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QUESTIONS ON PROVISIONAL COULOMB BRANCHES OF 3-DIMENSIONAL $\mathcal{N}=4$ GAUGE THEORIES

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ABSTRACT. This is a supplement to [Nak15], where an approach towards a mathematically rigorous definition of the Coulomb branch of a 3-dimensional $\mathcal{N}=4$ SUSY gauge theory was proposed. We ask questions on their expected properties, especially in relation to the corresponding Higgs branch, partly motivated by the interpretation of the level rank duality in terms of quiver varieties [Nak09] and the symplectic duality [BLPW14]. We study questions in a few examples.

INTRODUCTION

In [Nak15], we proposed an approach towards a mathematically rigorous definition of the Coulomb branch $\mathcal{M}_C$ of a 3-dimensional $\mathcal{N}=4$ SUSY gauge theory. Moreover various physically known examples and expected properties were reviewed. In this paper, we add questions on expected properties, especically in relation to the corresponding Higgs branch $\mathcal{M}_H$. Some are probably implicit in the physics literature, but we are motivated by (a) the interpretation of the level rank duality of affine Lie algebras of type A via quiver varieties, and also (b) the symplectic duality.

Recall when $\mathcal{M}_H$ is a quiver variety of affine type A, the corresponding $\mathcal{M}_C$ is also a quiver variety of affine type A [dBHOO97, dBHO$^+$97]. Under their relation to representation theory [Nak94], the pair is given by I. Frenkel's level rank duality [Fre82]. Then one can interpret representation theoretic statements as relation between $\mathcal{M}_H$ and $\mathcal{M}_C$. This idea was a source of inspiration in author's work [Nak09] on provisional double affine Grassmannian [BF10], and the joint work [BFN14].

Recall the symplectic duality [BLPW14] predicts pairs of symplectic manifolds $(\mathfrak{M},\mathfrak{M}^!)$ whose categories $\mathcal{O}$, $\mathcal{O}^!$ are Koszul dual to each other. Here the categories $\mathcal{O}$, $\mathcal{O}^!$ are certain full subcategories of modules of quantizations of $\mathfrak{M}$, $\mathfrak{M}^!$. Many examples of symplectic dual pairs appear as $(\mathfrak{M},\mathfrak{M}')=(\mathcal{M}_H,\mathcal{M}_C)$ for a gauge theory, e.g., the above quiver varieties of affine type A. If $\mathcal{O}$ and $\mathcal{O}^!$ are Koszul dual, one can deduce many relations between $\mathfrak{M}$ and $\mathfrak{M}^!$. Therefore it is natural to ask relations between $\mathcal{M}_H$ and $\mathcal{M}_C$. Note also that the Koszul duality is expected to be closely related to the level rank duality.

Note that $(\mathcal{M}_H, \mathcal{M}_C)$ are much more general: for example they may not have resolution of singularities nor a torus action with isolated fixed points, as required in the formulation of the symplectic duality. They have always $\mathbb{C}^\times$-action which scale the symplectic form by weight 2, but are not cone in general. Even more fundamentally, we have lots of examples where $\mathcal{M}_H$ is a point, while $\mathcal{M}_C$ are nontrivial. Therefore we decide to restrict ourselves to ask *basic* (or *naive*) questions which can be asked without introducing resolutions, torus action. We study these questions in a few examples.

Notation.

(1) We basically follow the notation in Part I [Nak15]. However we mainly use a complex reductive group instead of its maximal compact subgroup. Therefore we denote a reductive group by G, and its maximal compact by G_c. We only use the complex part of the hyper-Kähler moment map, for which we use the notation $\boldsymbol{\mu}$.
(2) We also change the notation for a compact Riemann surface from C to Σ to avoid a conflict with 'C' for the Coulomb branch.

1. More general target spaces

This section will be independent of other parts of the paper. The reader can skip it, but can be also considered as a brief review of the construction in [Nak15] and its generalization.

In [Nak15], we define a sheaf of a vanishing cycle on a moduli space associated with a complex symplectic representation $\mathbf{M}$ of a reductive group G. We consider their modification and generalization.

This construction, as well as the previous one in [Nak15], should be understood in the framework of shifted symplectic structures [PTVV13].[1] Because of the author's lack of ability, we cannot make it precise unfortunately.

1(i). **σ-models.** Let us first consider

- $(\mathbf{M}, \omega)$ is a (holomorphic) symplectic manifold with a $\mathbb{C}^\times$-action such that $t^*\omega = t^2\omega$ for $t \in \mathbb{C}^\times$.
- there is a Liouville form θ such that $d\theta = \omega$ and $t^*\theta = t^2\theta$.

Let Σ be a compact Riemann surface. We choose and fix a square root $K_\Sigma^{1/2}$ of the canonical bundle K_Σ. We consider the associated $\mathbb{C}^\times$-principal bundle $P_{K_\Sigma^{1/2}}$, that is $P_{K_\Sigma^{1/2}} \times_{\mathbb{C}^\times} \mathbb{C} \cong K_\Sigma^{1/2}$ when $\mathbb{C}^\times$ acts on $\mathbb{C}$ with weight 1.

We consider a $\mathbb{C}^\times$-equivariant C^∞-map $\Phi\colon P_{K_\Sigma^{1/2}} \to \mathbf{M}$, in other words a C^∞-section of a bundle $P_{K_\Sigma^{1/2}} \times_{\mathbb{C}^\times} \mathbf{M}$. Taking a local holomorphic trivialization of $P_{K_\Sigma^{1/2}}$, we see that the $(0,1)$-part of $\Phi^*\theta$ is a well-defined K_Σ-valued $(0,1)$-form,

[1]The author thanks Dominic Joyce for pointing out a relevance of [PTVV13] in our construction.

i.e., $(1,1)$-form on Σ: let $\{U_\alpha\}$ be an open cover of Σ, such that $P_{K_\Sigma^{1/2}}$ is trivialized over U_α. We denote the transition function by $g_{\alpha\beta}$. Then Φ is a collection $\{\Phi_\alpha\colon U_\alpha \to \mathbf{M}\}$ such that $\Phi_\alpha = g_{\alpha\beta}\cdot\Phi_\beta$ on $U_\alpha\cap U_\beta$, where $\cdot$ is the $\mathbb{C}^\times$-action on $\mathbf{M}$. We have $\overline{\partial}\Phi_\alpha = g_{\alpha\beta}\cdot\overline{\partial}\Phi_\beta$ as $g_{\alpha\beta}$ is holomorphic. Therefore $(\Phi_\alpha^*\theta)^{(0,1)} = \langle\theta,\overline{\partial}\Phi_\alpha\rangle = \langle\theta, g_{\alpha\beta}\cdot\overline{\partial}\Phi_\beta\rangle = g_{\alpha\beta}^2\langle\theta,\overline{\partial}\Phi_\beta\rangle$. This means that $\{\langle\theta,\overline{\partial}\Phi_\alpha\rangle\}$ is a K_Σ-valued $(0,1)$-form. This is $(\Phi^*\theta)^{(0,1)}$.

We integrate it over Σ:

$$\mathrm{CS}(\Phi) \overset{\text{def.}}{=} \int_\Sigma (\Phi^*\theta)^{(0,1)}.$$

This is the holomorphic Chern-Simons type integral in this setting. Moreover Φ is a critical point of CS if and only if Φ is a holomorphic section of $P_{K_\Sigma^{1/2}}\times_{\mathbb{C}^\times}\mathbf{M}$, i.e., a twisted holomorphic map from Σ to $\mathbf{M}$.

Let $\mathcal{F}$ be the space of fields, i.e., the space of all $\mathbb{C}^\times$-equivariant C^∞-maps $\Phi\colon P_{K_\Sigma^{1/2}}\to\mathbf{M}$.

We can consider $\varphi_{\mathrm{CS}}(\mathbb{C}_{\mathcal{F}})$, the sheaf of vanishing cycle with respect to CS on the moduli space of holomorphic sections of $K_{\Sigma^{1/2}}\times_{\mathbb{C}^\times}\mathbf{M}$ as in [Nak15, §7].

When Σ is an elliptic curve, we do not need to introduce a $\mathbb{C}^\times$-action as we have a nonvanishing holomorphic 1-form dz. We consider a genuine C^∞-map $\Phi\colon\Sigma\to\mathbf{M}$ and define

$$\mathrm{CS}(\Phi) = \int_\Sigma (\Phi^*\theta)^{(0,1)}\wedge dz.$$

Remark 1.1. Suppose that we have a $\mathbb{C}^\times\times\mathbb{C}^\times$-action on $\mathbf{M}$ such that $(t_1,t_2)^*\theta = t_1t_2\theta$. It corresponds to a *cotangent type* gauge theory in [Nak15]. We take a $\mathbb{C}^\times\times\mathbb{C}^\times$-bundle P' such that the associated bundle $P'_{\mathbb{C}^\times\times\mathbb{C}^\times}\mathbb{C}\cong K_\Sigma$, where $\mathbb{C}^\times\times\mathbb{C}^\times$ acts on $\mathbb{C}$ by $(t_1,t_2)z = t_1t_2z$. (In other words, we take two line bundles M_1, M_2 over Σ such that $M_1\otimes M_2 = K_\Sigma$.) Then a $\mathbb{C}^\times\times\mathbb{C}^\times$-equivariant C^∞-map $\Phi\colon P'\to\mathbf{M}$ gives a well-defined $(1,1)$-form $(\Phi^*\theta)^{(0,1)}$.

