2.2) in $(0,T) \times I$. Then, the following formulae hold on $(0,T) \times I$.*

1. *For any $\ell \in \mathbb{N}_0$, we have that*

$$\nabla_t\nabla_s^{\ell}\vec{\kappa} = -\nabla_s^{\ell+4}\vec{\kappa} + \lambda\nabla_s^{\ell+2}\vec{\kappa} + \sum_{\substack{[[a,b]]\leq[[\ell+2,3]]\\ c\leq\ell+2,\, b\, odd}} P_b^{a,c}(\vec{\kappa}) + \lambda \sum_{\substack{[[a,b]]\leq[[\ell,3]]\\ c\leq\ell,\, b\, odd}} P_b^{a,c}(\vec{\kappa})\,. \tag{3.13}$$

2. *For any $A, C \in \mathbb{N}_0$, $B, N, M \in \mathbb{N}$, B odd,*

$$\nabla_t \sum_{\substack{[[a,b]]\leq[[A,B]]\\ c\leq C,\, b\, odd}} P_b^{a,c}(\vec{\kappa}) = \sum_{\substack{[[a,b]]\leq[[A+4,B]]\\ c\leq C+4,\, b\, odd}} P_b^{a,c}(\vec{\kappa}) + \lambda \sum_{\substack{[[a,b]]\leq[[A+2,B]]\\ c\leq C+2,\, b\, odd}} P_b^{a,c}(\vec{\kappa})\,. \tag{3.14}$$

3. For any $m \in \mathbb{N}$

$$\begin{aligned}\nabla_t^m \vec{\kappa} - (-1)^m \nabla_s^{4m} \vec{\kappa} \qquad\qquad (3.15)\\ = \sum_{\substack{[[a,b]]\leq[[4m-2,3]]\\ c\leq 4m-2,\, b\, odd}} P_b^{a,c}(\vec{\kappa}) + \sum_{i=1}^{m} \lambda^i \sum_{\substack{[[a,b]]\leq[[4m-2i,1]]\\ c\leq 4m-2i,\, b\, odd}} P_b^{a,c}(\vec{\kappa}).\end{aligned}$$

4. For any $m \in \mathbb{N}$

$$\begin{aligned}\nabla_t^m f - (-1)^m \nabla_s^{4m-2} \vec{\kappa} \qquad\qquad (3.16)\\ = \sum_{\substack{[[a,b]]\leq[[4m-4,3]]\\ c\leq 4m-4,\, b\, odd}} P_b^{a,c}(\vec{\kappa}) + \sum_{i=1}^{m} \lambda^i \sum_{\substack{[[a,b]]\leq[[4m-2-2i,1]]\\ c\leq 4m-2-2i,\, b\, odd}} P_b^{a,c}(\vec{\kappa}).\end{aligned}$$

Proof. The proof is rather long and technical: equation (3.13) is proved in [2, Lemma 2.5], while for equations (3.14) to (3.16) see [3, Lemma 3.1]. □

The above lemma allows us to infer some information about the order reduction of the derivatives of the curvature vector at the boundary.

Lemma 3.6. *Suppose $f : [0,T) \times \bar{I} \to \mathbb{R}^n$ is a smooth regular solution to (2.2) in $(0,T) \times I$. At the boundary we have for $m \in \mathbb{N}$*

$$(-1)^{m+1} \nabla_s^{4m-2} \vec{\kappa} = \sum_{\substack{[[a,b]]\leq[[4m-4,3]]\\ c\leq 4m-4,\, b\, odd}} P_b^{a,c}(\vec{\kappa}) + \sum_{i=1}^{m} \lambda^i \sum_{\substack{[[a,b]]\leq[[4m-2-2i,1]]\\ c\leq 4m-2-2i,\, b\, odd}} P_b^{a,c}(\vec{\kappa}).$$

Proof. The statement follows from Lemma 3.2 and (3.16). □

3.3 Interpolation inequalities

Here let us recall important interpolation inequalities. To that end we need to introduce the following norms

$$\|\vec{\kappa}\|_{k,p} := \sum_{i=0}^{k} \|\nabla_s^i \vec{\kappa}\|_p \quad \text{with} \quad \|\nabla_s^i \vec{\kappa}\|_p := \mathcal{L}[f]^{i+1-1/p} \Big(\int_I |\nabla_s^i \vec{\kappa}|^p ds \Big)^{1/p},$$

as opposed to

$$\|\nabla_s^i \vec{\kappa}\|_{L^p} := \Big(\int_I |\nabla_s^i \vec{\kappa}|^p ds \Big)^{1/p}.$$

These norms are motivated by suitable scaling properties (see [3, § 4]).

