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On the fine behaviors of the eigenvalues of the linearized Gel'fand problem and its applications[1]

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Abstract

The purpose of this note is to overview our recent results concerning the linearlized eigenvalue problem for the Gel'fand problem. The main result is a second order estimate for the first m eigenvalues of the linearized Gel'fand problem associated to solutions which blow-up at m points. From this information, we determine some qualitative properties of the first m eigenfunctions.

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1 The Gel'fand problem

The Gel'fand problem is the following semilinear elliptic problem with exponential nonlinearity:

$$-\Delta u = \lambda e^{u} \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$ and $\lambda > 0$ is a real parameter. This problem appears in a wide variety of areas of mathematics such as the conformal embedding of a flat domain into a sphere, self-dual gauge field theories, equilibrium states of large number of vortices, stationary states of chemotaxis motion, and so forth. See [7, 8] for more about our motivation and further references.

Especially the asymptotic behavior of the solutions as $\lambda \downarrow 0$ was studied in detail. Let $G(x, y)$ be the Green function of $-\Delta$ in Ω with Dirichlet boundary condition. We divide the Green function into two parts as usual:

$$G(x, y) = \frac{1}{2\pi} \log |x - y|^{-1} + K(x, y), \tag{1.2}$$

$K(x, y)$ is called the regular part of $G(x, y)$ and $R(x) = K(x, x)$ is the Robin function. Using these functions we introduce a function over Ω^m, which is

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know as the *Hamiltonian* function of m vortices with equal intensities in the theory of 2-dimensional incompressible non-viscous fluid:

$$H^m(x_1,\dots,x_m) := \frac{1}{2}\sum_{j=1}^{m} R(x_j) + \frac{1}{2}\sum_{\substack{1\le j,h\le m\\ j\ne h}} G(x_j,x_h).$$

Concerning the Gel'fand problem, the following result now seems to be classical:

Theorem 1.1 ([11]). *Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be a sequence of positive values such that $\lambda_n \to 0$ as $n\to\infty$ and let $u_n = u_n(x)$ be a sequence of solutions of* (1.1) *for $\lambda=\lambda_n$. Then there exists some $m = 0,1,2,\cdots,+\infty$ and, along a subsequence,*

$$\lambda_n \int_\Omega e^{u_n}\, dx \to 8\pi m. \tag{1.3}$$

Moreover, the following behaviors of solutions appear in the limit $n\to\infty$:

(i) *If $m=0$, the sequence $\{u_n\}$ converges to 0 uniformly in Ω.*

(ii) *If $m=+\infty$ the entire blow-up occurs, i.e. $\inf_K u_n \to +\infty$ for any $K \Subset \Omega$.*

(iii) *If $0<m<\infty$ the solutions $\{u_n\}$ blow-up at m-points, that is, there is a set $\mathcal{S}=\{\kappa_1,\cdots,\kappa_m\}\subset\Omega$ of m distinct points and a subsequence of $\{u_n\}$ such that $\|u_n\|_{L^\infty(\omega)} = O(1)$ for any $\omega\Subset\overline{\Omega}\setminus\mathcal{S}$,*

$$u_n|_{\mathcal{S}} \to +\infty \qquad \text{as } n\to\infty,$$

and

$$u_n(x) \longrightarrow u_\infty(x) := \sum_{j=1}^{m} 8\pi G(x,\kappa_j) \tag{1.4}$$

locally uniformly in $\overline{\Omega}\setminus\{\kappa_1,\dots,\kappa_m\}$. Furthermore the blow-up points $\mathcal{S}=\{\kappa_1,\cdots,\kappa_m\}$ satisfy

$$\nabla H^m(\kappa_1,\dots,\kappa_m) = 0. \tag{1.5}$$

We note that a blow-up sequence of solutions for given $\mathcal{S}$ satisfying (1.5) really exists under appropriate assumptions on $\mathcal{S}$, see [1, 4, 5].

In this note we are concerned with more details about the case (iii) of Theorem 1.1. In the following we always assume that $\{u_n\}$ is a sequence of solutions to (1.1) with m blow-up points in the limit $n \longrightarrow \infty$.

2 The linealized eigenvalue problem of the Gel'fand problem

Our object in this note is the following eigenvalue problem:

$$-\Delta v=\mu\lambda_n e^{u_n}v \quad \text{in } \Omega, \qquad v=0 \quad \text{on } \partial\Omega, \tag{2.1}$$

where $\{u_n\}$ is a m-points blow-up sequence of solutions to (1.1). We are able to assume that there exists a sequence of eigenvalues $\mu_n^1\le\mu_n^2\le\mu_n^3\le\cdots$. We denote k-th eigenfunction of (2.1) corresponding to the eigenvalue μ_n^k as v_n^k.