1(ii). **Gauged σ-models.** Next suppose the following data are given:

- $(\mathbf{M},\omega)$ is a (holomorphic) symplectic manifold with a G-action preserving ω.
- there is a $\mathbb{C}^\times$-action commuting with the G-action, such that $t^*\omega = t^2\omega$ for $t\in\mathbb{C}^\times$.
- there is a G-invariant Liouville form θ such that $d\theta=\omega$ and $t^*\theta = t^2\theta$.

For a representation, the second $\mathbb{C}^\times$-action is the scaling one. Note that $\mathfrak{g}\ni\xi\mapsto -\theta(\xi^*)\in C^\infty(\mathbf{M})$ is a comoment map for the G-action on $\mathbf{M}$. Here $\xi^*\equiv\xi^*_{\mathbf{M}}$ is the vector field on $\mathbf{M}$ generated by $\xi\in\mathfrak{g}$.

We now consider a general G. We also fix a C^∞ principal G-bundle P. We then consider the fiber product $P\times_\Sigma P_{K_\Sigma^{1/2}}$, which is a principal $G\times\mathbb{C}^\times$-bundle over Σ.

Now a *field* consists of pairs

- $\overline{\partial} + A$: a partial connection on P,
- Φ : a C^∞ map $P \times_\Sigma P_{K_\Sigma^{1/2}} \to \mathbf{M}$, which is equivariant under $G \times \mathbb{C}^\times$-action.

The space of fields is denoted by $\mathcal{F}$ again.

We regard $\overline{\partial} + A$ as a collection of $\mathfrak{g}$-valued $(0,1)$-forms A_α such that $A_\alpha = -\overline{\partial} g'_{\alpha\beta} \cdot {g'_{\alpha\beta}}^{-1} + g'_{\alpha\beta} A_\beta {g'_{\alpha\beta}}^{-1}$, where $g'_{\alpha\beta}$ is the transition function for P. Similarly we regard Φ as a collection $\Phi_\alpha\colon U_\alpha \to \mathbf{M}$ such that $\Phi_\alpha = (g'_{\alpha\beta}, g_{\alpha\beta}) \cdot \Phi_\beta$, using the $G \times \mathbb{C}^\times$-action on $\mathbf{M}$. The term involving $\overline{\partial} g'_{\alpha\beta} \cdot {g'_{\alpha\beta}}^{-1}$ is absorbed by coupling with the moment map $\boldsymbol{\mu}$ on $\mathbf{M}$, as $\langle \xi, \boldsymbol{\mu} \rangle = -\theta(\xi^*)$. In fact, we have

$$\begin{aligned}
&\langle A_\alpha ; \boldsymbol{\mu}(\Phi_\alpha) \rangle \\
&= -\langle \overline{\partial} g'_{\alpha\beta} \cdot {g'_{\alpha\beta}}^{-1}, \boldsymbol{\mu}(\Phi_\alpha) \rangle + \langle g'_{\alpha\beta} A_\beta {g'_{\alpha\beta}}^{-1}, \boldsymbol{\mu}((g'_{\alpha\beta}, g_{\alpha\beta}) \cdot \Phi_\beta) \rangle \\
&= \langle \Phi_\alpha^* \theta, \overline{\partial} g'_{\alpha\beta} \cdot {g'_{\alpha\beta}}^{-1} \rangle + g_{\alpha\beta}^2 \langle A_\beta, \boldsymbol{\mu}(\Phi_\beta) \rangle,
\end{aligned}$$

where we have used the equivariance of the moment map in the second equality. Thus $\{ (\Phi_\alpha^* \theta)^{(0,1)} - \langle A_\alpha, \boldsymbol{\mu}(\Phi_\alpha) \rangle \}$ defines a well-defined $(1,1)$-form. We introduce the holomorphic Chern-Simons functional as

$$\mathrm{CS}(A, \Phi) \overset{\text{def.}}{=} \int_\Sigma (\Phi_\alpha^* \theta)^{(0,1)} - \langle A_\alpha, \boldsymbol{\mu}(\Phi_\alpha) \rangle.$$

Then (A, Φ) is a critical point of CS if and only if

- $\boldsymbol{\mu}(\Phi_\alpha) = 0$,
- $\{\Phi_\alpha\}$ is a holomorphic section of $(P \times_\Sigma P_{K_\Sigma^{1/2}}) \times_{G \times \mathbb{C}^\times} \mathbf{M}$. Here a holomorphic structure on P is given by $\overline{\partial} + A$.

Thus (A, Φ) is a twisted holomorphic map from Σ to the quotient stack $[\boldsymbol{\mu}^{-1}(0)/G]$.

When $\mathbf{M}$ is a symplectic representation of $\mathbf{M}$, this construction recovers the previous one in [Nak15, §7].

We consider the dual of the equivariant cohomology with compact support of $\mathcal{F}$ with vanishing cycle coefficient:

$$H^*_{c,\mathcal{G}(P)}(\mathcal{F}, \varphi_{\mathrm{CS}}(\mathbb{C}_\mathcal{F}))^*, \tag{1.2}$$

where $\mathcal{G}(P)$ is the complex gauge group denoted by $\mathcal{G}_\mathbb{C}(P)$ in [Nak15].

The main proposal in [Nak15] can be generalized in a straightforward manner:

- Take $\Sigma = \mathbb{P}^1$. Define a commutative product on $H^*_{c,\mathcal{G}(P)}(\mathcal{F}, \varphi_{\mathrm{CS}}(\mathbb{C}_\mathcal{F}))^*$ and consider an affine variety given by its spectrum. It should be the underlying affine variety of the moduli space of vacua of the gauged σ-model for $(G, \mathbf{M}, \omega)$ [HKLR87]. In particular, it is expected to satisfy various properties claimed in the physics literature.

2. Symplectic leaves and transversal slices

Let us consider the case $\mathbf{M}$ is a symplectic representation of G.

Question 2.1. (1) Are there only finitely many symplectic leaves in the Coulomb branch $\mathcal{M}_C$?

(2) Are symplectic leaves and transversal slices Coulomb branches of gauge theories ?

(3) Is there a 'natural' order-reversing bijection between symplectic leaves of the Coulomb branch $\mathcal{M}_C$ and the Higgs branch $\mathcal{M}_H$? Suppose (2) is true. Are gauge theories for symplectic leaves and transversal slices of $\mathcal{M}_C$ and $\mathcal{M}_H$ interchanged under the bijection ?

For quiver gauge theories of type ADE, Higgs branches are quiver varieties while Coulomb branches are slices in affine Grassmannian. See [Nak15, §3(ii)]. (This statement is true only the *dominance* condition is satisfied. See §5(iii).) In this case it is well-known that both leaves are parametrized by dominant weights, and the answer is yes. See §5(iii) for detail.

Also the question (3) is one of requirements in the definition of the symplectic duality [BLPW14, §10].[2] Properties (1),(2) are known for Higgs branches $\mathcal{M}_H$, as we will review below. Therefore (1),(2) for $\mathcal{M}_C$ are natural to ask. The second question in (3) *does* make sense thanks to this fact.

In fact, the symplectic duality requires an order-reversing bijection on posets of *special* symplectic leaves, but we ignore a subtle difference between special and arbitrary symplectic leaves at this moment. Properties (1),(2) make sense even without relation to $\mathcal{M}_H$.

The naturality in (2) is vague, but the author cannot make it precise. In fact, we will find examples where (3) fail in §5(ii) below. It means that an ad-hoc bijection for (2) may be unnatural. Therefore we should regard (2) are also false for these. These examples will be related to *un*special leaves, hence this question must be corrected in future.

Also the question (3) is a little imprecise as Coulomb branches are affine algebraic varieties, and leaves are usually only quasi-affine. A typical example is the n^{th} symmetric product $S^n\mathbb{C}^2$ and its standard stratification $\bigsqcup S_\nu\mathbb{C}^2$ parametrized by partitions ν of n. The stratum $S_\nu\mathbb{C}^2$ is an open subvariety in $S^{\nu_1}\mathbb{C}^2 \times S^{\nu_2}\mathbb{C}^2 \times \cdots$ for $\nu = (1^{\nu_1}2^{\nu_2}\dots)$. The latter is the Coulomb branch of a certain gauge theory. See §5(ii) below. Note also that $S^{\nu_1}\mathbb{C}^2 \times S^{\nu_2}\mathbb{C}^2 \times \cdots \to \overline{S_\nu\mathbb{C}^2}$ is a finite birational morphism, but is *not* an isomorphism. Therefore, as the best, we can ask whether the closure of a symplectic leaf is an image of a finite birational morphism from the Coulomb branch of a gauge theory. It is not clear (at least to the author) whether this determine the gauge theory uniquely or not. We ignore this complexity and simply ask whether a symplectic leaf is the Coulomb branch hereafter.

An important special case of (2) is

[2]The author thanks Tom Braden who explained the statement in talks by at Kyoto and Boston.