Lemma 3.7. *Let $f : I \to \mathbb{R}^n$ be a smooth regular curve. Then for all $k \in \mathbb{N}$, $p \geq 2$ and $0 \leq i < k$ we have*

$$\|\nabla_s^i \vec{\kappa}\|_p \leq C \|\vec{\kappa}\|_2^{1-\alpha} \|\vec{\kappa}\|_{k,2}^{\alpha},$$

with $\alpha = (i + \frac{1}{2} - \frac{1}{p})/k$ and $C = C(n,k,p)$.

Proof. A proof of this fact is hinted at in [4, Lemma 2.4] and [5, Lemma 5]. A detailed proof is given in [3, Lemma 4.1]. □

Corollary 3.8. *Let $f : I \to \mathbb{R}^n$ be a smooth regular curve. Then for all $k \in \mathbb{N}$ we have*

$$\|\vec{\kappa}\|_{k,2} \leq C(\|\nabla_s^k \vec{\kappa}\|_2 + \|\vec{\kappa}\|_2)\,,$$

with $C = C(n,k)$.

Proof. If follows by the above lemma and an induction argument: see [3, Corollary 4.2]. □

Lemma 3.9. *For any $a, c \in \mathbb{N}_0$, $b \in \mathbb{N}$, $b \geq 2$, $c \leq k-1$ we find*

$$\int_I |P_b^{a,c}(\vec{\kappa})|\, ds \leq C\mathcal{L}[f]^{1-a-b}\|\vec{\kappa}\|_2^{b-\gamma}\|\vec{\kappa}\|_{k,2}^{\gamma}\,,$$

with $\gamma = (a + \frac{1}{2}b - 1)/k$ and $C = C(n,k,b)$. Further if $A, B, M \in \mathbb{N}$, $M \geq 2$ with $A + \frac{1}{2}B < 2k+1$, then for any $\epsilon \in (0,1)$

$$\sum_{\substack{[[a,b]]\leq[[A,B]]\\ c\leq k-1\\ b\in[2,M]}} \int_I |P_b^{a,c}(\vec{\kappa})| \leq \epsilon \int_I |\nabla_s^k \vec{\kappa}|^2 ds + C\epsilon^{-\frac{\overline{\gamma}}{2-\overline{\gamma}}} \max\{1, \|\vec{\kappa}\|_{L^2}^2\}^{\frac{M-\overline{\gamma}}{2-\overline{\gamma}}}$$

$$+ C\min\{1, \mathcal{L}[f]\}^{1-A-\frac{B}{2}} \max\{1, \|\vec{\kappa}\|_{L^2}\}^M + C\|\vec{\kappa}\|_{L^2}^2\,,$$

with $\overline{\gamma} = (A + \frac{1}{2}B - 1)/k$ and $C = C(n,k,A,B)$.

Note that the right-hand side of the second inequality depends only on the *lower bound* of the length of the curve.

Proof. For the first claim one uses Hölder inequality and Lemma 3.7. The second claim follows with Young inequality. See [3, Lemma 4.3] for details. □

In the more recent work [2] the authors were able to sharpen the above estimate in the sense that, under suitable conditions, one is able to allow for the case where $c = k$.

Lemma 3.10. *Let $f : I \to \mathbb{R}^n$ be a smooth regular curve and $\ell \in \mathbb{N}_0$. If $A, B \in \mathbb{N}$ with $B \geq 2$ and $A + \frac{1}{2}B < 2\ell + 5$ then we have*

$$\sum_{\substack{[[a,b]]\leq[[A,B]]\\ c\leq \ell+2,\, 2\leq b}} \int_I |P_b^{a,c}(\vec{\kappa})|\, ds \leq C\min\{1, \mathcal{L}([f])\}^{1-2A-B} \max\{1, \|\vec{\kappa}\|_2\}^{2A+B} \max\{1, \|\vec{\kappa}\|_{\ell+2,2}\}^{\overline{\gamma}}\,, \tag{3.17}$$

and for any $\varepsilon \in (0,1)$

$$\sum_{\substack{[[a,b]]\leq[[A,B]]\\ c\leq \ell+2,\, 2\leq b}} \int_I |P_b^{a,c}(\vec{\kappa})| \leq \varepsilon \int_I |\nabla_s^{\ell+2} \vec{\kappa}|^2 ds + C\varepsilon^{-\frac{\overline{\gamma}}{2-\overline{\gamma}}} \max\{1, \|\vec{\kappa}\|_{L^2}^2\}^{\frac{2A+B}{2-\overline{\gamma}}} \tag{3.18}$$

$$+ C\min\{1, \mathcal{L}[f]\}^{1-A-\frac{B}{2}} \max\{1, \|\vec{\kappa}\|_{L^2}\}^{2A+B}\,,$$

with $\overline{\gamma} = (A + \frac{1}{2}B - 1)/(\ell+2)$ and $C = C(n,\ell,A,B)$.

Proof. It follows from Lemma 3.9 and a careful use of the Cauchy- Schwarz inequality: see [2, Lemma 3.5] for more details. □

The following estimates are also useful in the proof of long-time existence.