We define a diagonal matrix $D:=\operatorname{diag}[d_1,d_1,d_2,d_2,\cdots,d_m,d_m]$, where d_j is a constant given by

$$d_j=\frac{1}{8}\exp\left\{4\pi R(\kappa_j)+4\pi\sum_{1\le i\le m,\, i\ne j}G(\kappa_j,\kappa_i)\right\}(>0). \tag{2.2}$$

Previously we get the following behavior of μ_n^k:

Theorem 2.1 ([8]). *For $\lambda_n\to 0$, it holds that*

$$\mu_n^k=-\frac{1}{2}\frac{1}{\log\lambda_n}+o\left(\frac{1}{\log\lambda_n}\right)(\to 0),\ \textit{for}\ 1\le k\le m, \tag{2.3}$$
$$\mu_n^k=1-48\pi\eta^{(2m+1-s)}\lambda_n+o(\lambda_n)(\to 1),\ \textit{for}\ m+1\le k\ (=: m+s)\le 3m,$$
$$\mu_n^k>1,\ \textit{for}\ k\ge 3m+1$$

where η^k $(k=1,\cdots,2m)$ is the k-th eigenvalue of the matrix $D(\mathrm{Hess}H^m)D$ at $(\kappa_1,\cdots,\kappa_m)$.

We use these to calculate the Morse index of u_n for $n\gg 1$. Actually we are able to get the following estimate easily from the above behaviors of $\{\mu_n^k\}$:

$$m+\mathrm{ind}_M\{-H^m(\kappa_1,\cdots,\kappa_m)\}\le \mathrm{ind}_M(u_n), \tag{2.4}$$
$$\mathrm{ind}_M^*(u_n)\le m+\mathrm{ind}_M^*\{-H^m(\kappa_1,\cdots,\kappa_m)\}. \tag{2.5}$$

where

$$\mathrm{ind}_M(u_n)=\#\{k\in\mathbb{N}\,;\,\mu_n^k<1\},\quad \mathrm{ind}_M^*(u_n)=\#\{k\in\mathbb{N}\,;\,\mu_n^k\le 1\}.$$

are the Morse index and the augmented Morse index of u_n, respectively.

$$\mathrm{ind}_M\{-H^m(\kappa_1,\cdots,\kappa_m)\},\quad \mathrm{ind}_M^*\{-H^m(\kappa_1,\cdots,\kappa_m)\}$$

are the Morse index and the augmented Morse index of the $-H^m$, that is, the numbers of the negative and non-positive eigenvalues of Hessian of $-H^m$ at $(\kappa_1, \cdots, \kappa_m)$, respectively. These results are a generalization of the results in [6] obtained for the case $m = 1$.

Recently we have refined the case $1 \le k \le m$ as follows:

Theorem 2.2 ([7]). For each $k \in \{1, \ldots, m\}$, the followings hold:

(i) Let (h_{ij}) be the matrix given by

$$h_{ij} = \begin{cases} R(\kappa_i) + 2\sum_{\substack{1 \le l \le m \\ l \ne i}} G(\kappa_l, \kappa_i), & \text{for } i = j, \\ -G(\kappa_i, \kappa_j), & \text{for } i \ne j, \end{cases}.$$

and Λ^k be the k-th eigenvalue of (h_{ij}), assuming $\Lambda^1 \le \cdots \le \Lambda^m$. Then

$$\mu_n^k = -\frac{1}{2}\frac{1}{\log \lambda_n} + \left(2\pi\Lambda^k - \frac{3\log 2 - 1}{2}\right)\frac{1}{(\log \lambda_n)^2} + o\left(\frac{1}{(\log \lambda_n)^2}\right) \tag{2.6}$$

as $n \longrightarrow +\infty$.

(ii) Suppose v_n^k is normalized as

$$\|v_n^k\|_\infty = 1.$$

Then there exists a k-th eigenvector $\mathfrak{c}^k = (c_1^k, \ldots, c_m^k) \in [-1, 1]^m \subset \mathbb{R}^m$ ($\mathfrak{c}^k \ne \mathbf{0}$) of (h_{ij}) and a subsequence satisfying

$$\frac{v_n^k}{\mu_n^k} \longrightarrow \sum_{j=1}^m 8\pi c_j^k G(\cdot, \kappa_j) \quad \text{locally uniformly in } \overline{\Omega} \setminus \{\kappa_1, \ldots, \kappa_m\}. \tag{2.7}$$

We note that it seems difficult to realize the matrix (h_{ij}) (and D) in the case $\#\mathcal{S} = 1$ considered in [6].