Question 2.2. Suppose the Higgs branch $\mathcal{M}_H = \mathbf{M}/\!\!/\!\!/G$ consists of only $\mathbf{M}^G$.[3] Is the corresponding Coulomb branch $\mathcal{M}_C$ a smooth symplectic manifold ? Is converse true also ?

2(i). **Higgs branch.** Let us recall a well-known result on symplecitc leaves on the Higgs branch $\mathbf{M}/\!\!/\!\!/G = \boldsymbol{\mu}^{-1}(0)/\!\!/G$.

We consider $\mathbf{M}/\!\!/\!\!/G$ as a set of closed G-orbits in $\boldsymbol{\mu}^{-1}(0)$. It has a natural stratification given by conjugacy classes of stabilizers (see [Nak94, §6], which is based on [SL91]):

$$\mathbf{M}/\!\!/\!\!/G = \bigsqcup_{(\widehat{G})} (\mathbf{M}/\!\!/\!\!/G)_{(\widehat{G})}, \tag{2.3}$$

where a stratum $(\mathbf{M}/\!\!/\!\!/G)_{(\widehat{G})}$ consists of orbits through points $x \in \mathbf{M}/\!\!/\!\!/G$ whose stablizer $\operatorname{Stab}_G(x)$ is conjugate to $\hat{G}$. Note that $\widehat{G}$ is a reductive group as x has a closed orbit.

Let us describe $(\mathbf{M}/\!\!/\!\!/G)_{(\widehat{G})}$ as a symplectic reduction. (See [SL91, Th. 3.5] for detail.) Let $\mathbf{M}^{\widehat{G}} = \{m \in \mathbf{M} \mid gm = m \text{ for } g \in \hat{G}\}$ denote the fixed point locus. It is a symplectic vector subspace. Let $N_G(\widehat{G})$ denote the normalizer of $\widehat{G}$ in G. We have an action of ${}^{N_G(\widehat{G})}/{}_{\widehat{G}}$ on $\mathbf{M}^{\widehat{G}}$. Then we have

$$(\mathbf{M}/\!\!/\!\!/G)_{(\widehat{G})} = (\mathbf{M}^{\widehat{G}}/\!\!/\!\!/{}^{N_G(\widehat{G})}/{}_{\widehat{G}})_{(\{e\})}. \tag{2.4}$$

Here the subscript $(\{e\})$ means ${}^{N_G(\widehat{G})}/{}_{\widehat{G}}$-orbits through points whose stabilizer is conjugate to the identity group $\{e\}$, i.e., free orbits. In particular, $(\mathbf{M}^{\widehat{G}}/\!\!/\!\!/{}^{N_G(\widehat{G})}/{}_{\widehat{G}})_{(\{e\})}$ is symplectic. We do *not* claim that it is a symplectic leaf as it may not be connected in general. This is a subtle point, which we do not understand well. See the example 5(iv).

A transversal slice to a stratum $(\mathbf{M}/\!\!/\!\!/G)_{\widehat{G}}$ is also described as a symplectic reduction. (See [SL91, §2]. See also [Nak01, §3] and [CB03].) Let $m \in \boldsymbol{\mu}^{-1}(0)$ such that $\operatorname{Stab}_G(m) = \widehat{G}$. We take the orbit Gm through m and its tangent space T_mGm at m. The latter is an isotropic subspace. We consider the symplectic normal space $\widehat{\mathbf{M}} \overset{\text{def.}}{=} (T_mGm)^\omega/T_mGm$, which has a natural symplectic structure. It is naturally a representation of $\widehat{G}$. Then $\mathbf{M}/\!\!/\!\!/G$ and $\widehat{\mathbf{M}}/\!\!/\!\!/\widehat{G}$ are locally isomorphic around $[m]$ and $[0]$. Under the local isomorphism the stratum $(\mathbf{M}/\!\!/\!\!/G)_{(\widehat{G})}$ is mapped to $(\widehat{\mathbf{M}}/\!\!/\!\!/\widehat{G})_{(\widehat{G})}$. This is nothing but the fixed subspace $T \overset{\text{def.}}{=} \widehat{\mathbf{M}}^{\widehat{G}}$. We take a complementary subspace $T^\perp$ of T in $\widehat{\mathbf{M}}$. The transversal slice is given by

$$T^\perp/\!\!/\!\!/\widehat{G}. \tag{2.5}$$

[3]The author thanks Alexander Braverman to point out a mistake in an earlier version.

2(ii). **Coulomb branch.** Combining Question 2.1 with (2.4, 2.5), we arrive at the following:

Question 2.6. Strata (resp. transversal slices) of $\mathcal{M}_C$ are the Coulomb branches of gauge theories $\mathrm{Hyp}(T^\perp) /\!/\!/\!/ \widehat{G}$ (resp. $\mathrm{Hyp}(\mathbf{M}^{\widehat{G}}) /\!/\!/\!/ N_G(\widehat{G})/\widehat{G}$) ?

We have the following naive construction: Consider morphisms between Higgs branches.

$$\mathbf{M}^{\widehat{G}} /\!/\!/ N_G(\widehat{G})/\widehat{G} \to \mathbf{M} /\!/\!/ G \to T^\perp /\!/\!/ \widehat{G}.$$

They induces morphisms between moduli spaces of twisted holomorphic maps from a Riemann surface Σ. Taking the dual of cohomology groups with vanishing cycle coefficients, we get homomorphisms in the same direction. They conjecturally respect the multiplication, and hence induce morphisms on the spectrum. Thus morphisms between Coulomb branches goes in the opposite direction, and the strata and slices are interchanged.

In the proposed 'definition' of Coulomb branches, the affine GIT quotient $\mathbf{M}/\!/\!/G = \boldsymbol{\mu}^{-1}(0)/\!/G$ is not enough: we need the quotient stack $[\boldsymbol{\mu}^{-1}(0)/G]$. But we do not think this makes a crucial difference.

3. Complete intersection and conical $\mathbb{C}^\times$-action

For a coweight λ of G, we define

$$\Delta(\lambda) \overset{\text{def.}}{=} -\sum_{\alpha\in\Delta^+} |\langle\alpha,\lambda\rangle| + \frac{1}{4}\sum_{\mu} |\langle\mu,\lambda\rangle| \dim \mathbf{M}(\mu), \tag{3.1}$$

where Δ^+ is the set of positive roots, and μ runs over the set of weights of $\mathbf{M}$. Here $\mathbf{M}(\mu)$ is the weight space.

The gauge theory $\mathrm{Hyp}(\mathbf{M}) /\!/\!/\!/ G$ is said *good* if $2\Delta(\lambda) > 1$ for any $\lambda \neq 0$, and *ugly* if $2\Delta(\lambda) \geq 1$ and not good. (Note $2\Delta(\lambda) \in \mathbb{Z}$ as weights appear in pairs for a symplectic representation.) The monopole formula predicts that $2\Delta(\lambda)$ corresponds to a weight of the $\mathbb{C}^\times$-action on $\mathcal{M}_C$. (See [Nak15, §4].) Therefore

- $\mathrm{Hyp}(\mathbf{M})/\!/\!/\!/G$ is 'good' or 'ugly' if and only if the $\mathbb{C}^\times$-action on the Coulomb branch $\mathcal{M}_C$ is *conical.*

Recall that a $\mathbb{C}^\times$-action is conical if the coordinate ring $\mathbb{C}[\mathcal{M}_C]$ is only with nonnegative weights, and the zero weight space is 1-dimensional consisting of constant functions. Similarly $\mathrm{Hyp}(\mathbf{M}) /\!/\!/\!/ G$ is good if and only if the $\mathbb{C}^\times$-aciton is conical and there is no function of weight 1.

On the other hand, it is observed in [Nak15, §2(iv)] that $2\Delta(\lambda) \geq 1$ looks similar to a complete intersection criterion of $\boldsymbol{\mu} = 0$ for the Higgs branch by Crawley-Boevey [CB01] for quiver gauge theories. The following question was asked:

Question 3.2. Is the followings true ?

The $\mathbb{C}^\times$-action of the Coulomb branch $\mathcal{M}_C$ is conical only if the level set $\boldsymbol{\mu}^{-1}(0)$ of the moment map equation for the Higgs branch $\mathcal{M}_H$ is complete intersection in $\mathbf{M}$ of dim $= \dim \mathbf{M} - \dim G$.

We assume $\mathbf{M}$ is a faithful representation of G.

Since $\mathbf{M}/\!\!/\!\!/G$ depends on the image $G \to \mathrm{Sp}(\mathbf{M})$, the faithfulness assumption is a natural requirement. If $G \to \mathrm{Sp}(\mathbf{M})$ has a positive dimensional kernel, $\boldsymbol{\mu}^{-1}(0)$ has dimension larger than $\dim \mathbf{M} - \dim G$. If $G \to \mathrm{Sp}(\mathbf{M})$ has only finite kernel, the complete intersection property does not matter, but we have the following example: Suppose that $\mathbb{C}^\times$ acts on $\mathbf{N} = \mathbb{C}$ by weight $N \neq 0$, and take $\mathbf{M} = \mathbf{N} \oplus \mathbf{N}^*$. The level set $\boldsymbol{\mu}^{-1}(0)$ is the same for the case $N = 1$ and is not irreducible, but it is good if $N > 1$ and ugly if $N = 1$. On the other hand, the Coulomb branch $\mathcal{M}_C$ *does* depends on N. One can see from the monopole formula as $\mathcal{M}_C = \mathbb{C}^2/(\mathbb{Z}/N\mathbb{Z})$ with the $\mathbb{C}^\times$-action induced from $t\cdot(x,y) = (tx, ty)$ for $(x,y) \in \mathbb{C}^2$.