Lemma 3.11. *Assume that* $\|\vec{\kappa}\|_{L^2} \leq C$. *If* $\|\nabla_t^m(\vec{\kappa} + \langle\zeta,\tau\rangle\tau)\|_{L^2} \leq C$, *for some* $m \in \mathbb{N}$, *then it follows that*

$$\|\nabla_s^i \vec{\kappa}\|_{L^2} \leq C, \qquad \text{for all } 0 \leq i \leq 4m.$$

The constant C *depends on* λ, n, m, ζ, *and on the lower bound on* $\mathcal{L}[f]$.

Proof. Here we give a proof of the statement only for $m = 1$. Let $\vec{\phi} = \nabla_t(\vec{\kappa} + \langle\zeta,\tau\rangle\tau)$. Using (4.3) below and $(\sum_{i=1}^q a_i)^2 \leq q \sum_i^q a_i^2$ we can write

$$\begin{aligned}
\|\nabla_s^4\vec{\kappa}\|_{L^2}^2 &\leq 2\|\nabla_s^4\vec{\kappa} + \vec{\phi}\|_{L^2}^2 + 2\|\vec{\phi}\|_{L^2} \\
&\leq C\int_I \sum_{\substack{[[a,b]]\leq[[4,6]] \\ c\leq 2,\, b\, even}} |P_b^{a,c}(\vec{\kappa})|ds + C\lambda^2 \int_I \sum_{\substack{[[a,b]]\leq[[4,2]] \\ c\leq 2,\, b\, even}} |P_b^{a,c}(\vec{\kappa})|ds \\
&\quad + C|\zeta|^2 \int_I \sum_{\substack{[[a,b]]\leq[[6,2]] \\ c\leq 3,\, b\, even}} |P_b^{a,c}(\vec{\kappa})|ds + C|\zeta|^2 \int_I \lambda^2|\nabla_s\vec{\kappa}|^2 ds + C \\
&\leq \epsilon(1+|\zeta|^2)\int_I |\nabla_s^4\vec{\kappa}|^2 ds + C(\zeta,\epsilon),
\end{aligned}$$

where we have used Lemma 3.9 in the last inequality. Choosing ϵ appropriately yields $\|\nabla_s^4\vec{\kappa}\|_{L^2} \leq C$. Again with Lemma 3.9 one obtains bounds for the derivatives of lower order and the claim for $m = 1$ follows.

The case $m \geq 2$ can be proved with similar arguments. □

So far we have derived bounds for the normal component of the derivatives of the curvature. The following lemmata indicate how to gain control over the whole derivative.

Lemma 3.12. *We have the identities*

$$\begin{aligned}
\partial_s\vec{\kappa} &= \nabla_s\vec{\kappa} - |\vec{\kappa}|^2\tau, \\
\partial_s^m\vec{\kappa} &= \nabla_s^m\vec{\kappa} + \tau \sum_{\substack{[[a,b]]\leq[[m-1,2]] \\ c\leq m-1 \\ b\in[2,2[\frac{m+1}{2}]],even}} P_b^{a,c}(\vec{\kappa}) + \sum_{\substack{[[a,b]]\leq[[m-2,3]] \\ c\leq m-2 \\ b\in[3,2[\frac{m}{2}]+1],odd}} P_b^{a,c}(\vec{\kappa}) \qquad \textit{for } m \geq 2\,.
\end{aligned}$$

Proof. The first claim is obtained directly using that

$$\partial_s\vec{\kappa} = \nabla_s\vec{\kappa} + \langle\partial_s\vec{\kappa},\tau\rangle\tau = \nabla_s\vec{\kappa} - |\vec{\kappa}|^2\tau\,.$$

The second claim follows by induction. See [3, Lemma 4.5]. □

Lemma 3.13. *Given* $m \geq 1$, *assume that* $\|\nabla_s^m\vec{\kappa}\|_{L^2} \leq C$ *and* $\|\vec{\kappa}\|_{L^2} \leq C$. *Then we have that*

$$\|\partial_s^l\vec{\kappa}\|_{L^2} \leq C \quad \textit{for } 0 \leq l \leq m.$$

The constant C *depends on* n, m *and on the lower bound on* $\mathcal{L}[f]$.

Proof. It follows from Lemma 3.12 and Lemma 3.9. See [3, Lemma 4.6] for details. □

4 A proof of long-time existence

In this section we illustrate a new proof of the long-time existence result as formulated in Theorem 2.1 and under the assumption that $\lambda > 0$. As already stated in the introduction, our aim is to convey main ideas and avoid technicalities (which are carefully explained in [3] for a different but strictly related Ansatz).

Proof of Theorem 2.1. In the following C denotes a generic constant that may vary from line to line. We will explicitly write down what the constant depends on.

A short-time existence result gives that the solution exists in a small time interval. We assume by contradiction that the solution of (2.4) does not exist globally. Let $0 < T < \infty$ be the maximal time.

