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Analytic approaches to the Möbius energy: History and recent topics

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1 Introduction

We consider the Möbius energy

$$\mathcal{E}(\boldsymbol{f})=\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^{2}}\left(\frac{1}{\|\boldsymbol{f}(s_{1})-\boldsymbol{f}(s_{2})\|_{\mathbb{R}^{n}}^{2}}-\frac{1}{\mathscr{D}(\boldsymbol{f}(s_{1}),\boldsymbol{f}(s_{2}))^{2}}\right)ds_{1}ds_{2}$$

defined for a closed curve $\boldsymbol{f}:\mathbb{R}/\mathcal{L}\mathbb{Z}\rightarrow\mathbb{R}^{n}$. Here $\mathcal{L}$ is the total length of the closed curve, s_{i}'s are arc-length parameters, and $\mathscr{D}$ is the intrinsic distance (*i.e.*, the distance along the curve) between two points $\boldsymbol{f}(s_{1})$ and $\boldsymbol{f}(s_{2})$. This energy was originally proposed by O'Hara [15] in 1991 for $n=3$ as one of energies of knots. Indeed he introduced the energies

$$\mathcal{E}_{(\alpha,p)}(\boldsymbol{f})=\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^{2}}\left(\frac{1}{\|\boldsymbol{f}(s_{1})-\boldsymbol{f}(s_{2})\|_{\mathbb{R}^{n}}^{\alpha}}-\frac{1}{\mathscr{D}(\boldsymbol{f}(s_{1}),\boldsymbol{f}(s_{2}))^{\alpha}}\right)^{p}ds_{1}ds_{2},$$

which are called *O'Hara's* energy. The density contains the negative power of "distance", which implies that a minimizer, if exists, is the "canonical configuration" of knots among the given knot type, even though it makes the analysis hard.

It is easy to see that $\mathcal{E}_{(\alpha,p)}$ is scale-invariant if $\alpha p=2$, including our energy $\mathcal{E}=\mathcal{E}_{(2,1)}$. In mid-1990's, Freedman-He-Wang [7] showed that $\mathcal{E}$ has the invariance not only under scaling but also under Möbius transformations. Since then, it has been called the *Möbius energy.*

In this article the author firstly surveys fundamental results on the Möbius energy:

- the existence of minimizers in the class of prime knots ([7]),
- the Kusner-Sullivan conjecture ([14]),

- the bi-Lipschitz continuity ([17, 2]),
- the regularity of critical points ([7, 8, 19, 6]),
- the gradient flow ([8, 3, 4, 5])

without proofs.

Secondary the author presents recent progress of $\mathcal{E}$, which is based on a series of papers [10]–[13] with Ishizeki. Blatt [2] found the proper domain of the energy: $\mathcal{E}(\boldsymbol{f}) < \infty$ if and only if $\boldsymbol{f}$ is bi-Lipschitz and belongs to the fractional Sobolev space $H^{3/2}(\mathbb{R}/\mathcal{L}\mathbb{Z}) \cap W^{1,\infty}(\mathbb{R}/\mathcal{L}\mathbb{Z})$. Consequently we may assume the existence of the unit tangent vector $\boldsymbol{\tau}(s) = \boldsymbol{f}'(s)$ almost everywhere. By use of the unit tangent vector field along the curve, the energy may be decomposed into three parts:

$$\mathcal{E}(\boldsymbol{f}) = \mathcal{E}_1(\boldsymbol{f}) + \mathcal{E}_2(\boldsymbol{f}) + 4,$$

where

$$\begin{aligned}
\mathcal{E}_i(\boldsymbol{f}) &= \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \mathscr{M}_i(\boldsymbol{f})\, ds_1 ds_2, \\
\mathscr{M}_1(\boldsymbol{f}) &= \frac{\|\boldsymbol{\tau}(s_1) - \boldsymbol{\tau}(s_2)\|_{\mathbb{R}^n}^2}{2\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2}, \\
\mathscr{M}_2(\boldsymbol{f}) &= \frac{2}{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^4} \\
&\quad \times \det \begin{pmatrix} \boldsymbol{\tau}(s_1)\cdot\boldsymbol{\tau}(s_2) & (\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2))\cdot\boldsymbol{\tau}(s_1) \\ (\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2))\cdot\boldsymbol{\tau}(s_2) & \|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2 \end{pmatrix}.
\end{aligned}$$