In [Nak15] it was asked two conditions in Question 3.2 are equivalent. But it turns out that there are counter examples for the converse.

4. Poisson and intersection homology groups

Let X be a Poisson variety and let $HP_*(X)$ denote its Poisson homology group, defined in [ES10]. We are interested in the case X is affine and the degree 0 part $HP_0(X)$. It is known that $HP_0(X)$ is the quotient of $\mathbb{C}[X]$ by the linear span of all brackets. The following is a $\mathcal{M}_H/\mathcal{M}_C$ version of conjecture of Proudfoot [Pro14]:

Question 4.1. Suppose $\mathrm{Hyp}(\mathbf{M}) /\!\!/\!\!/ G$ is good.

(1) Are there natural isomorphisms of graded vector spaces ?

$$HP_0(\mathcal{M}_H) \cong IH^*(\mathcal{M}_C), \quad HP_0(\mathcal{M}_C) \cong IH^*(\mathcal{M}_H).$$

Here the grading of the left hand sides are given by the $\mathbb{C}^\times$-action.

(2) Let $\mathcal{A}_\hbar(\mathcal{M}_H)$, $\mathcal{A}_\hbar(\mathcal{M}_C)$ be the quantization of $\mathbb{C}[\mathcal{M}_H]$, $\mathbb{C}[\mathcal{M}_C]$ respectively. Are there natural isomorphisms of graded vector spaces ?

$$HH_0(\mathcal{A}_\hbar(\mathcal{M}_H)) \cong IH^*_{\mathbb{C}^\times}(\mathcal{M}_C), \quad HH_0(\mathcal{A}_\hbar(\mathcal{M}_C)) \cong IH^*_{\mathbb{C}^\times}(\mathcal{M}_H).$$

Here HH_0 denote the zero-th Hochschild homology group, i.e., the quotient of the quantization by the span of its commutator.

The quantization $\mathcal{A}_\hbar(\mathcal{M}_H)$ is defined under the assumption that $\mathbf{M}$ is of cotangent type, i.e., $\mathbf{M} = \mathbf{N} \oplus \mathbf{N}^*$ for a G-module $\mathbf{N}$. Then we consider the ring $\mathcal{D}_\hbar(\mathbf{N})$ of $\hbar$-differential operators on $\mathbf{N}$, and define $\mathcal{A}_\hbar(\mathcal{M}_H)$ as its quantum hamiltonian reduction by G. The Coulomb branch $\mathcal{M}_C$ is expected to have a quantization always as a $\mathbb{C}^\times$-equivariant homology group in (1.2), where $\mathbb{C}^\times$ acts on $\mathcal{F}$ through the $\mathbb{C}^\times$-action on $\Sigma = \mathbb{P}^1$. Here $\hbar$ appears as $H^*_{\mathbb{C}^\times}(\mathrm{pt}) = \mathbb{C}[\hbar]$.

This question is well formulated *except* the meaning of the naturality, which we do not discuss here. If we do not assume the good condition, we easily find couter-examples anyway, as we have many cases with $\mathcal{M}_H = \{0\}$, while $\mathcal{M}_C$ is nontrivial. In this case, $\mathcal{M}_C$ is expected to be smooth, hence $IH^*(\mathcal{M}_C) \cong H^*(\mathcal{M}_C)$, $HP_0(\mathcal{M}_C) \cong H^{\dim \mathcal{M}_C}(\mathcal{M}_C)$. These cohomology groups are often nontrivial.

In order to exclude these cases, we have assumed the good condition.

5. (Counter)Examples

5(i). **Quiver gauge theories of type** ADE. Let us consider a quiver gauge theory (see [Nak15, 2(iv)]). We follow the notation there. We first suppose $W = 0$. Note that scalars in $G = \prod_{i\in Q_0} \mathrm{GL}(V_i)$ act trivially on $\mathbf{M}$. We need $\mathbf{M}$ is a faithful representation in Question 3.2 for the Higgs branch. Also the embedding $\lambda\colon \mathbb{C}^\times \to G$ of scalars gives $\Delta(\lambda) = 0$. Therefore $\mathcal{M}_C$ is never conical. Therefore we replace G by $\prod_{i\in Q_0} \mathrm{GL}(V_i)/\mathbb{C}^\times$. We also assume that the support $\{i \in Q_0 \mid \dim V_i \neq 0\}$ of $\dim V$ is connected, as $\mathbf{M}$ is not faithful otherwise. We write the dimension vector $\dim V$ by $\mathbf{v}$.

We first assume that the quiver $Q = (Q_0, Q_1)$ is of type ADE. The Coulomb branch $\mathcal{M}_C$ is the moduli space of centered ADE monopoles on $\mathbb{R}^3$ where $\mathbf{v}$ is the corresponding monopole charge ([HW97] for type A, [Ton99] in general[4]). This is a smooth symplectic manifold. By [Don84, Hur89, Jar98], it is the same as the moduli space of centered based rational maps from $\mathbb{P}^1$ to the flag manifold of type ADE. The definition in [BFN15] produces $\mathcal{M}_C$ as this moduli space.

On the other hand, consider the Higgs branch $\mathcal{M}_H$. It is known that any element in $\boldsymbol{\mu}^{-1}(0)$ is automatically nilpotent for ADE quivers [Lus90]. Therefore the only closed orbit in $\boldsymbol{\mu}^{-1}(0)$ is 0. (See also [Nak94, Prop. 6.7].) Therefore the answer to Question 2.2 is yes.

Proposition 5.1. *$\boldsymbol{\mu}^{-1}(0)$ is complete intersection of dimension $\dim \mathbf{M} - \dim G$ if and only if $\mathbf{v}$ is a positive root.*

Proof. By the criterion in [CB01, Th. 1.1], $\boldsymbol{\mu}^{-1}(0)$ is complete intersection of $\dim = \dim \mathbf{M} - \dim G$ if and only if

$$2 - \langle \mathbf{v}, \mathbf{C}\mathbf{v}\rangle \geq \sum_k (2 - \langle \beta^{(k)}, \mathbf{C}\beta^{(k)}\rangle) \tag{5.2}$$

for any decomposition $\mathbf{v} = \sum \beta^{(k)}$ such that $\beta^{(k)}$ is a positive root (or equivalently nonzero positive vector). Here $\mathbf{C} = (2\delta_{ij} - a_{ij})$ with a_{ij}, the number of edges (regardless of orientation) between i and j if $i \neq j$, and its twice if $i = j$. (The latter does not occur for type ADE.) Since we are assuming Q is of type ADE, the right hand side is always 0. On the other hand, the left hand side is nonpositive, and is zero if and only if $\mathbf{v}$ is a positive root. $\square$

[4]The author thanks Vasily Pestun who explained this statement to a collaborator of [BFN15]. It is compatible with his work with Nekrasov of 4*d* quiver gauge theories [NP12].

Next let us study $\Delta(\lambda)$ in (3.1). We take a maximal torus $T = \prod_i T(V_i)/\mathbb{C}^\times$, where $T(V_i)$ is the diagonal subgroup of $\mathrm{GL}(V_i)$ and consider $\lambda\colon \mathbb{C}^\times \to T$. According to weights on $\bigoplus_i V_i$, we decompose $V = V^{(1)} \oplus V^{(2)} \oplus \cdots$ such that $\lambda(t)$ acts on $t^{\lambda^{(k)}}$ on $V^{(k)}$. Set $\beta^k = (\beta_i^{(k)}) = (\dim V_i^{(k)})$. Then

$$(5.3)\qquad \begin{aligned} 2\Delta(\lambda) &= \sum_{k<l} |\lambda^{(k)} - \lambda^{(l)}| \left(\sum_i -2\beta_i^{(k)}\beta_i^{(l)} + \sum_{i,j} a_{ij}\beta_i^{(k)}\beta_j^{(l)} \right) \\ &= -\sum_{k<l} |\lambda^{(k)} - \lambda^{(l)}| \langle \beta^{(k)}, \mathbf{C}\beta^{(l)} \rangle. \end{aligned}$$

Suppose that $\mathbf{v}$ is not a positive root. We decompose $\mathbf{v}$ as sum $\sum \beta^{(k)}$ of positive roots $\beta^{(k)}$ so that any combination $\beta^{(k)} + \beta^{(l)}$ is not a root. Then $4 \le \langle \beta^{(k)} + \beta^{(l)}, \mathbf{C}(\beta^{(k)} + \beta^{(l)}) \rangle = \langle \beta^{(k)}, \mathbf{C}\beta^{(k)} \rangle + \langle \beta^{(l)}, \mathbf{C}\beta^{(l)} \rangle + 2\langle \beta^{(k)}, \mathbf{C}\beta^{(l)} \rangle = 4 + 2\langle \beta^{(k)}, \mathbf{C}\beta^{(l)} \rangle$ for $k \ne l$. Therefore $\langle \beta^{(k)}, \mathbf{C}\beta^{(l)} \rangle \ge 0$. Hence $2\Delta(\lambda) \le 0$, so the gauge theory is not good or ugly. Therefore the answer to Question 3.2 is yes.