First Step: $|f_- - f_+| \leq \mathcal{L}[f] \leq C(W_\lambda(f_0), \lambda, \zeta)$ and $\int_I |\vec{\kappa}|^2\, ds \leq C(W_\lambda(f_0), \zeta)$ for $t \in (0, T)$.
We observe that the steepest descent property of the flow gives a natural bound on the L^2-norm of the curvature vector as follows. Since $W_\lambda(f(t)) \leq W_\lambda(f_0)$ for all $t \in [0, T)$ (recall Lemma 3.3), we have that

$$\frac{1}{2}\int_I |\vec{\kappa}|^2\, ds \leq \frac{1}{2}\int_I |\vec{\kappa}|^2\, ds - \int_I \langle \vec{\kappa}, \zeta\rangle\, ds + \left|\int_I \langle \vec{\kappa}, \zeta\rangle\, ds\right| \leq W_\lambda(f_0) + \left|[\langle \tau, \zeta\rangle]_0^1\right|.$$

A similar argument gives

$$\mathcal{L}[f(t)] \leq \frac{1}{\lambda}\left(W_\lambda(f(t)) + \int_I \langle \vec{\kappa}, \zeta\rangle ds\right) \leq \frac{1}{\lambda}\left(W_\lambda(f_0) + |[\langle \tau, \zeta\rangle]_0^1|\right) \leq C(W_\lambda(f_0), \lambda, \zeta). \quad (4.1)$$

The bound from below on the length of the curve is straightforward.

Strategy of the second Step:
Next, we will try to get uniform upper bounds for the L^2-norms of the curvature and its derivatives $\nabla_s^m \vec{\kappa}$, for an increasing sequence of natural numbers $m \in \mathbb{N}$. This is meaningful because Lemma 3.13 implies that every time that we can bound the L^2-norm of the curvature (which we have done in the first step) and the L^2-norm of one of its derivatives $\nabla_s^m \vec{\kappa}$ then we get (by interpolation) L^2-bounds on all derivatives of lower order $\partial_s^l \vec{\kappa}$, $0 \leq l \leq m$.

Our strategy is to apply Lemma 3.4 with $\vec{\phi} = \nabla_t^m(\vec{\kappa} + \langle \zeta, \tau\rangle \tau)$ for $m = 1, 2, \ldots$, and use Gronwall Lemma and interpolation inequalities to get upper bounds for the L^2-norm of $\vec{\phi}$. That this procedure yields the desidered estimates on the derivatives of the curvature has been already proven in Lemma 3.11 and uses the fact that $\vec{\phi}$ behaves like $\nabla_s^{4m}\vec{\kappa}$, with $m \in \mathbb{N}$ (recall (3.15)).

This is not the only reason for our choice of $\vec{\phi}$. Due to the boundary condition on the curvature vector (cf. Lemma 3.2), we have that ϕ is zero at the boundary so that we can work with (3.9). It turns out that again the boundary conditions (this time we use the fact that the end-points of the curve are kept fixed at the boundary, cf. Lemma 3.6) imply a sufficient order reduction at the boundary for the remaining boundary term $[\langle \nabla_s \vec{\phi}, \nabla_s^2 \vec{\phi}\rangle]_0^1$ to be non-problematic.

We will now work out through most relevant details of the first step ($m = 1$).

Case $m=1$: $\sup_{t\in(0,T)}\|\nabla_t(\vec{\kappa}+\langle\zeta,\tau\rangle\tau)\|_{L^2}\leq C(W_\lambda(f_0),\lambda,f_0,\zeta,f_-,f_+,n)$

Let $\vec{\phi}=\nabla_t(\vec{\kappa}+\langle\zeta,\tau\rangle\tau)$. We start from (3.9) with this choice of $\vec{\phi}$. The main idea is that the term $\int_I|\nabla_s^2\vec{\phi}|^2$ on the left-hand side can control the right-hand side. More precisely, we show that this integral behaves like $\int_I|\nabla_s^6\vec{\kappa}|^2$ and that this term can absorb the worst order terms appearing on the right-hand side. Adding $\frac{1}{2}\int_I|\vec{\phi}|^2 ds$ to both sides of (3.9) we find

$$\frac{d}{dt}\frac{1}{2}\int_I|\vec{\phi}|^2 ds+\frac{1}{2}\int_I|\vec{\phi}|^2 ds+\int_I|\nabla_s^2\vec{\phi}|^2 ds\leq\frac{1}{2}\int_I|\vec{\phi}|^2 ds+|[\langle\nabla_s\vec{\phi},\nabla_s^2\vec{\phi}\rangle]_0^1|$$
$$+|\int_I\langle Y,\vec{\phi}\rangle ds|+\frac{1}{2}|\int_I|\vec{\phi}|^2\langle\vec{\kappa},\vec{V}\rangle ds|,$$