This was recently shown by our research group [10]. The first decomposed energy $\mathcal{E}_1$ is an analogue of the Gagliardo semi-norm of $\boldsymbol{\tau}$ in the fractional Sobolev space $H^{1/2}(\mathbb{R}/\mathcal{L}\mathbb{Z})$. This implies the domain of $\mathcal{E}$ is $H^{3/2}(\mathbb{R}/\mathcal{L}\mathbb{Z}) \cap W^{1,\infty}(\mathbb{R}/\mathcal{L}\mathbb{Z})$, as shown by Blatt. The integrand $\mathscr{M}_2$ of second one has the determinant structure, which implies a cancellation of integrand. Since the last part "4" is an absolute constant, we can ignore it when considering variational problems. This fact shortens the derivation of variational formulae, and enables us to find their "good" estimates in several functional spaces [11]. Furthermore we find the L^2-gradient of each decomposed energy which contains the fractional Laplacian $(-\Delta_s)^{3/2}$ as the principal term [13].

In the final section we focus on the Möbius invariance of the decomposed energy $\mathcal{E}_i$ studied in [10, 12], in particular the invariance with respect to a sphere whose center is on the image of $\boldsymbol{f}$. This case was discussed in [12] but not in [10]. As a consequence we can show that right circles are only minimizers of $\mathcal{E}_1$ in the class $C^{1,1}$ with the minimum value $2\pi^2$. This seems to be related to the fact that the first eigenvalue of $(-\Delta_s)^{3/2}$ is $\left(\dfrac{2\pi}{\mathcal{L}}\right)^3$.

2 The existence of minimizers

We begin with an exercise. Let S^1 be a right circle (*i.e.*, a circle with a constant curvature) whose center is at origin, radius is r immersed in the x_1x_2-plane:

$$\boldsymbol{f}(s) = r\left(\cos\frac{s}{r}, \sin\frac{s}{r}, 0, \cdots, 0\right).$$

Since

$$\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_1)\|_{\mathbb{R}^n}^2 = 2r^2\left(1 - \cos\frac{s_1 - s_2}{r}\right),$$

and since

$$\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))^2 = (s_1 - s_2)^2 \quad \text{when} |s_1 - s_2| \leqq \pi r,$$

it is not difficult to see

$$\mathcal{E}(\boldsymbol{f}) = 4.$$

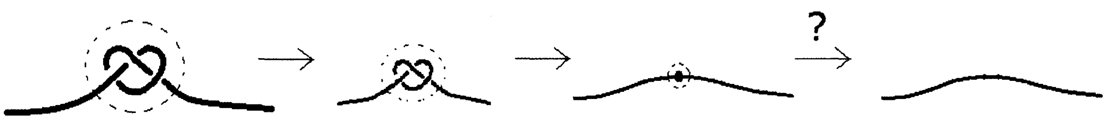
Freedman-He-Wang [7] showed the following.

Theorem 2.1 **([7])** *It holds that $\mathcal{E}(\boldsymbol{f}) \geqq 4$. The equality holds if and only of $\boldsymbol{f}$ is a right circle.*

Several proofs of the above theorem are known. We will revisit it in § 7.

Remark 2.1 Abrams-Cantarella-Fu-Ghomi-Howard [1] generalized this fact. Let $\alpha \geqq 1$, $p \in (0, 2 + \frac{1}{\alpha})$. Then right circles are the only global minimizer of $\mathcal{E}_{(\alpha,p)}$.