Let us continue to study $\Delta(\lambda)$.

Proposition 5.4. $\mathrm{Hyp}(\mathbf{M}) /\!\!/\!\!/ G$ *is never good.*

Proof. Take a decomposition $\mathbf{v} = \beta^1 + \beta^2$, $\beta^1 = \mathbf{v} - \alpha_i$, $\beta^2 = \alpha_i$, we find $2\Delta(\lambda) = -|\lambda^{(1)} - \lambda^{(2)}|(\langle \alpha_i, \mathbf{C}\mathbf{v} \rangle - 2)$. We have $2\Delta(\lambda) > 1$ for λ of this form if and only if $\langle \alpha_i, \mathbf{C}\mathbf{v} \rangle \le 0$. But this is never possible unless $\mathbf{v} = 0$. □

Suppose Q is of type A and $\mathbf{v}$ is a positive root. Then G is a torus and $2\Delta(\lambda) > 0$ unless $\lambda = 0$. Therefore $\mathrm{Hyp}(\mathbf{M}) /\!\!/\!\!/ G$ is ugly. It is also possible to check $\mathrm{Hyp}(\mathbf{M}) /\!\!/\!\!/ G$ is ugly if Q is of type D and $\mathbf{v}$ is positive root as follows.

Suppose Q is of type D_ℓ. We consider the case $\mathbf{v} = (12\dots2\begin{smallmatrix}1\\ \\1\end{smallmatrix})$, other cases are similar. We represent a coweight λ as an integer vector $(\lambda^1, \lambda^2_1, \lambda^2_2, \dots, \lambda^{\ell-2}_1, \lambda^{\ell-2}_2, \lambda^{\ell-1}, \lambda^\ell)$. We have

$$\begin{aligned} 2\Delta(\lambda) = &-2\sum_{p=2}^{\ell-2} |\lambda^p_1 - \lambda^p_2| + |\lambda^1 - \lambda^2_1| + |\lambda^1 - \lambda^2_2| + \sum_{p=2}^{\ell-3}\sum_{a,b=1}^{n} |\lambda^p_a - \lambda^{p+1}_b| \\ &+ \sum_{p=\ell-1,\ell} |\lambda^{\ell-2}_1 - \lambda^p| + |\lambda^{\ell-2}_2 - \lambda^p|. \end{aligned}$$

We use

$$
\begin{aligned}
2|\lambda_1^2 - \lambda_2^2| &\le |\lambda^1 - \lambda_1^2| + |\lambda^1 - \lambda_2^2| + \frac{1}{2}\sum_{a,b=1}^{2} |\lambda_a^2 - \lambda_b^3|,\\
2|\lambda_1^p - \lambda_2^p| &\le \frac{1}{2}\sum_{a,b=1}^{2} |\lambda_a^{p-1} - \lambda_b^p| + \frac{1}{2}\sum_{a,b=1}^{2} |\lambda_a^p - \lambda_b^{p+1}| \quad (p = 3,\dots,\ell-3),\\
2|\lambda_1^{\ell-2} - \lambda_2^{\ell-2}| &\le \frac{1}{2}\sum_{a,b=1}^{2} |\lambda_a^{\ell-3} - \lambda_b^{\ell-2}| + \sum_{p=\ell-1,\ell} |\lambda_1^{\ell-2} - \lambda^p| + |\lambda_2^{\ell-2} - \lambda^p|.
\end{aligned}
$$

(In fact, $p = \ell - 1$ is enough in the second sum in the last equality.) Taking sum, we find $2\Delta(\lambda) \ge 0$ and the equality holds if and only if all entries of λ are the same, i.e., it is zero as a coweight of $\prod_i \mathrm{GL}(V_i)/\mathbb{C}^\times$. Therefore it is ugly.

We do not know for exceptional cases, though it is a finite check.

5(ii). **Affine types.** Next suppose the underlying graph is of type *affine ADE* with $W = 0$. In this case, the Coulomb branch $\mathcal{M}_C$ is conjecturally the moduli space of calorons, in other words, instantons on $\mathbb{R}^3 \times S^1$. The charge is again given by $\mathbf{v}$. More precisely we need two modifications: a) We take the symplectic reduction by $\mathbb{C}^\times$, the action induced from $\mathbb{C}^\times$-action on the base $\mathbb{R}^3 \times S^1 = \mathbb{C}^\times \times \mathbb{C}$. b) We take the Uhlenbeck partial compactification like in case of $\mathbb{R}^4$. Therefore $\mathcal{M}_C$ has a stratification

$$
\mathcal{M}_C(\mathbf{v}) = \bigsqcup \mathcal{M}_C^\circ(\mathbf{v} - |\nu|\delta) \times S_\nu(\mathbb{R}^3 \times S^1)_c, \tag{5.5}
$$

where δ is the primitive imaginary root vector, $\mathcal{M}_C^\circ(\mathbf{v} - |\nu|\delta)$ denotes the moduli space of *genuine* calorons with charge $\mathbf{v} - |\nu|\delta$, and $S_\nu(\mathbb{R}^3 \times S^1)$ is a stratum of the symmetric product of $\mathbb{R}^3 \times S^1$ given by a partition ν, and $S_\nu(\mathbb{R}^3 \times S^1)_c$ means the symplectic reduction by the $\mathbb{C}^\times$-action. Therefore the strata are parametrized by partitions ν such that $\mathbf{v} - |\nu|\delta$ is nonnegative.

This result remains true for Jordan quiver. It corresponds to U(1)-calorons, and there is no genuine calorons. Therefore the Coulomb branch is expected to be $S^{\dim V}(\mathbb{R}^3 \times S^1)_c = \bigsqcup S_\nu(\mathbb{R}^3 \times S^1)_c$. (It is confirmed in [BFN15].)

Moreover

(a) The stratum in (5.5), replaced by $\mathcal{M}_C(\mathbf{v} - |\nu|\delta) \times \prod S^{\nu_k}(\mathbb{R}^3 \times S^1)_c$ for $\nu = (1^{\nu_1} 2^{\nu_2} \dots)$, is the Coulomb branch of the quiver gauge theory, where the graph is the disjoint union of the original Q and copies of the Jordan quiver with the dimension vector $\mathbf{v} - |\nu|\delta$, ν_1, ν_2, ... respectively.

(b) The transversal slice is the product of *centered* ADE instanton moduli spaces on $\mathbb{R}^4$ with instanton numbers ν_1, ν_2, ... If we ignore the *centered* condition, it is the Coulomb branch of the quiver gauge theory, where the quiver is the disjoint union of copies of the original Q with the dimension vectors $(\mathbf{v}, \mathbf{w}) = \underbrace{(\delta, \delta_{0i}), \dots}_{\nu_1 \text{ times}}, \underbrace{(2\delta, \delta_{0i}), \dots}_{\nu_2 \text{ times}}, \dots$. See [Nak15, 1(vii) and 3(i)].

Remark 5.6. For affine type A, the moduli space of calorons is isomorphic to the moduli space of framed locally free parabolic sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$. (See [CH10, CH08] and [Tak15].) This is expected to be true for any affine type if we replace locally free parabolic sheaves by principal G_{ADE}-bundles with parabolic structures. The definition of [BFN15] gives this moduli space for affine type A, and conjecturally in general.

Let us consider $\mathcal{M}_H = \boldsymbol{\mu}^{-1}(0)/\!\!/G$. A closed G-orbit corresponds to a semisimple representation of the preprojective algebra of type affine ADE. By [CB01], we have the classification of simple representations: they are either S_i (one dimensional at the vertex i) or have dimension δ. In the latter case, $\mathcal{M}_H$ is a particular case of Kronheimer's construction [Kro89], and is $\mathbb{R}^4/\Gamma$ for a finite subgroup $\Gamma \subset \mathrm{SL}(2)$ corresponding to the affine ADE graph. The origin corresponds to a direct sum of S_i's, but any point except the origin is a simple representation. Since semisimple representation is a direct sum of simple representations, we have a stratification

$$\mathcal{M}_H(\mathbf{v}) = \bigsqcup S_\nu(\mathbb{R}^4 \setminus \{0\}/\Gamma), \tag{5.7}$$

where ν is a partition such that $\mathbf{v} - |\nu|\delta$ is nonnegative. Here we have a direct sum of S_i's corresponding to $\mathbf{v} - |\nu|\delta$. It corresponds to the 'origin' in $\mathbb{R}^4$.

A semisimple representation of the preprojective algebra for the Jordan quiver is a pair of commuting semisimple matrices. Hence $\boldsymbol{\mu}^{-1}(0)/\!\!/G = S^n(\mathbb{C}^2) = \bigsqcup S_\nu(\mathbb{R}^4)$. Therefore we have the same result for the Jordan quiver.

(c) The stratum in (5.7), replaced by $S^{\nu_1}(\mathbb{R}^4/\Gamma) \times S^{\nu_2}(\mathbb{R}^4/\Gamma) \times \cdots$, is the Higgs branch of the quiver gauge theory in (b) above. The quiver is the disjoint union of copies of Q, the dimension vectors are $(\mathbf{v}, \mathbf{w}) = (\nu_1\delta, \delta_{0i}), (\nu_2\delta, \delta_{0i}), \dots$.