From the conclusion (ii), we are able to show that $v_n^k \longrightarrow 0$ outside the blow-up set. On the other hand, we know that $c_j^k = 0$ implies $v_n^k \longrightarrow 0$ locally uniformly near κ_j, see [8, Proposition 2.11]. Therefore we introduce the following definition:

Definition 2.3. *We say that an eigenfunction v_n^k concentrates at $\kappa_j \in \Omega$ if* *there exists* $\kappa_{j,n} \to \kappa_j$ *such that*

$$\left|v_n^k(\kappa_{j,n})\right| \geq C > 0 \quad \textit{for } n \textit{ large.} \tag{2.8}$$

As we see later, c_j^k is obtained as the limit of $v_n^k(x_{j,n})$ as $n \longrightarrow \infty$ for the sequence satisfying $x_{j,n} \longrightarrow \kappa_j$ and $u_n(x_{j,n}) \longrightarrow \infty$. Therefore it holds that

$$v_n^k \text{ concentrates at } \kappa_j \text{ if and only if } c_j^k \neq 0, \tag{2.9}$$

that is, we are able to count the number of peaks of v_n^k from the number of non-zero components of $\mathfrak{c}^k$ as an application of this work.

Remark 2.4. We note that the behavior (2.7) for *some* $\mathfrak{c}^k \in [-1,1]^m \subset \mathbb{R}^m(\mathfrak{c}^k \neq \mathbf{0})$ is obtained in the previous work [8, Proposition 2.5 and 2.13]. In this work we clarify the origin of $\mathfrak{c}^k$ from the *fine* behavior of eigenvalues.

3 On the scaling argument and the behavior of eigenfunctions

In this section we sketch the proof of (2.7) and introduce the scaling argument necessary to get it.

Fix $0 < R \ll 1$ satisfying

$$B_{2R}(\kappa_i) \Subset \Omega \text{ for } i = 1, \ldots, m \text{ and } B_R(\kappa_i) \cap B_R(\kappa_j) = \emptyset \text{ if } i \neq j.$$

Choose a sequence $\{x_{j,n}\}$ for each $\kappa_j \in \mathcal{S}$ satisfying

$$x_{j,n} \longrightarrow \kappa_j, \quad u_n(x_{j,n}) = \max_{B_R(x_{j,n})} u_n(x) \longrightarrow \infty.$$

From the Green representation formula and the behavior (1.4) of u_n, we get

$$\begin{aligned} \frac{v_n^k(x)}{\mu_n^k} &= \int_\Omega G(x,y)\lambda_n e^{u_n} v_n^k dy \\ &= \sum_{j=1}^m \int_{B_R(x_{j,n})} G(x,y)\lambda_n e^{u_n} v_n^k dy + O(\lambda_n). \end{aligned}$$

Since $G(x, x_{j,n})$ is smooth far from $x_{j,n}$, Taylor's theorem

$$\begin{aligned} G(x,y) &= G(x,x_{j,n}) + (y - x_{j,n}) \cdot \nabla_y G(x, x_{j,n}) + s(x, \eta, y - x_{j,n}), \\ s(x,\eta,y-x_{j,n}) &= \frac{1}{2} \sum_{1 \leq \alpha,\beta \leq 2} G_{y_\alpha y_\beta}(x,\eta)(y - x_{j,n})_\alpha (y - x_{j,n})_\beta, \\ \eta &= \eta(j,n,y) \in B_R(x_{j,n}), \end{aligned}$$

guarantees that

$$\int_{B_R(x_{j,n})} G(x,y)\lambda_n e^{u_n} v_n^k dy$$
$$= G(x,x_{j,n}) \int_{B_R(x_{j,n})} \lambda_n e^{u_n} v_n^k dy + \nabla_y G(x,x_{j,n}) \cdot \int_{B_R(x_{j,n})} (y-x_{j,n})\lambda_n e^{u_n} v_n^k dy$$
$$+ \frac{1}{2} \sum_{1\le\alpha,\beta\le 2} \int_{B_R(x_{j,n})} (y-x_{j,n})_\alpha (y-x_{j,n})_\beta G_{y_\alpha y_\beta}(x,\eta)\lambda_n e^{u_n} v_n^k dy$$
$$=: \gamma_{j,n}^0 G(x,x_{j,n}) + \gamma_{j,n}^1 \cdot \nabla_y G(x,x_{j,n}) + \gamma_{j,n}^2.$$

So we need to see the behaviors of $\gamma_{j,n}^0$, $\gamma_{j,n}^1$, and $\gamma_{j,n}^2$ to get (2.7).

To this purpose we rescale the solution u_n and eigenfunction v_n^k around $x_{j,n}$. Let $\delta_{j,n}$ be a parameter determined by the relation

$$\lambda_n e^{u_n(x_{j,n})} \delta_{j,n}^2 = 1 \tag{3.1}$$

and set

$$\widetilde{u}_{j,n}(\widetilde{x}) := u_n(\delta_{j,n}\widetilde{x} + x_{j,n}) - u_n(x_{j,n}) \quad \text{in } B_{\frac{R}{\delta_{j,n}}}(0),$$
$$\widetilde{v}_{j,n}^k(\widetilde{x}) := v_n^k(\delta_{j,n}\widetilde{x} + x_{j,n}) \quad \text{in } B_{\frac{R}{\delta_{j,n}}}(0).$$