Now consider the minimizing problem in each knot type. Since the energy $\mathcal{E}$ is non-negative, there exists a minimizing sequence. Its convergence, however, is not trivial. The scaling-invariance of energy $\mathcal{E}$ might allows the pull-tight phenomena along the sequence.



pull-tight

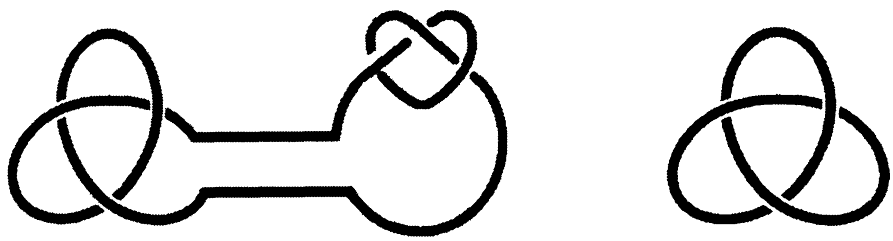
If such a phenomena occurs, the limit knot, if it exists, is not in the same knot type as sequence. The energy behavior along the pull-tight for $\mathcal{E}_{(\alpha,p)}$ is studied by O'Hara [16].

Theorem 2.2 **([16])** *Let a knot K_ε be a connected sum of K and a small tangle T_ε. The difference of energy $D(\varepsilon) = \mathcal{E}_{(\alpha,p)}(K_\varepsilon) - \mathcal{E}_{(\alpha,p)}(K)$ behaves as follows in a pull-tight process $T_\varepsilon \to \{a\ point\}$:*

- *$D(\varepsilon)$ blows up when $\alpha p > 2$.*
- *$D(\varepsilon)$ converges a positive constant when $\alpha p = 2$.*
- *$D(\varepsilon)$ vanishes $\alpha p < 2$.*

We call the cases $\alpha p > 2$, $= 2$, and < 2 respectively *subcritical*, *critical*, and *supercritical*. A pull-tight means the disappearance of a tangle. The above results implies that a pull-tight may happen in critical and supercritical cases. This shows that the argument of minimizing sequence fails in these cases. Nevertheless the argument works for the Möbius energy in *prime* knot type. This remarkable result was proven by Freedman-He-Wang [7].

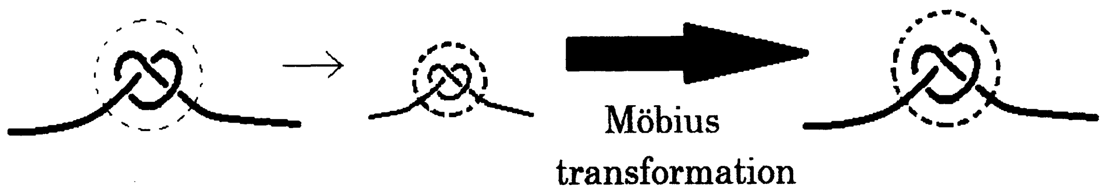
Definiton 2.1 Let $n = 3$. A knot is a *composite knot* if it is a connected sum of two non-trivial knots. A *prime knot* is a knot which is not composite.



A composite knot A prime knot

Theorem 2.3 ([7]) *There exists a minimizer of $\mathcal{E}$ on each* prime *knot type.*

The key is how to avoid the pull-tight along the minimizing sequence. If the knot is prime, we can enlarge the tangle by the inversion with respect to a sphere near the shrinking tangle without changing energy level.



Freedman-He-Wang's procedure

They passed the limit of minimizing sequence together with, if necessary, the enlarging tangle, and showed the limit knot is a minimizer in the given knot type. If the knot is composite, two tangles might shrink simultaneously. Hence such a method does not work. See next section.

The argument of minimizing sequence works in the subcritical case. Indeed O'Hara [17] showed the following result.

Theorem 2.4 ([17]) *Let $n = 3$. There exists a minimizer (under rescaling) for any knot types if and only if $\alpha p > 2$.*

3 The Kusner-Sullivan conjecture

Let $n = 3$, and $[K]$ be a knot type. We denote

$$\mathcal{E}([K]) = \inf_{f \in [K]} \mathcal{E}(\boldsymbol{f}).$$

Note that $\mathcal{E}([K])$ exists, since the energy density is non-negative.