(d) The transversal slice is the product

$$\mathcal{M}_H(\mathbf{v} - |\nu|\delta) \times \underbrace{S^1_0(\mathbb{R}^4) \times \dots}_{\nu_1 \text{ times}} \times \underbrace{S^2_0(\mathbb{R}^4) \times \dots}_{\nu_2 \text{ times}} \times \cdots .$$

Here $S^n_0(\mathbb{R}^4)$ is the nth *centered* symmetric power, i.e., $\{(x_1, \dots, x_n) \bmod S_n \mid \sum x_i = 0\}$. If we ignore the *centered* condition, it is the Higgs branch of the quiver gauge theory in (a) above. The quiver is the union of Q and copies of the Jordan quiver, the dimension vector is $\mathbf{v} - |\nu|\delta, \underbrace{\delta, , \dots}_{\nu_1 \text{ times}}, \underbrace{2\delta, \dots}_{\nu_2 \text{ times}}, \dots$.

In fact, for (d), we can put the centered condition, namely we impose that endomorphisms of $\mathbb{C}^k$ in $\mathbf{M}$ are trace-free. Since Coulomb branches are unchanged if we add trivial representations to $\mathbf{M}$, we can do the same for (a).

Comparing two stratifications, we find that answers to Question 2.1(1),(2) are yes, but (3) is no. The most natural order-reversing bijection is $\nu \to \nu^t$, but the dimension vectors do not match in (a), (d) and (b), (c) respectively. Even ignoring the ordering, it seems we do not have a 'natural' bijection compatible with the change of dimension vectors. In fact, *special* leaves have only $\nu = (1^\bullet)$,[5] hence we have an order-reversing bijection on special leaves.

These examples presents a difficulty to generalize the description of Coulomb branches for general type quiver gauge theories. It is a partial compactification of the space of based maps from $\mathbb{P}^1$ to the corresponding Kac-Moody flag manifold. We need to add 'defects' which correspond to semisimple representations of the corresponding preprojective algebra. Such a partial compactification has not been studied before, as far as the author knows.

The answer to Question 3.2 is yes thanks to the following:

Proposition 5.8. (1) *$\boldsymbol{\mu}^{-1}(0)$ is complete intersection of dimension* $\dim \mathbf{M} - \dim G$ *if and only if* $\mathbf{v}$ *is either α, $\delta - \alpha$ or δ, where α is a positive root of the root system of finite ADE type, obtained from Q by removing the 0-vertex.*

(2) *If* $\mathrm{Hyp}(\mathbf{M})/\!\!/\!\!/G$ *is good or ugly,* $\mathbf{v}$ *is either of the above form.*

(3) *If* $\mathrm{Hyp}(\mathbf{M})/\!\!/\!\!/G$ *is good,* $\mathbf{v}$ *is* δ.

Proof. (1) We can use the criterion (5.2) above. If $\beta^{(k)}$ is an imaginary (resp. a real) root, $\langle \beta^{(k)}, \mathbf{C}\beta^{(k)}\rangle = 0$ (resp. $= 2$). Also positive roots are $n\delta + \alpha$ ($n \geq 0$), $n\delta - \alpha$ ($n > 0$), $n\delta$ ($n > 0$) for a positive root α of the underlying finite root system. We see that dimension vectors α, $\delta - \alpha$, δ give complete intersection.

(2) We use the same argument as in finite type case using (5.3). If $\mathbf{v}$ is not a positive root, we have a decomposition $\mathbf{v} = \sum \beta^{(k)}$ such that $\langle \beta^{(k)}, \mathbf{C}\beta^{(l)}\rangle \geq 0$ for $k \neq l$ as before: We have $\langle \beta, \mathbf{C}\beta\rangle \geq 0$. It is equal to 0 (resp. 2 if and only β is an imaginary (resp. a real) root. Therefore $2\Delta(\lambda) \leq 0$, so the gauge theory is not good or ugly.

If $\mathbf{v} = n\delta + \alpha$ for a positive root α of the underlying finite root system and $n > 0$, we take the decomposition $\beta^{(1)} = n\delta$, $\beta^{(2)} = \alpha$. Then $\langle \beta^{(1)}, \mathbf{C}\beta^{(2)}\rangle = 0$. Similarly, for $\mathbf{v} = n\delta - \alpha$ for $n > 1$, we take $\beta^{(1)} = (n-1)\delta$, $\beta^{(2)} = \delta - \alpha$ to deduce a contradiction. For $\mathbf{v} = n\delta$ with $n > 1$, consider $\beta^{(1)} = (n-1)\delta$, $\beta^{(2)} = \delta$.

(3) If $\mathbf{v} = \alpha$ or $\delta - \alpha$, it is a quiver gauge theory of type ADE. Therefore Proposition 5.4 implies that it is not good. $\square$

For affine type A, we see that $\mathrm{Hyp}(\mathbf{M})/\!\!/\!\!/G$ is good if $\mathbf{v} = \delta$. It is probably true in general. Assuming it, we have $\mathcal{M}_H = \mathbb{C}^2/\Gamma$, where Γ is a finite subgroup of $\mathrm{SL}(2)$ corresponding to Q. In view of Question 4.1, it is interesting to compute $IH^*(\mathcal{M}_C)$, $HP_0(\mathcal{M}_C)$.

5(iii). **Quiver gauge theories with $W \neq 0$.** Let us turn to quiver gauge theories with $W \neq 0$. We take two Q_0-graded vector spaces V, W. We take

[5]The author thanks Ben Webster for an explanation of this result.

$G = \prod_i \mathrm{GL}(V_i)$ unlike the case $W = 0$. When $W \neq 0$, we can deform and take (partial) resolution of the Coulomb branch. But we set the parameter to be 0, and consider the most singular Coulomb branch. We denote the dimension vector $\dim W$ by $\mathbf{w}$.

Assume the underlying graph is of type ADE. In [Nak15, 3(ii)], it was conjectured that the Coulomb branch is the moduli space of S^1-equivariant ADE-instantons on $\mathbb{R}^4$ where $\mathbf{w}$ (resp. $\mathbf{w} - \mathbf{C}\mathbf{v}$) corresponds to the coweight λ (resp. μ) : $S^1 \to G_{ADE,c}$ giving the S^1-action on the fiber at 0 (resp. ∞). Here $\mathbf{C}$ is the Cartan matrix. This can be true only if $\mu = \mathbf{w} - \mathbf{C}\mathbf{v}$ is dominant, as the S^1-action on the fiber corresponds to a *dominant* coweight.

After reading the preprint [BDG15], which appears in arXiv shortly after [Nak15] and then looking at the original physics literature [CK98], the author understand that the conjecture must be corrected. Namely the Coulomb branch is the moduli space of singular ADE-monopoles on $\mathbb{R}^3$. In order to connect with the previous conjecture, let us recall that singular monopoles are S^1-equivariant instantons on the Taub-NUT space [Kro85]. The Taub-NUT space and $\mathbb{R}^4$ are both $\mathbb{C}^2$ as a holomorphic symplectic manifold, but the Riemannian metrics are different. It is expected that moduli spaces of S^1-equivariant instantons on two spaces are isomorphic as holomorphic symplectic manifolds when μ is dominant, but the Taub-NUT case has more general moduli spaces, as we do not need to assume μ is dominant. (It corresponds to the monopole charge.)[6] The definition in [BFN15] produces a certain moduli space of bundles over $\mathbb{P}^1$, which conjecturally isomorphic to the moduli space of singular monopoles.

Let us describe the stratification on $\mathcal{M}_C$. As in the case with $W = 0$, we need to consider the Uhlenbeck partial compactification of the moduli space of instantons. Since we are considering S^1-equivariant instantons, the bubbling only occurs at 0, the S^1-fixed point in the Taub-NUT space. (A bubbled instanton is defined over $\mathbb{R}^4$.) The remained S^1-equivariant instanton has a different dominant coweight for the S^1-action on the fiber at 0. But the coweight for ∞ is unchanged. Therefore

$$\mathcal{M}_C(\mu, \lambda) = \bigsqcup_{\substack{\lambda' : \text{ dominant} \\ \mu \le \lambda' \le \lambda}} \mathcal{M}_C^\circ(\mu, \lambda'),$$

where $\mathcal{M}_C^\circ(\mu, \lambda')$ denote the moduli space of genuine S^1-equivariant instantons with 1) the monopole charge μ, and 2) the coweight λ' for the S^1-action at 0. For example, $\lambda' = \lambda$ is the open stratum, and the minimum $\lambda' \geq \mu$ is the closed stratum.

The stratum (resp. the transversal slice) is the Coulomb branch of the quiver gauge theory of the same type with the dimension vectors given by (μ, λ') (resp. (λ', λ)).

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[6]The author thanks Sergey Cherkis for his explanation on instantons on (multi-)Taub-NUT spaces, a.k.a. bow varieties over years.