with $Y=(\nabla_t+\nabla_s^4)\vec{\phi}$. Using on the term $\int_I|\nabla_s^2\vec{\phi}|^2$ the elementary inequality

$$|a+b|^2\geq|a|^2+|b|^2-2|a||b|\geq\frac{1}{2}|a|^2-|b|^2$$

with $a=-\nabla_s^6\vec{\kappa}$, $b=\nabla_s^2\vec{\phi}+\nabla_s^6\vec{\kappa}$ we infer

$$\begin{aligned}&\frac{d}{dt}\frac{1}{2}\int_I|\vec{\phi}|^2 ds+\frac{1}{2}\int_I|\vec{\phi}|^2 ds+\frac{1}{2}\int_I|\nabla_s^6\vec{\kappa}|^2 ds\\&\leq\int_I|\nabla_s^2\vec{\phi}+\nabla_s^6\vec{\kappa}|^2 ds+\frac{1}{2}\int_I|\vec{\phi}|^2 ds+|[\langle\nabla_s\vec{\phi},\nabla_s^2\vec{\phi}\rangle]_0^1|+|\int_I\langle Y,\vec{\phi}\rangle ds|+\frac{1}{2}|\int_I|\vec{\phi}|^2\langle\vec{\kappa},\vec{V}\rangle ds|\\&=I+II+III+IV+V.\end{aligned}\tag{4.2}$$

By interpolation inequality we show that each of the terms I, II, III, IV and V can be controlled by $\int_I|\nabla_s^6\vec{\kappa}|^2$.

For this we need first to make some computations. Using (3.15), (3.12) and (3.3) we can write

$$\begin{aligned}\vec{\phi}&=\nabla_t\vec{\kappa}+\langle\zeta,\tau\rangle\nabla_t\tau\\&=-\nabla_s^4\vec{\kappa}+\sum_{\substack{[[a,b]]\leq[[2,3]]\\c\leq 2,b\,odd}}P_b^{a,c}(\vec{\kappa})+\lambda\sum_{\substack{[[a,b]]\leq[[2,1]]\\c\leq 2,b\,odd}}P_b^{a,c}(\vec{\kappa})+\langle\zeta,\tau\rangle(\sum_{\substack{[[a,b]]\leq[[3,1]]\\c\leq 3,b\,odd}}P_b^{a,c}(\vec{\kappa})+\lambda\nabla_s\vec{\kappa}).\end{aligned}$$

Since λ is a fixed positive constant from now on we will not write separately the terms multiplied by (powers of) λ. With this notation the terms $P_b^{a,c}(\vec{\phi})$ have coefficients bounded by some constant depending on λ. We write

$$\vec{\phi}=-\nabla_s^4\vec{\kappa}+\sum_{\substack{[[a,b]]\leq[[2,3]]\\c\leq 2,b\,odd}}P_b^{a,c}(\vec{\kappa})+\langle\zeta,\tau\rangle\sum_{\substack{[[a,b]]\leq[[3,1]]\\c\leq 3,b\,odd}}P_b^{a,c}(\vec{\kappa}).\tag{4.3}$$

Then, using (3.12) again, we obtain

$$\nabla_s\vec{\phi}=-\nabla_s^5\vec{\kappa}+\sum_{\substack{[[a,b]]\leq[[3,3]]\\c\leq 3,b\,odd}}P_b^{a,c}(\vec{\kappa})+\langle\zeta,\tau\rangle\sum_{\substack{[[a,b]]\leq[[4,1]]\\c\leq 4,b\,odd}}P_b^{a,c}(\vec{\kappa})+\langle\zeta,\vec{\kappa}\rangle\sum_{\substack{[[a,b]]\leq[[3,1]]\\c\leq 3,b\,odd}}P_b^{a,c}(\vec{\kappa}).\tag{4.4}$$

Moreover using that $\langle \zeta, \partial_s \vec{\kappa} \rangle = \langle \zeta, \nabla_s \vec{\kappa} \rangle - |\vec{\kappa}|^2 \langle \zeta, \tau \rangle$ (see Lemma 3.12) we can write