Then it holds that

$$-\Delta \widetilde{u}_{j,n} = e^{\widetilde{u}_{j,n}} \text{ in } B_{\frac{R}{\delta_{j,n}}}(0), \qquad \widetilde{u}_{j,n} \le \widetilde{u}_{j,n}(0) = 0 \text{ in } B_{\frac{R}{\delta_{j,n}}}(0) \tag{3.2}$$

and

$$-\Delta \tilde{v}_{j,n}^k = \mu_n^k e^{\tilde{u}_{j,n}} \tilde{v}_{j,n}^k, \quad \text{in } \widetilde{x} \in B_{\frac{R}{\delta_{j,n}}}(0), \qquad \|\tilde{v}_{j,n}^k\|_{L^\infty\left(B_{\frac{R}{\delta_{j,n}}}(0)\right)} \le 1. \tag{3.3}$$

We note that there exists $d_j > 0$ such that

$$\delta_{j,n} = d_j \lambda_n^{\frac{1}{2}} + o(\lambda_n^{\frac{1}{2}}) (\longrightarrow 0) \tag{3.4}$$

from Y.Y.Li's estimate ([10], see also [9, Corollary 4.3]). We also note that d_j is know to given by (2.2), see [7, Proposition 3.4].

Now assuming

$$\mu_n^k \longrightarrow \mu_\infty^k \in \mathbb{R},$$

we reach the following problem at the limit $n \longrightarrow \infty$:

$$-\Delta U = e^U \quad \text{in } \mathbb{R}^2, \quad U(\widetilde{x}) \le U(0) \tag{3.5}$$

and

$$-\Delta V=\mu_\infty^k e^U V, \quad \|V\|_{L^\infty(\mathbb{R}^2)}\le 1. \tag{3.6}$$

From Chen-Li's result [3], we see

$$U(\tilde{x})=\log\frac{1}{\left(1+\frac{|\tilde{x}|^2}{8}\right)^2}$$

and we get

$$\tilde{u}_{j,n}(\tilde{x})\to U(\tilde{x}) \quad \text{in } C^{2,\alpha}_{loc}(\mathbb{R}^2).$$

On the other hand, we can show

$$\tilde{v}^k_{j,n}\longrightarrow V^k_j \quad \text{in } C^{2,\alpha}_{loc}\left(\mathbb{R}^2\right)$$

holds for a subsequence, where V_j^k is a solution of (3.6)([8, Proposition 2.2]). Since we know that there exists $j\in\{1,\cdots m\}$ such that $V_j^k\not\equiv 0$ ([8, Proposition 2.11]), we see that μ_∞^k is an eigenvalue of the linealized eigenvalue problem (3.6) for (3.5). These eigenvalues (and the eigenfunctions) are studied in [6] and we know

$$\mu_\infty^k=\frac{l(l+1)}{2} \quad \text{for some } l=0,1,2,\cdots.$$

In this note, we are interested in the case $\mu_\infty^k=0$ and for this case the eigenfunction is known to

$$V_j^k\equiv \text{const.}\ (=: c_j^k)\in[-1,1].$$

Consequently we are able to confirm that

$$\gamma^0_{j,n}=\int_{B_R(x_{j,n})}\lambda_n e^{u_n}v_n^k=\int_{B_{\frac{R}{\delta_{j,n}}}(0)}e^{\tilde{u}_{j,n}}\tilde{v}^k_{j,n}\longrightarrow c_j^k\int_{\mathbb{R}^2}e^U=8\pi c_j^k.$$

Similarly we see

$$\begin{aligned}\gamma^1_{j,n}&=\int_{B_R(x_{j,n})}(y-x_{j,n})\lambda_n e^{u_n}v_n^k=\delta_{j,n}\int_{B_{\frac{R}{\delta_{j,n}}}(0)}\tilde{y}e^{\tilde{u}_{j,n}}\tilde{v}^k_{j,n}\\&=\delta_{j,n}\left\{\int_{\mathbb{R}^2}\tilde{y}e^{\tilde{U}}\tilde{V}_j^k+o(1)\right\}=o(\delta_{j,n})=o\left(\lambda_n^{\frac{1}{2}}\right)\end{aligned}$$

and, taking $\varepsilon \in (0,1)$,

$$\left|\int_{B_R(x_{j,n})} G_{y_\alpha y_\beta}(x,\eta)(y-x_{j,n})_\alpha (y-x_{j,n})_\beta \lambda_n e^{u_n} v_n^k dy\right|$$
$$\leq CR^\varepsilon \int_{B_R(x_{j,n})} |y-x_{j,n}|^{2-\varepsilon} \lambda_n e^{u_n} dy$$
$$= CR^\varepsilon \delta_{j,n}^{2-\varepsilon} \int_{B_{\frac{R}{\delta_{j,n}}}(0)} |\widetilde{y}|^{2-\varepsilon} e^{\widetilde{u}_{j,n}} d\widetilde{y} = O\left(\delta_{j,n}^{2-\varepsilon}\right) = o\left(\lambda_n^{\frac{1}{2}}\right).$$

For simplify the presentation, we omitted here some additional argument necessary to the process $n \longrightarrow \infty$, see [8, Proposition 2.5 and 2.13] for details.