Kusner and Sullivan [14] investigated the energy $\mathcal{E}$ for various knots numerically, and proposed the following conjecture.

Conjecture 3.1 (The Kusner-Sullivan conjecture [14])

1. *There does not exist minimizers of composite knot type.*
2. *Assume that $\boldsymbol{f} \in [K]$ is composite, and it is a connected sum $\boldsymbol{f}_1 \sharp \boldsymbol{f}_2$, $\boldsymbol{f}_i \in [K_i]$. It holds that*
$$\mathcal{E}([K]) = \mathcal{E}([K_1]) + \mathcal{E}([K_2]).$$

As far as the author knows, this is still open.

4 The bi-Lipschitz continuity

Since our energy was introduced to determine the "canonical" configuration of knots, we expect that the finiteness of energy suggests some regularity of curve. Indeed we have the bi-Lipschitz estimate fot the curve with finite energy, which means that the curve cannot bend rapidly.

Since we use the arc-length parameter,

$$\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n} \leqq \mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))$$

is trivial. *That is,* $\boldsymbol{f}$ is Lipschitz continuous with the Lipschitz constant 1. The finiteness of $\mathcal{E}$ implies the converse estimate.

Theorem 4.1 ([17, 2]) *If $\mathcal{E}(\boldsymbol{f}) < M$, then there exists $\lambda = \lambda(M) > 0$ such that*

$$\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n} \geqq \lambda \mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2)).$$

We call these estimates the *bi-Lipschitz continuity.* In what follows we assume that $\boldsymbol{f}$ is bi-Lipschitz.

5 The regularity of critical points

The bi-Lipschitz continuity holds for all curves with finite energy. The criticality of energy derives more informations about regularity. Several results are known. Freedman-He-Wang [7] and He [8] showed the regularity of local minimizers.

Theorem 5.1 ([7, 8]) *Local minimizers of $\mathcal{E}$ with respect to $L^\infty(\mathbb{R}/\mathcal{L}\mathbb{Z})$ topology are smooth.*

Reiter [19] proved the regularity not only for local minimizers but for critical points.

Theorem 5.2 ([19]) *Any critical points of $\mathcal{E}$ in $W^{2,2}(\mathbb{R}/\mathcal{L}\mathbb{Z})$ are smooth.*

Recently Blatt-Reiter-Shikorra [6] improved the assumption of the previous result.

Theorem 5.3 ([6]) *Any critical points of $\mathcal{E}$ with finite energy are smooth.*

The finiteness of energy implies not only the bi-Lipschitz continuity but also the integrability of (fractional) derivatives.

Definiton 5.1 (Sobolev-Slobodeckij space) *For $j \in \mathbb{N} \cup \{0\}$, and $\alpha \in (0,1)$, $W^{j+\alpha,p}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$ is defined as*

$$W^{j+\alpha,p}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n) = \{\boldsymbol{f} \in W^{j,p}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n) \mid [\boldsymbol{f}^{(j)}]_{\alpha,p} < \infty\},$$

$$[\boldsymbol{f}^{(j)}]_{\alpha,p} = \left(\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{-\frac{\mathcal{L}}{2}}^{\frac{\mathcal{L}}{2}} \frac{\|\boldsymbol{f}^{(j)}(s_1+s_2) - \boldsymbol{f}^{(j)}(s_1)\|_{\mathbb{R}^n}^p}{|s_2|^{\alpha p+1}} ds_2 ds_1 \right)^{\frac{1}{p}}$$

with the norm

$$\|\boldsymbol{f}\|_{W^{j+\alpha,p}} = \|\boldsymbol{f}\|_{W^{j,p}} + [\boldsymbol{f}^{(j)}]_{\alpha,p}.$$

When $p = 2$, we denote $W^{j+\alpha,2}(\mathbb{R}/\mathcal{L}\mathbb{Z})$ by $H^{j+\alpha}(\mathbb{R}/\mathcal{L}\mathbb{Z})$.

The following result is due to Blatt [2].