$$\begin{aligned}
\nabla_s^2 \vec{\phi} &= -\nabla_s^6 \vec{\kappa} + \sum_{\substack{[[a,b]] \leq [[4,3]] \\ c \leq 4, b \text{ odd}}} P_b^{a,c}(\vec{\kappa}) + \langle \zeta, \tau \rangle \sum_{\substack{[[a,b]] \leq [[5,1]] \\ c \leq 5, b \text{ odd}}} P_b^{a,c}(\vec{\kappa}) \\
&\quad + (\langle \zeta, \nabla_s \vec{\kappa} \rangle - |\vec{\kappa}|^2 \langle \zeta, \tau \rangle) \sum_{\substack{[[a,b]] \leq [[3,1]] \\ c \leq 3, b \text{ odd}}} P_b^{a,c}(\vec{\kappa}) + 2\langle \zeta, \vec{\kappa} \rangle \sum_{\substack{[[a,b]] \leq [[4,1]] \\ c \leq 4, b \text{ odd}}} P_b^{a,c}(\vec{\kappa}) \\
&= -\nabla_s^6 \vec{\kappa} + (1 + \langle \zeta, \tau \rangle) \sum_{\substack{[[a,b]] \leq [[5,1]] \\ c \leq 5, b \text{ odd}}} P_b^{a,c}(\vec{\kappa}) \\
&\quad + \langle \zeta, \nabla_s \vec{\kappa} \rangle \sum_{\substack{[[a,b]] \leq [[3,1]] \\ c \leq 3, b \text{ odd}}} P_b^{a,c}(\vec{\kappa}) + 2\langle \zeta, \vec{\kappa} \rangle \sum_{\substack{[[a,b]] \leq [[4,1]] \\ c \leq 4, b \text{ odd}}} P_b^{a,c}(\vec{\kappa}). \qquad (4.5)
\end{aligned}$$

We are now ready to prove with the interpolation inqualities that the terms I, II and V in (4.2) can be controlled by $\int_I |\nabla_s^6 \vec{\kappa}|^2 ds$. For example, by (4.5) we know that

$$\nabla_s^2 \vec{\phi} + \nabla_s^6 \vec{\kappa} = (1 + \langle \zeta, \tau \rangle) \sum_{\substack{[[a,b]] \leq [[5,1]] \\ c \leq 5, b \text{ odd}}} P_b^{a,c}(\vec{\kappa}) + \text{lower order terms},$$

and one observes that

$$\begin{aligned}
\int_I |(1 + \langle \zeta, \tau \rangle) \sum_{\substack{[[a,b]] \leq [[5,1]] \\ c \leq 5, b \text{ odd}}} P_b^{a,c}(\vec{\kappa})|^2 ds &\leq C(\zeta) \int_I \sum_{\substack{[[a,b]] \leq [[10,2]] \\ c \leq 5, b \text{ even}}} |P_b^{a,c}(\vec{\kappa})| \\
&\leq C(\zeta)\, \epsilon \int_I |\nabla_s^6 \vec{\kappa}|^2 ds + C_\epsilon(\zeta, W(f_0), f_-, f_+, n)
\end{aligned}$$

by Lemma 3.9 with $k = 6$, $A = 10$, $B = 2$ and the bounds obtained in the first step. Proceeding similarly for the other terms we get

$$I + II + V \leq \epsilon \int_I |\nabla_s^6 \vec{\kappa}|^2 ds + C_\epsilon(\zeta, W(f_0), \lambda, f_-, f_+, n).$$

The most critical terms are III and IV. Let us first consider the boundary term $III := |[\langle \nabla_s \vec{\phi}, \nabla_s^2 \vec{\phi} \rangle]_0^1|$. In view of Lemma 3.6 and (4.5), at the boundary we have

$$\nabla_s^2 \vec{\phi} = (1 + \langle \zeta, \tau \rangle) \sum_{\substack{[[a,b]] \leq [[5,1]] \\ c \leq 5, b \text{ odd}}} P_b^{a,c}(\vec{\kappa}) + \text{lower order terms}.$$

Using (4.4) and neglecting for simplicity all lower order terms in the expressions for $\nabla_s \vec{\phi}$ and $\nabla_s^2 \vec{\phi}$ we derive (mimicking the proof of [2, Lemma 3.6])

$$III = |[\langle \nabla_s^5 \vec{\kappa}, (1 + \langle \zeta, \tau \rangle) \sum_{\substack{[[a,b]] \leq [[5,1]] \\ c \leq 5, b \text{ odd}}} P_b^{a,c}(\vec{\kappa}) \rangle]_0^1|$$

$$\leq \int_0^1 |\partial_s \langle \nabla_s^5 \vec{\kappa}, (1+\langle \zeta, \tau \rangle) \sum_{\substack{[[a,b]]\leq[[5,1]]\\ c\leq 5\,, b\, odd}} P_b^{a,c}(\vec{\kappa}) \rangle| ds$$

$$\leq C(\zeta) \int | \sum_{\substack{[[a,b]]\leq[[11,2]]\\ c\leq 6\,, b\, even}} P_b^{a,c}(\vec{\kappa})| ds + \int_I |\langle \zeta, \vec{\kappa} \rangle| | \sum_{\substack{[[a,b]]\leq[[10,2]]\\ c\leq 5\,, b\, even}} P_b^{a,c}(\vec{\kappa})| ds$$

$$\leq C(\zeta) \int \sum_{\substack{[[a,b]]\leq[[11,2]]\\ c\leq 6\,, b\, even}} |P_b^{a,c}(\vec{\kappa})| ds + C(\zeta) \int_I \sum_{\substack{[[a,b]]\leq[[10,3]]\\ c\leq 5\,, b\, odd}} |P_b^{a,c}(\vec{\kappa})| ds.$$