4 On the improvement of the behavior of the eigenvalues

The sharp formula (2.6) is from the determination of the following constant L:

$$\frac{1}{\mu_n^k} = -2\log\lambda_n + L + o(1). \tag{4.1}$$

Indeed, this leads

$$\mu_n^k = \frac{1}{-2\log\lambda_n + L + o(1)} = -\frac{1}{2\log\lambda_n} \cdot \frac{1}{1 - \frac{L+o(1)}{2\log\lambda_n}}$$
$$= -\frac{1}{2\log\lambda_n}\left\{1 + \frac{L+o(1)}{2\log\lambda_n} + o\left(\frac{1}{\log\lambda_n}\right)\right\}$$
$$= -\frac{1}{2\log\lambda_n} - \frac{L}{4} \cdot \frac{1}{(\log\lambda_n)^2} + o\left(\frac{1}{(\log\lambda_n)^2}\right).$$

It is easy to see that we get (2.3) from this if L is not specified. In the following we sketch how to get

$$L = -8\pi\Lambda^k + 2(3\log 2 - 1).$$

The constant L in (4.1) comes from two formula:

$$u_n(x_{j,n}) = -2\log\lambda_n - 2\log d_j + o(1), \tag{4.2}$$

where d_j is the number given in (2.2), and

$$\left\{\frac{1}{\mu_n^k} - u_n(x_{j,n})\right\}\gamma_{j,n}^0 + 16\pi c_j^k = -(8\pi)^2\sum_{i=1}^m g_{ji}c_i^k + o(1), \qquad (4.3)$$

where

$$g_{ji} = \begin{cases} \sum_{\substack{1\le h\le m \\ h\neq j}} G(\kappa_j,\kappa_h), & \text{for } j=i, \\ -G(\kappa_j,\kappa_i), & \text{for } j\neq i. \end{cases}$$

Concerning this point, previously we used

$$u_n(x_{j,n}) = -2\log\lambda_n + O(1)$$

and

$$\left\{\frac{1}{\mu_n^k} - u_n(x_{j,n})\right\}\gamma_{j,n}^0 = O(1).$$

Here we eliminate $u_n(x_{j,n})$ from these and get

$$\left\{\frac{1}{\mu_n^k} + 2\log\lambda_n + O(1)\right\}\gamma_{j,n}^0 = O(1).$$

We know that $\gamma_{j,n}^0 \longrightarrow 8\pi c_j^k$ and there exists at least one $j = 1,\cdots,m$ such that $c_j^k \neq 0$ since $\mathbf{c}^k = (c_1^k,\ldots,c_m^k) \neq \mathbf{0}$. Therefore we get

$$\frac{1}{\mu_n^k} = -2\log\lambda_n + O(1).$$

Similarly, eliminating $u_n(x_{j,n})$ from (4.2) and (4.3), we get

$$\left\{\frac{1}{\mu_n^k} + 2\log\lambda_n + 2\log d_j + o(1)\right\}\gamma_{j,n}^0 + 16\pi c_j^k = -(8\pi)^2\sum_{i=1}^m g_{ji}c_i^k + o(1),$$

that is,

$$\begin{aligned}\frac{1}{\mu_n^k} &= -2\log\lambda_n - 2 - 2\log d_j - \frac{8\pi\sum_{i=1}^m g_{ji}c_i^k}{c_j^k} + o(1) \\ &= -2\log\lambda_n + 2(3\log 2 - 1) - \frac{8\pi\sum_{i=1}^m h_{ji}c_i^k}{c_j^k} + o(1).\end{aligned}$$

Here we assume $c_j^k \neq 0$ for simplicity. Since this formula exists for each $j = 1,\cdots,m$. We are able to get

$$\frac{\sum_{i=1}^m h_{ji}c_i^k}{c_j^k} = \frac{\sum_{i=1}^m h_{li}c_i^k}{c_l^k}$$

for $j \neq l$. This means that there exists a constant Λ^k such that

$$\sum_{i=1}^{m} h_{ji} c_i^k = \Lambda^k c_j^k$$

for every $j = 1, \cdots, m$, that is, Λ^k is an eigenvalue of the matrix (h_{ji}), see [7] for details. Obviously $\mathfrak{c}^k = (c_1^k, \ldots, c_m^k)$ is an eigenvector of (h_{ji}).

Finally we are able to conclude that Λ^k is the k-th eigenvalue of (h_{ji}) because $\mu_n^1 \leq \cdots \leq \mu_n^k$.