Theorem 5.4 ([2]) *The finiteness $\mathcal{E}(\boldsymbol{f}) < \infty$ implies the bi-Lipschitz continuity of $\boldsymbol{f}$ and $\boldsymbol{f} \in H^{3/2}(\mathbb{R}/\mathcal{L}\mathbb{Z}) \cap W^{1,\infty}(\mathbb{R}/\mathcal{L}\mathbb{Z})$. The converse is also true,* i.e., *If $\boldsymbol{f}$ is bi-Lipschitz and belongs to $H^{3/2}(\mathbb{R}/\mathcal{L}\mathbb{Z}) \cap W^{1,\infty}(\mathbb{R}/\mathcal{L}\mathbb{Z})$, then $\mathcal{E}(\boldsymbol{f})$ is finite.*

This shows the proper domain of $\mathcal{E}$.

Remark 5.1 In fact Blatt [2] considered the general cases. Let $(\alpha, p) \in (0,\infty)^2$ satisfy $\alpha p \geqq 2$, $\alpha \geqq 1$, $(\alpha - 2)p < 1$. Then $\mathcal{E}_{(\alpha,p)}(\boldsymbol{f})$ is finite if and only if $\boldsymbol{f} \in W^{\frac{(2+p)\alpha-1}{2\alpha}, 2\alpha}(\mathbb{R}/\mathcal{L}\mathbb{Z}) \cap W^{1,\infty}(\mathbb{R}/\mathcal{L}\mathbb{Z})$ and bi-Lipschitz continuous.

6 The gradient flow

Let $\delta_{L^2}\mathcal{E}$ be the L^2-gradient:

$$\langle \delta_{L^2}\mathcal{E}(\boldsymbol{f}), \boldsymbol{\phi}\rangle_{L^2(\mathbb{R}/\mathcal{L}\mathbb{Z})} = \frac{d}{d\varepsilon}\mathcal{E}(\boldsymbol{f}+\varepsilon\boldsymbol{\phi})\bigg|_{\varepsilon=0}.$$

For the explicit formula of the L^2-gradient, see [13]. Consider the L^2-gradient flow:

$$\partial_t \boldsymbol{f} = -\delta_{L^2}\mathcal{E}(\boldsymbol{f}).$$

The local existence and uniqueness was shown by He [8] for smooth initial data, and improved by Blatt [3] for the initial curve in the little Hölder space.

Theorem 6.1 **([8])** *Let the knot $\boldsymbol{f}_0$ be smooth. The there exists a unique local solution to the L^2-gradient flow with $\boldsymbol{f}(0) = \boldsymbol{f}_0$*

Theorem 6.2 **([3])** *Let $\boldsymbol{f}_0 \in h^{2+\alpha}(\mathbb{R}/\mathcal{L}\mathbb{Z})$, where $h^{2+\alpha}(\mathbb{R}/\mathcal{L}\mathbb{Z})$ is the little Hölder space of order $2+\alpha$. Then there exists a unique local solution to the L^2-gradient flow with $\boldsymbol{f}(0) = \boldsymbol{f}_0$.*

Blatt [3] also showed the global existence near local minimizers.

Theorem 6.3 **([3])** *Let $\boldsymbol{f}_*$ be a local minimizer of $\mathcal{E}$ in $C^k(\mathbb{R}/\mathcal{L}\mathbb{Z})$ for some $k \in \{0\}\cup\mathbb{N}$. Assume $\|\boldsymbol{f}(0)-\boldsymbol{f}_*\|_{C^{2+\beta}} \ll 1$. Then the L^2-gradient flow with the initial knot $\boldsymbol{f}(0)$ exists globally in time. $\boldsymbol{f}(t)$ has the limit $\lim_{t\to\infty}\boldsymbol{f}(t) = \boldsymbol{f}_\infty$, and $\boldsymbol{f}_\infty$ is a critical point satisfying $\mathcal{E}(\boldsymbol{f}_*) = \mathcal{E}(\boldsymbol{f}_\infty)$.*