Let us consider the Higgs branch $\mathcal{M}_H = \boldsymbol{\mu}^{-1}(0)$. It is a quiver variety of type ADE, and the stratification is given [Nak94, Prop. 6.7] as in §2(i):

$$\mathcal{M}_H(\mu, \lambda) = \bigsqcup_{\substack{\mu' \,:\, \text{dominant} \\ \mu \le \mu' \le \lambda}} \mathcal{M}_H^\circ(\mu', \lambda),$$

where $\mathcal{M}_H^\circ(\mu', \lambda) = \mathfrak{M}_0^{\text{reg}}(\mathbf{v}^{(0)}, \mathbf{w})$ with $\lambda = \mathbf{w}$, $\mu' = \mathbf{w} - \mathbf{C}\mathbf{v}^{(0)}$ in the notation in [Nak94]. For example, $\mu' = \lambda$ is the closed stratum and the maximum $\mu' \le \lambda$ is the open stratum.

The stratum (resp. the transversal slice) is the Higgs branch of the quiver gauge theory of the same type with the dimension vectors given by (μ', λ) (resp. (μ, μ')).

Comparing two stratifications, we find that the answer to Question 2.1 is yes. The bijection is simply given by $\lambda' = \mu'$.

There should be a similar stratification for quiver gauge theories of affine types, where $\mathcal{M}_C$ is a moduli space of $\mathbb{Z}/\ell\mathbb{Z}$-equivariant instantons on the Taub-NUT space, where ℓ is the level of $\dim W$. But the author is not familiar enough with such a moduli space, and in particular, it is not clear whether a puzzle on the affine Dynkin diagram automorphism group raised in [Nak15, 3(ii)] is clarified or not. When μ is dominant and Q is of type A, we can consider equivariant instantons on $\mathbb{R}^4$ instead. Then we get quiver varieties of affine type A. Then answers to Question 2.1(1),(2) are yes, while we have the same phenomenon as in $W = 0$ case for (3).

Let us turn to Question 3.2. The answer is yes thanks to

Proposition 5.9. *Suppose Q is finite or affine type.*

(1) *If* $\operatorname{Hyp}(\mathbf{M}) /\!\!/\!\!/ G$ *is good or ugly,* $\boldsymbol{\mu}^{-1}(0)$ *is complete intersection of dimension* $\dim \mathbf{M} - \dim G$.

(2) *If* $\operatorname{Hyp}(\mathbf{M}) /\!\!/\!\!/ G$ *is good,* $\mathbf{w} - \mathbf{C}\mathbf{v}$ *is dominant.*

Proof. Crawley-Boevey's criterion of the complete intersection property of $\boldsymbol{\mu}^{-1}(0)$ can be modified by the trick adding a new vertex ∞. Then instead of (5.2), we have

$$\langle \mathbf{v}, 2\mathbf{w} - \mathbf{C}\mathbf{v}\rangle \ge \langle \mathbf{v}^{(0)}, 2\mathbf{w} - \mathbf{C}\mathbf{v}^{(0)}\rangle + \sum_k (2 - \langle \beta^{(k)}, \mathbf{C}\beta^{(k)}\rangle) \tag{5.10}$$

for any decomposition $\mathbf{v} = \mathbf{v}^{(0)} + \sum_k \beta^{(k)}$ into positive vectors such that $\beta^{(k)}$ is nonzero. ($\dim V^{(0)}$ could be zero.) (See [Nak09, Th. 2.15] for a similar deduction.) It is also equivalent to the inequality holds for $\mathbf{v} = \mathbf{v}^{(0)} + \sum_k \beta^{(k)}$ such that $\mathbf{w} - \mathbf{C}\mathbf{v}^{(0)}$ is a weight of the highest weight representation $V(\mathbf{w})$ with highest weight $\mathbf{w}$, and $\beta^{(k)}$ is a positive root.

If Q is of finite type ADE, $\langle \beta^{(k)}, \mathbf{C}\beta^{(k)}\rangle = 2$ for any k. Therefore (5.10) is equivalent to

$$\langle \beta, \mathbf{w} - \mathbf{C}\mathbf{v}\rangle \ge -\frac{1}{2}\langle \beta, \mathbf{C}\beta\rangle \tag{5.11}$$

for any $\beta \le \mathbf{v}$ such that $\mathbf{w} - \mathbf{C}(\mathbf{v} - \beta)$ is a weight of $V(\mathbf{w})$.

Taking a positive root β, we have $\langle \beta, \mathbf{w} - \mathbf{C}\mathbf{v}\rangle \ge -1$. Next suppose $\beta = \beta^{(1)} + \beta^{(2)} + \cdots$ such that $\beta^{(k)}$ is a positive root and $\beta^{(k)} + \beta^{(l)}$ is not a root. Then $\langle \beta^{(k)}, \mathbf{C}\beta^{(l)}\rangle \ge 0$ as in §5(i). Therefore $\langle \beta, \mathbf{C}\beta\rangle \ge 2\#\{\beta^{(k)}\}$. Therefore if $\langle \beta^{(k)}, \mathbf{w} - \mathbf{C}\mathbf{v}\rangle \ge -1$ for any k, we have $\langle \beta, \mathbf{w} - \mathbf{C}\mathbf{v}\rangle \ge -\#\{\beta^{(k)}\} \ge -\frac{1}{2}\langle \beta, \mathbf{C}\beta\rangle$. Hence it is enough to suppose (5.11) is true for an arbitrary positive root β.

Suppose Q is of affine type. If $\beta^{(k)} = \delta$ in (5.10), we absorb it into $\mathbf{v}^{(0)}$. Since $\mathbf{C}\delta = 0$, the first term of the right hand side of (5.10) increases $2\langle \delta, \mathbf{w}\rangle$, which is ≥ 2. On the other hand, the second term decreases by 2. Therefore it is enough to assume (5.10) when all $\beta^{(k)}$ is a real root. Then the same argument as above shows that it is enough to assume $\langle \beta, \mathbf{w} - \mathbf{C}\mathbf{v}\rangle \ge -1$ for an arbitrary positive real root β.

On the other hand, let us take $\lambda\colon \mathbb{C}^\times \to T = \prod_i T(V_i)$ as in (5.3). Then if $\lambda(t)$ acts on $t^{\lambda^{(k)}}$ on $V^{(k)}$,

$$
\begin{aligned}
2\Delta(\lambda) &= \sum_{k<l} |\lambda^{(k)} - \lambda^{(l)}| \left(\sum_i -2\beta_i^{(k)}\beta_i^{(l)} + \sum_{i,j} a_{ij}\beta_i^{(k)}\beta_j^{(l)} \right) \\
&\qquad + \sum_k |\lambda^{(k)}| \sum_i \beta_i^{(k)} \dim W_i \\
&= -\sum_{k<l} |\lambda^{(k)} - \lambda^{(l)}| \langle \beta^{(k)}, \mathbf{C}\beta^{(l)}\rangle + \sum_k |\lambda^{(k)}| \langle \beta^{(k)}, \mathbf{w}\rangle
\end{aligned} \tag{5.12}
$$

with $\beta^{(k)} = \dim V^{(k)}$. We take $\beta^{(1)} = \beta$ a positive real root, and $\beta^{(2)} = \mathbf{v} - \beta$. Furthermore assume $\lambda^{(1)} \ge \lambda^{(2)} \ge 0$. Then

$$2\Delta(\lambda) = \lambda^{(1)}\langle \beta, \mathbf{w} - \mathbf{C}(\mathbf{v} - \beta)\rangle + \lambda^{(2)}\langle \mathbf{w} + \mathbf{C}\beta, \mathbf{v} - \beta\rangle.$$

For good or ugly cases, as $2\Delta(\lambda) \ge 1$ for any λ, we have $\langle \beta, \mathbf{w} - \mathbf{C}(\mathbf{v} - \beta)\rangle \ge 1$, i.e., $\langle \beta, \mathbf{w} - \mathbf{C}\mathbf{v}\rangle \ge -1$. For good cases, $\langle \beta, \mathbf{w} - \mathbf{C}\mathbf{v}\rangle \ge 0$. □

The converses of (1),(2) are probably true.

Suppose $\mathrm{Hyp}(\mathbf{M}) /\!\!/\!\!/ G$ is good. By Proposition 5.9(2), $\mathbf{w} - \mathbf{C}\mathbf{v}$ is dominant, hence $\mathcal{M}_C$ is expected to be a slice in the affine Grassmannian when Q is finite type, as we explained in the beginning of this subsection. When Q is affine type, we still need to solve a puzzle in [Nak15, 3(ii)], but is the Uhlenbeck partial compactification of an instanton moduli space on $\mathbb{R}^4/(\mathbb{Z}/\ell\mathbb{Z})$ as a first approximation. Then $IH^*(\mathcal{M}_C)$ is a weight space of a finite dimensional irreducible representation of the Lie algebra $\mathfrak{g}$ corresponding to Q by geometric Satake correspondence. This is so when Q is of finite type. If Q is affine type, this is the statement of a conjecture in [BF10], geometric Satake correspondence for the affine Lie algebra $\mathfrak{g}_{\mathrm{aff}}$.