4.1 Derivation of (4.2)

The formula (4.2) was essentially proved by C. C. Chen and C.-S. Lin [2, Estimate D] from the Green's representation formula:

$$\begin{aligned}
u_n(x_{j,n}) &= \int_{\Omega} G(x_{j,n}, y) \lambda_n e^{u_n(y)}\, dy \\
&= \frac{1}{2\pi} \int_{B_R(x_{j,n})} \log |x_{j,n} - y|^{-1} \lambda_n e^{u_n(y)}\, dy \\
&\quad + \int_{B_R(x_{j,n})} K(x_{j,n}, y) \lambda_n e^{u_n(y)}\, dy \\
&\quad + \sum_{\substack{1 \leq i \leq m \\ i \neq j}} \int_{B_R(x_{i,n})} G(x_{j,n}, y) \lambda_n e^{u_n(y)}\, dy \\
&\quad + \int_{\Omega \setminus \bigcup_{i=1}^m B_R(x_{i,n})} G(x_{j,n}, y) \lambda_n e^{u_n(y)}\, dy \\
&= -\frac{\sigma_{j,n}}{2\pi} \log \delta_{j,n} + \frac{1}{2\pi} \int_{B_{\frac{R}{\delta_{j,n}}}(0)} \log |\tilde{y}|^{-1} e^{\tilde{u}_{j,n}(\tilde{y})}\, d\tilde{y} \\
&\quad + 8\pi \Big\{ R(x_{j,n}) + \sum_{\substack{1 \leq i \leq m \\ i \neq j}} G(x_{j,n}, x_{i,n}) \Big\} + o(1),
\end{aligned}$$

where

$$\sigma_{j,n} := \lambda_n \int_{B_R(x_{j,n})} e^{u_n}\, dx.$$

We are able to confirm that

$$\frac{1}{2\pi} \int_{B_{\frac{R}{\delta_{j,n}}}(0)} \log |\tilde{y}|^{-1} e^{\tilde{u}_{j,n}(\tilde{y})}\, d\tilde{y} \to \frac{1}{2\pi} \int_{\mathbb{R}^2} \log |\tilde{y}|^{-1} e^{U(\tilde{y})}\, d\tilde{y} = -6 \log 2 \qquad (4.4)$$

and

$$8\pi\Big\{R(x_{j,n}) + \sum_{\substack{1\le i\le m\\ i\neq j}} G(x_{j,n}, x_{i,n})\Big\} \longrightarrow 2\log(8d_j). \tag{4.5}$$

On the other hand, we know $\sigma_{j,n} \longrightarrow 8\pi$ from Theorem 1.1. Therefore we get the following formula from the relation (3.1):

$$u_n(x_{j,n}) = -\frac{\sigma_{j,n}}{\sigma_{j,n}-4\pi}\log\lambda_n - 2\log d_j + o(1) \tag{4.6}$$

$$= -2\log\lambda_n + \frac{\sigma_{j,n}-8\pi}{\sigma_{j,n}-4\pi}\log\lambda_n - 2\log d_j + o(1). \tag{4.7}$$

To get (4.2), we need to know more precise behavior of $\sigma_{j,n}$ along $\lambda_n \longrightarrow 0$. To this purpose the following one obtained in [2, (3.56)] is sufficient:

$$\sigma_{j,n} = 8\pi + o\left(\lambda_n\right). \tag{4.8}$$

We note that a weaker version

$$\sigma_{j,n} = 8\pi + o\left(\lambda_n^{\frac{1}{2}}\right),$$

which is also sufficient for our purpose, can be obtained rather easily, see [13].

4.2 Derivation of (4.3)

To get the formula (4.3) we use the Green's theorem for u_n and $\frac{v_n^k}{\mu_n^k}$ around $B_R(x_{j,n})$:

$$\int_{\partial B_R(x_{j,n})}\left\{\frac{\partial u_n}{\partial\nu}\frac{v_n^k}{\mu_n^k} - u_n\frac{\partial}{\partial\nu}\left(\frac{v_n^k}{\mu_n^k}\right)\right\}d\sigma = \int_{B_R(x_{j,n})}\left\{\Delta u_n\frac{v_n^k}{\mu_n^k} - u_n\Delta\frac{v_n^k}{\mu_n^k}\right\}dx \tag{4.9}$$

The choice of u_n and $\frac{v_n^k}{\mu_n^k}$ seems to be a kind of trick. Indeed we know the behaviors of u_n and $\frac{v_n^k}{\mu_n^k}$ far from $\mathcal{S} = \{\kappa_1, \cdots, \kappa_m\}$, see (1.4) and (2.7). Therefore the left-hand side of (4.9) has limit in the process $n \longrightarrow \infty$.