Using (3.18) and the bounds obtained in the first step, and estimating the neglected lower order terms in a similar manner, we obtain

$$III \leq \epsilon \int_I |\nabla_s^6 \vec{\kappa}|^2 ds + C_\epsilon(\zeta, W(f_0), \lambda, f_-, f_+, n).$$

Next let us consider the term $IV := | \int_I \langle Y, \vec{\phi} \rangle | ds$. It turns out that

$$Y = (\nabla_t + \nabla_s^4)\vec{\phi} = (\nabla_t + \nabla_s^4)(\nabla_t \vec{\kappa}) + (\nabla_t + \nabla_s^4)(\langle \zeta, \tau \rangle \nabla_t \tau) = Q_1 + Q_2$$

is of lower order than expected. This fact has to do with the structure of the pde (2.2) and is best visualized by equation (3.15) with $m = 1$. Let us take a closer look at each term. Using (3.15) with $m = 2$ and (3.13) with $\ell = 0$ we immediately infer

$$Q_1 = (\nabla_t + \nabla_s^4)(\nabla_t \vec{\kappa}) = \sum_{\substack{[[a,b]]\leq[[6,3]]\\ c\leq 6\,, b\, odd}} P_b^{a,c}(\vec{\kappa}).$$

For Q_2 we observe that with (3.3) and (3.12) we can write

$$\begin{aligned} Q_2 &= (\nabla_t + \nabla_s^4)(\langle \zeta, \tau \rangle \nabla_t \tau) = \langle \zeta, \nabla_s \vec{V} \rangle \nabla_s \vec{V} + \langle \zeta, \tau \rangle \nabla_t \nabla_s \vec{V} + \nabla_s^4(\langle \zeta, \tau \rangle \nabla_s \vec{V}) \\ &= \langle \zeta, \nabla_s \vec{V} \rangle \nabla_s \vec{V} + \langle \zeta, \tau \rangle (\nabla_t \nabla_s \vec{V} + \nabla_s^5 \vec{V}) \\ &\quad + \langle \zeta, \partial_s^3 \vec{\kappa} \rangle \nabla_s \vec{V} + 4\langle \zeta, \partial_s^2 \vec{\kappa} \rangle \nabla_s^2 \vec{V} + 6\langle \zeta, \partial_s \vec{\kappa} \rangle \nabla_s^3 \vec{V} + 4\langle \zeta, \vec{\kappa} \rangle \nabla_s^4 \vec{V}. \end{aligned} \tag{4.6}$$

At a first sight in the equation above the worst order terms seem to be $\langle \zeta, \tau \rangle (\nabla_t \nabla_s \vec{V} + \nabla_s^5 \vec{V})$. However, this is not the case since there is a cancellation. Indeed, writing

$$\vec{V} = -\nabla_s^2 \vec{\kappa} + \sum_{\substack{[[a,b]]\leq[[0,3]]\\ c\leq 0\,, b\, odd}} P_b^{a,c}(\vec{\kappa}) = \sum_{\substack{[[a,b]]\leq[[2,1]]\\ c\leq 2\,, b\, odd}} P_b^{a,c}(\vec{\kappa})$$

and using (3.7), (3.13), and (3.14) we get

$$\begin{aligned} \nabla_t \nabla_s \vec{V} + \nabla_s^5 \vec{V} &= \nabla_s \nabla_t \vec{V} + \langle \vec{\kappa}, \vec{V} \rangle \nabla_s \vec{V} + [\langle \vec{\kappa}, \vec{V} \rangle \nabla_s \vec{V} - \langle \nabla_s \vec{V}, \vec{V} \rangle \vec{\kappa}] + \nabla_s^5 \vec{V} \\ &= \nabla_s^7 \vec{\kappa} + \sum_{\substack{[[a,b]]\leq[[5,3]]\\ c\leq 5\,, b\, odd}} P_b^{a,c}(\vec{\kappa}) - \nabla_s^7 \vec{\kappa} = \sum_{\substack{[[a,b]]\leq[[5,3]]\\ c\leq 5\,, b\, odd}} P_b^{a,c}(\vec{\kappa}). \end{aligned}$$

With Lemma 3.12 one sees that the rest of the terms in (4.6) are of lower order than Q_1. More precisely,

$$\langle \zeta, \nabla_s \vec{V}\rangle \nabla_s \vec{V} + \langle \zeta, \partial_s^3 \vec{\kappa}\rangle \nabla_s \vec{V} + 4\langle \zeta, \partial_s^2 \vec{\kappa}\rangle \nabla_s^2 \vec{V} + 6\langle \zeta, \partial_s \vec{\kappa}\rangle \nabla_s^3 \vec{V} + 4\langle \zeta, \vec{\kappa}\rangle \nabla_s^4 \vec{V}$$
$$= \sum_{i=0}^{3} \langle \zeta, \nabla_s^i \vec{\kappa}\rangle \sum_{\substack{[[a,b]]\leq[[6-i,1]]\\ c\leq 6-i\,, b\, odd}} P_b^{a,c}(\vec{\kappa}) + \langle \zeta, \tau\rangle \sum_{\substack{[[a,b]]\leq[[5,3]]\\ c\leq 5\,, b\, odd}} P_b^{a,c}(\vec{\kappa})\,.$$