On the other hand, $\mathcal{M}_H$ is a quiver variety. In particular, $\mathcal{M}_H$ has a symplectic resolution $\tilde{\mathcal{M}}_H \to \mathcal{M}_H$. It is conjectured that $HP_0(\mathcal{M}_H) \cong H^{\dim \tilde{\mathcal{M}}_H}(\tilde{\mathcal{M}}_H)$

in [ES14]. The right hand side is a weight space of a finite dimensional or integrable irreducible representation of $\mathfrak{g}$ or $\mathfrak{g}_{\mathrm{aff}}$ by [Nak94]. Therefore, modulo a conjecture in [ES14], we have $HP_0(\mathcal{M}_H) \cong IH^*(\mathcal{M}_C)$, the first isomorphism in Question 4.1. We also expect $HP_0(\mathcal{M}_C) \cong IH^*(\mathcal{M}_H)$, where the right hand side is the multiplicity of the finite dimensional (resp. integrable) irreducible representation $L(\mathbf{w} - \mathbf{C}\mathbf{v})$ of $\mathfrak{g}$ (resp. $\mathfrak{g}_{\mathrm{aff}}$) in the (resp. affine) Yangian $Y(\mathfrak{g})$ (resp. $Y(\mathfrak{g}_{\mathrm{aff}})$) [Nak01, §15]. It is interesting to study $HP_0(\mathcal{M}_C)$.

5(iv). SU(2) **gauge theories with fundamental matters.** Consider $(G, \mathbf{N}) = (\mathrm{SL}(2), (\mathbb{C}^2)^{\oplus N})$ with $\mathbf{M} = \mathbf{N} \oplus \mathbf{N}^*$ for $N = 0, 1, 2, \ldots$. In this case, the Coulomb branch is a complex surface

$$y^2 = x^2 z - z^{N-1} \quad \text{if } N \geq 1, \quad y^2 = x^2 z + x \quad \text{if } N = 0. \tag{5.13}$$

See [SW97]. This is of cotangent type, and the definition in [BFN15] reproduces the above at least for $N \neq 1, 2, 3$. The degrees for the $\mathbb{C}^\times$-action are $\deg x = N-2$, $\deg y = N - 1$, $\deg z = 2$.

Let us study the Higgs branch $\mathbf{M}/\!\!/\!\!/G$. We use the standard notation for quiver varieties: $i\colon \mathbb{C}^N \to \mathbb{C}^2$, $j\colon \mathbb{C}^2 \to \mathbb{C}^N$ with the SL(2)-action by $g\cdot(i,j) = (gi, jg^{-1})$. The moment map $\boldsymbol{\mu}(i,j)$ is the trace-free part of ij. If $N = 0$, we have $\mathbf{M}/\!\!/\!\!/G = \{0\}$ by a trivial reason. If $N = 1$, it is not trivial, but not difficult to check the following:

Lemma 5.14. *Suppose* $N = 1$.

Then $\boldsymbol{\mu} = 0$ *implies either* $i = 0$ *or* $j = 0$. *In particular, the only closed* SL(2)*-orbit in* $\boldsymbol{\mu}^{-1}(0)$ *is just* 0. *Therefore* $\mathbf{M}/\!\!/\!\!/G = \{0\}$.

Therefore $\boldsymbol{\mu}^{-1}(0)$ is not complete intersection of dim $= \dim \mathbf{M} - \dim G = 1$. The degree of x is -1, hence $\mathcal{M}_C$ is not conical.

On the other hand, since the Higgs branch has only single point, $\mathcal{M}_C(G, \mathbf{N})$ should have only one stratum, i.e., it is a nonsingular symplectic manifold. It is not difficult to check that (5.13) is indeed so when $N = 0, 1$. Therefore Question 2.2 is affirmative.

Next consider the case $N \geq 2$.

Proposition 5.15. *Assume* $N \geq 2$.

(1) $\boldsymbol{\mu}^{-1}(0)$ *is a complete intersection in* $\mathbf{M}$ *of* dim $= \dim \mathbf{M} - 3 = 4N - 3$. *It is irreducible if* $N \geq 3$ *and has two irreducible components if* $N = 2$.

(2) $\mathbf{M}/\!\!/\!\!/G = \boldsymbol{\mu}^{-1}(0)/\!\!/G$ *has singularity only at* 0. $\mathbf{M}/\!\!/\!\!/G \setminus \{0\}$ *is irreducible if* $N > 2$ *and has two irreducible components if* $N = 2$.

Since the gauge theory $\mathrm{Hyp}(\mathbf{M}) /\!\!/\!\!/\!\!/ G$ is good or ugly if and only if $N \geq 3$, so the answer to Question 3.2 is yes, but $N = 2$ is a counter-example to its converse.

Proof. (1) From $\boldsymbol{\mu}(i,j) = 0$, we have $ij = \zeta\,\mathrm{id}$ for some $\zeta \in \mathbb{C}$. Suppose $\zeta \neq 0$. A standard argument shows that the stabilizer is trivial. Therefore the differential of $\boldsymbol{\mu}$ is surjective, hence $\boldsymbol{\mu}^{-1}(0)$ is a complete intersection. If we take the quotient of $ij = \zeta\,\mathrm{id}$ by $\mathrm{GL}(2)$, it is a quiver variety $\mathfrak{M}_\zeta$ and is smooth and irreducible and forms a smooth family over $\zeta \neq 0$. (In fact, it is a semisimple coadjoint orbit in $\mathfrak{gl}(N)$ with eigenvalues λ with multiplicity 2 and 0 with multiplicity $N-2$.)

If i is surjective, we can form the (GIT) quotient even across $\zeta = 0$ and get a smooth family over $\zeta \in \mathbb{C}$. The same is true if j is injective. Therefore on the open subset either of i or j is rank 2, $\boldsymbol{\mu}^{-1}(0)$ is a smooth irreducible variety of $\dim = 4N-3$.

Therefore consider the case when both i, j have rank ≤ 1. This can happen only when $\zeta = 0$. Let us write

$$i = \begin{pmatrix} i_{11} & i_{12} & \cdots & i_{1N} \\ i_{21} & i_{22} & \cdots & i_{2N} \end{pmatrix}, \quad j = \begin{pmatrix} j_{11} & j_{12} \\ \vdots & \vdots \\ j_{N1} & i_{N2} \end{pmatrix}.$$

Since i is rank 1, the upper row and the lower row are the same up to constant multiple. The same is true for columns of j. Then the equation $ij = 0$ is a single scalar equation. Therefore it forms an irreducible variety of dimension $2N+1$. Since $2N+1 \leq 4N-3$ and the equality holds if and only if $N=2$, the assertion follows.

(2) We suppose $(i,j) \in \boldsymbol{\mu}^{-1}(0)$ corresponds to a singular point. We assume that it has a closed $\mathrm{SL}(2)$-orbit. By the above argument, we may suppose both i, j have rank 1. If $\operatorname{Im} i \cap \operatorname{Ker} j = \{0\}$, the stabilizer is trivial. Therefore it gives a smooth point in $\mathbf{M}/\!\!/\!\!/G$. If $\operatorname{Im} i = \operatorname{Ker} j$, we can find a one-parameter subgroup $\lambda\colon \mathbb{C}^\times \to \mathrm{SL}(2)$ such that $\lambda \cdot (i,j) \to (0,0)$. Therefore it cannot have a closed orbit.

There are indeed points $\operatorname{Im} i \cap \operatorname{Ker} j = \{0\}$ in $\boldsymbol{\mu}^{-1}(0)$. They form a smooth variety of dimension $2N-2$. When $N=2$, it gives the second irreducible component. $\square$

This example shows that a stratum $(\mathbf{M}/\!\!/\!\!/G)_{(\widehat{G})}$ may not be connected in general. In fact, we have $\widehat{G} = \{e\}$ in the above example, and $(\mathbf{M}/\!\!/\!\!/G)_{(\widehat{G})}$ is an open stratum and has two connected components. Therefore $\mathbf{M}/\!\!/\!\!/G$ actually has three strata. It is interesting to note that (5.13) with $N=2$ also has three strata, a smooth locus and two singular points $x = \pm 1$, $y = z = 0$. Thus answers to Question 2.1(1) and the bijection part of (3) are yes even in this case.

(2) and the second half of (3) are *not* clear as stated: The transversal slice $T^\perp/\widehat{G}$, for $\mathcal{M}_H$ at points in either of two components of $(\mathbf{M}/\!\!/\!\!/G)_{\{e\}}$, is $\{0\}/\{e\}$. It is the Higgs branch of the trivial gauge theory $\mathrm{Hyp}(\{0\})/\!\!/\!\!/\!\!/\{e\}$. The corresponding Coulomb branch is just a single point, and strata for two singular points may be identified with this Coulomb branch. However it is not clear (at least to the author) whether we can naturally view each component of $(\mathbf{M}/\!\!/\!\!/G)_{\{e\}}$ is the Higgs branch of a gauge theory $\mathrm{Hyp}(\mathbf{M}')/\!\!/\!\!/\!\!/G'$ for some $(G', \mathbf{M}')$. Say, are $(G', \mathbf{M}')$ different for two components ? Similarly it is not clear whether the transversal slices to singular points $x = \pm 1$, $y = z = 0$ in $\mathcal{M}_C$ (both A_1 type) can be

naturally identified with the Coulomb branch of a gauge theory $\mathrm{Hyp}(\mathbf{M}') /\!\!/\!\!/ G'$. It is desirable to understand this phenomenon better.

For Question 4.1, $HP_0(\mathcal{M}_C)$, $IH^*(\mathcal{M}_C)$ are easy to compute as $\mathcal{M}_C$ is of type D_N singularity. We do not know $IH^*(\mathcal{M}_H)$, $HP_0(\mathcal{M}_H)$.

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