In fact, we have

$$\int_{\partial B_R(x_{j,n})}\left\{\frac{\partial u_n}{\partial\nu}\frac{v_n^k}{\mu_n^k} - u_n\frac{\partial}{\partial\nu}\left(\frac{v_n^k}{\mu_n^k}\right)\right\}d\sigma$$

$$\to (8\pi)^2\sum_{h=1}^m\sum_{i=1}^m c_i^k\int_{\partial B_R(\kappa_j)}\left\{\frac{\partial}{\partial\nu}G(x,\kappa_h)G(x,\kappa_i) - G(x,\kappa_h)\frac{\partial}{\partial\nu}G(x,\kappa_i)\right\}d\sigma. \tag{4.10}$$

It is easy to see that

$$\int_{\partial B_R(\kappa_j)} \left\{ \frac{\partial}{\partial\nu} G(x,\kappa_h) G(x,\kappa_i) - G(x,\kappa_h) \frac{\partial}{\partial\nu} G(x,\kappa_i) \right\} d\sigma$$
$$= \begin{cases} 0, & h = i \\ -G(\kappa_j,\kappa_i)\delta_h^j + G(\kappa_j,\kappa_h)\delta_j^i, & h \neq i, \end{cases}$$

where $\delta_a^b = 1$ if $a = b$ and $\delta_a^b = 0$ else.

Therefore, from (4.10) we have

$$\int_{\partial B_R(x_{j,n})} \left\{ \frac{\partial u_n}{\partial\nu} \frac{v_n^k}{\mu_n^k} - u_n \frac{\partial}{\partial\nu} \left(\frac{v_n^k}{\mu_n^k} \right) \right\} d\sigma$$
$$= (8\pi)^2 \sum_{h=1}^{m} \sum_{\substack{1 \le i \le m \\ i \neq h}} c_i^k \left\{ -G(\kappa_j,\kappa_i)\delta_h^j + G(\kappa_j,\kappa_h)\delta_j^i \right\} + o(1)$$
$$= (8\pi)^2 \sum_{i=1}^{m} g_{ji} c_i^k + o(1).$$

On the other hand, we are able to apply the scaling argument to the right-hand side of (4.9). Indeed the following holds:

$$\int_{B_R(x_{j,n})} \left\{ \Delta u_n \frac{v_n^k}{\mu_n^k} - u_n \Delta \frac{v_n^k}{\mu_n^k} \right\} dx$$
$$= -\frac{1}{\mu_n^k} \lambda_n \int_{B_R(x_{j,n})} e^{u_n} v_n^k \, dx + \lambda_n \int_{B_R(x_{j,n})} e^{u_n} v_n^k u_n \, dx$$
$$= -\frac{1}{\mu_n^k} \lambda_n \int_{B_R(x_{j,n})} e^{u_n} v_n^k \, dx + u_n(x_{j,n}) \lambda_n \int_{B_R(x_{j,n})} e^{u_n} v_n^k \, dx$$
$$+ \int_{B_{\frac{R}{\delta_{j,n}}}(0)} e^{\tilde{u}_{j,n}} \tilde{v}_{j,n}^k \tilde{u}_{j,n} \, d\widetilde{x}$$
$$= -\left(\frac{1}{\mu_n^k} - u_n(x_{j,n}) \right) \gamma_{j,n}^0 + \int_{B_{\frac{R}{\delta_{j,n}}}(0)} e^{\tilde{u}_{j,n}} \tilde{v}_{j,n}^k \tilde{u}_{j,n} \, d\widetilde{x}.$$

Here

$$\int_{B_{\frac{R}{\delta_{j,n}}}(0)} e^{\tilde{u}_{j,n}} \tilde{v}_{j,n}^k \tilde{u}_{j,n} \, d\widetilde{x} \longrightarrow \int_{\mathbb{R}^2} e^U V_j^k U \, d\widetilde{x} = -16\pi c_j^k.$$

Consequently we get (4.3).

5 Examples of $\mathfrak{c}^k = (c_1^k, \ldots, c_m^k) \neq 0$

Let us fix an integer $m \geq 2$ and Ω be an annulus $\{x \in \mathbb{R}^2; (0<)a < |x| < 1\}$. In [12] there was constructed a m-mode solution u_n to (1.1), i.e. a solution which is invariant with respect to a rotation of $\frac{2\pi}{m}$ in $\mathbb{R}^2$,

$$u(r,\theta) = u\left(r, \left(\theta + \frac{2\pi}{m}\right)\right).$$

This solution blows-up at m points $\kappa_1 = \cdots = \kappa_m$ which are located on a circle concentric with the annulus and are vertices of a regular polygon with m sides. So we can assume that $\kappa_1 = (r_0, 0)$, $\kappa_2 = r_0\left(\cos\frac{2\pi}{m}, \sin\frac{2\pi}{m}\right), \ldots, \kappa_m = r_0\left(\cos\frac{2(m-1)\pi}{m}, \sin\frac{2(m-1)\pi}{m}\right)$ for some $r_0 \in (a,1)$.