The bound for IV follows using (3.18). For instance, using (4.3) and again looking only at the worst order terms, we see that

$$IV \leq \int_I |\langle \nabla_s^4 \vec{\kappa}, \sum_{\substack{[[a,b]]\leq[[6,3]]\\ c\leq 6\,, b\, odd}} P_b^{a,c}(\vec{\kappa})\rangle| ds \leq \int_I \sum_{\substack{[[a,b]]\leq[[10,4]]\\ c\leq 6\,, b\, even}} |P_b^{a,c}(\vec{\kappa})| ds$$
$$\leq \epsilon \int_I |\nabla_s^6 \vec{\kappa}|^2 ds + C_\epsilon(\zeta, W(f_0), f_-, f_+, n),$$

by (3.18) with $A = 10$, $B = 4$ and $\ell = 4$.

Putting all estimates together and choosing ϵ appropriately we finally get

$$\frac{d}{dt}\frac{1}{2}\int_I |\vec{\phi}|^2 ds + \frac{1}{2}\int_I |\vec{\phi}|^2 ds \leq C(\zeta, W(f_0), \lambda, f_-, f_+, n)$$

and a Gronwall Lemma gives our claim that $\|\vec{\phi}\|_{L^2} \leq C(\zeta, W(f_0), f_0, \lambda, f_-, f_+, n)$.

Next it is left to the reader to show with similar arguments as outlined so far that

$$\sup_{t\in(0,T)} \|\nabla_t^m(\vec{\kappa} + \langle \zeta, \tau\rangle \tau)\|_{L^2} \leq C(m, W_\lambda(f_0), \lambda, f_0, \zeta, f_-, f_+, n) \text{ for } m \in \mathbb{N}, m \geq 2.$$

Application of Lemma 3.11 and Lemma 3.13 yields that

$$\|\partial_s^l \vec{\kappa}\|_{L^2}, \|\nabla_s^l \vec{\kappa}\|_{L^2} \leq C(n, l, \lambda, W_\lambda(f_0), f_0, \zeta, f_-, f_+)$$

for any $l \in \mathbb{N}_0$.

Final steps: From now one proceeds exactly as in [3, §5, Step 6- Step 9]. There it is shown how to gain control of the L^∞-estimates of the above vectors by mean of embedding theory. Then, after deriving upper (and lower) bounds of the arc-length element $|\partial_x f|$ and its derivatives, it is shown how we can get L^∞-estimates of the curvature vector and its derivatives with respect to the original parametrization. Once this is achieved we are able to extend the solution smoothly up to the maximal time T and then by a short-time existence result even beyond T. This gives a contradiction, hence $T = \infty$. □

Remark 4.1. The statement of Theorem 2.1 is very similar in its structure to the related results given in [4, Theorem 3.2, Theorem 3.3] (elastic flow for closed curves with penalization of length reps. subject to fixed length), [5, Theorem 1] (elastic flow for open curves subject to clamped boundary conditions and with penalization of length), [1, Theorem 3.1] (elastic flow for open curves subject to hinged/natural boundary conditions and subject to fixed length), [2, Theorem 1.1] (elastic flow for open curves subject to clamped boundary conditions and fixed length). All these works share the same strategy of proof depicted in this paper. The first step is common to all cited references: indeed, the bound from above (and in the case of fixed length also from below) of the L^2-norm of the curvature vector and a control of the length of the curve are crucial in order to be able to apply interpolation inequalities and embedding theory. The second step differs from paper to paper mostly by the choice of vector field $\vec{\phi}$: here the idea is to find a vector field that contains information about $\nabla_s^m \vec{\kappa}$ and that allows for order reduction of the term $Y = (\nabla_t + \nabla_s^4)\vec{\phi}$ and of the boundary terms showing in equation (3.8). If the curves are closed (i.e. periodic) then one can take $\vec{\phi} = \nabla_s^m \vec{\kappa}$ (see [4]; see also [1] where the curves are open but the boundary terms in (3.8) disappear due to the choice of hinged boundary conditions). For open curves it is often convenient to use $\vec{\phi} = \nabla_t^m f$ (see [5] and [3]). In [2], where also derivatives of λ are involved in the computations, the authors choose $\vec{\phi} = \nabla_t f$ in the first step and then $\vec{\phi} = \nabla_s^{4m} \vec{\kappa}$ for $m \in \mathbb{N}$. Note that considering derivatives in multiple of four is, in some sense, like taking one derivative with respect to time.

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