Since $G(x,\kappa_1)$ is symmetric with respect to the x_1-axis, we get $G(\kappa_j,\kappa_1) = G(\kappa_{m-j+2},\kappa_1)$, $j = 2,..,m$. Similarly the value $G(\kappa_i,\kappa_j)$ depends only on the distance between κ_i and κ_j. Since Ω is an annulus, the Robin function $R(x)$ is radial, so that $R(\kappa_1) = \cdots = R(\kappa_m) = R$.

Here we set $G_i := G(\kappa_i,\kappa_1)$ and $R_l := R + 4\sum_{h=2}^{l} G_h$ for simplicity. Then the matrix h_{ij} becomes as follows:
for $m = 2l$ $(l = 1, 2, \cdots)$,

$$(h_{ij}) = \begin{pmatrix} R_l + 2G_{l+1} & -G_2 & -G_3 & \ldots & -G_{l+1} & \ldots & -G_2 \\ -G_2 & R_l + 2G_{l+1} & -G_2 & \ldots & \ldots & \ldots & -G_3 \\ & \ldots & & & & & \\ -G_2 & -G_3 & \ldots & \ldots & \ldots & \ldots & R_l + 2G_{l+1} \end{pmatrix},$$

and for $m = 2l+1$ $(l = 1, 2, \cdots)$,

$$(h_{ij}) = \begin{pmatrix} R_l & -G_2 & -G_3 & \ldots & -G_l & -G_l & \ldots & -G_2 \\ -G_2 & R_l & -G_2 & \ldots & \ldots & \ldots & \ldots & -G_3 \\ & \ldots & & & & & & \\ -G_2 & -G_3 & \ldots & \ldots & \ldots & \ldots & -G_2 & R_l \end{pmatrix},$$

A straightforward computation shows the following facts:

- *m=3*

  $\lambda_1 = R + 2G_2$, $\mathfrak{c}_1 = (1,1,1)$.

  $\lambda_2 = \lambda_3 = R + 5G_2$, $\mathfrak{c}_2 = (1,-1,0)$, $\mathfrak{c}_3 = (1,0,-1)$.

- *m=4*

  $\lambda_1 = R + 2G_2 + G_3$, $\mathfrak{c}_1 = (1,1,1,1)$.

$\lambda_2 = \lambda_3 = R + 4G_2 + 3G_3$, $\mathfrak{c}_2 = (1, 0, -1, 0)$, $\mathfrak{c}_3 = (0, 1, 0, -1)$.
$\lambda_4 = R + 6G_2 + G_3$, $\mathfrak{c}_4 = (1, -1, 1, -1)$.

- *m=5*

  $\lambda_1 = R + 2G_2 + 2G_3$, $\mathfrak{c}_1 = (1, 1, 1, 1, 1)$,

  $\lambda_2 = \lambda_3 = R + \frac{9-\sqrt{5}}{2}G_2 + \frac{9+\sqrt{5}}{2}G_3$,
  $\mathfrak{c}_2 = (1, \frac{-1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, -1, 0)$, $\mathfrak{c}_3 = (1, -1, \frac{-1-\sqrt{5}}{2}, 0, \frac{1+\sqrt{5}}{2})$,

  $\lambda_4 = \lambda_5 = R + \frac{9+\sqrt{5}}{2}G_2 + \frac{9-\sqrt{5}}{2}G_3$.
  $\mathfrak{c}_4 = (1, \frac{-1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}, -1, 0)$, $\mathfrak{c}_5 = (1, -1, \frac{-1+\sqrt{5}}{2}, 0, \frac{1-\sqrt{5}}{2})$.

- *m=6*

  $\lambda_1 = R + 2G_2 + 2G_3 + G_4$, $\mathfrak{c}_1 = (1, 1, 1, 1, 1, 1)$,

  $\lambda_2 = \lambda_3 = R + 3G_2 + 5G_3 + 3G_4$,
  $\mathfrak{c}_2 = (1, 0, -1, -1, 0, 1)$, $\mathfrak{c}_3 = (1, 1, 0, -1, -1, 0)$

  $\lambda_4 = \lambda_5 = R + 5G_2 + 5G_3 + G_4$,
  $\mathfrak{c}_4 = (1, -1, 0, -1, 1, 0)$, $\mathfrak{c}_5 = (1, 0, 1, -1, 0, 1)$

  $\lambda_6 = R + 6G_2 + 2G_3 + 3G_4$,
  $\mathfrak{c}_6 = (1, -1, 1, -1, 1, -1)$

- $\cdots$

In general, it is easy to see that the first eigenvalue of (h_{ij}) is $\Lambda^1 = R + 2\sum_{h=2}^{l} G_h + G_{l+1}$ for $m = 2l$ and $R + 2\sum_{h=2}^{l} G_h$ for $m = 2l + 1$ which is simple. It is easy to see that the eigenspace corresponding to Λ^1 is spanned by $\mathfrak{c}^1 = (1, 1, \cdots, 1)$.

Unfortunately we are not yet able to get further information on the multiplicity of the eigenvalues even for these cases and we will leave this for future work, see also [7, Remark 5.1].

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