Blatt showed the following *Lojasiewicz-Simon gradient estimate* in his papar. Let $\boldsymbol{f}_*$ be a critical point. Then there exist $\theta \in [0, \frac{1}{2}]$, $\sigma > 0$, $c > 0$ such that $\|\boldsymbol{f} - \boldsymbol{f}_*\|_{H^3} \leqq \sigma$ implies

$$|\mathcal{E}(\boldsymbol{f}) - \mathcal{E}(\boldsymbol{f}_*)|^{1-\theta} \leqq c\|\delta_{L^2}\mathcal{E}(\boldsymbol{f})\|_{L^2}$$

The assertion of global existence follows from Łojasiewicz's argument. It is still open that the limit curve $\boldsymbol{f}_\infty$ is the image of some Möbius transformation of $\boldsymbol{f}_*$ or not.

For the case of $n = 2$, then the L^2-gradient flow exists globally in time, and converges to a right circle, *i.e.*, to a global minimizer.

Theorem 6.4 **([4])** *Let $\boldsymbol{f}(0)$ be a planar curve. Then there exists a global solution to the L^2-gradient flow with the initial knot $\boldsymbol{f}(0)$ such that*

$$\boldsymbol{f}(t) \to S^1 \textit{(a right circle) as } t \to \infty.$$

Blatt kindly informed the author his result of gradient flow for the subcritical case.

Theorem 6.5 ([5]) *For $\mathcal{E}_{(1,p)}$ with $p > 2$ it holds that*

- *there exists a global solution of length-constraint-gradient flow for any smooth initial knots,*
- *the flow converges to a critical point.*

7 The decomposition of the Möbius energy and its consequences

Let $\mathscr{F}(\boldsymbol{f})$ be a geometric quantity determined by the closed curve $\boldsymbol{f}$, and let $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ be functions from $\mathbb{R}/\mathcal{L}\mathbb{Z}$ to $\mathbb{R}^n$. We use δ and δ^2 to mean

$$\delta\mathscr{F}(\boldsymbol{f})[\boldsymbol{\phi}] = \frac{d}{d\varepsilon}\mathscr{F}(\boldsymbol{f}+\varepsilon\boldsymbol{\phi})\Big|_{\varepsilon=0},$$

$$\delta^2\mathscr{F}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}] = \frac{d^2}{d\varepsilon_1 d\varepsilon_2}\mathscr{F}(\boldsymbol{f}+\varepsilon_1\boldsymbol{\phi}+\varepsilon_2\boldsymbol{\psi})\Big|_{\varepsilon_1=\varepsilon_2=0}.$$

Calculating the first and second variational formulae formally, then we have

$$\delta\mathcal{E}(\boldsymbol{f})[\boldsymbol{\phi}] = \iint\left(\frac{***}{\|\boldsymbol{f}(s_1)-\boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^4}+\cdots\right)ds_1ds_2,$$

$$\delta^2\mathcal{E}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}] = \iint\left(\frac{***}{\|\boldsymbol{f}(s_1)-\boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^8}+\cdots\right)ds_1ds_2.$$

These contain a lot of terms which are not integrable. He [8] obtained these formulae as integration of Cauchy's principal value:

$$\lim_{\varepsilon\to+0}\iint_{|s_1-s_2|>\varepsilon}\cdots\, ds_1ds_2.$$

Ishizeki [9] studies the absolute integrability under the regular class $C^{3+\alpha}$.

Theorem 7.1 ([9]) *If $\boldsymbol{f} \in C^{3+\alpha}(\mathbb{R}/\mathcal{L}\mathbb{Z})$, then the above integrands are absolutely integrable.*

Her result was improved for the proper domain by using the following decomposition theorem. The Lipshitz continuity of $\boldsymbol{f}$ implies the existence of unit tangent vector $\boldsymbol{\tau}$ almost everywhere by Rademacher's theorem.

Theorem 7.2 ([10]) *Let $\boldsymbol{\tau}(s) = \boldsymbol{f}'(s)$ be the unit tangent vector. Assume that $\boldsymbol{f}$ satisfies the bi-Lipshitz estimate and that it belongs to $H^{3/2}(\mathbb{R}/\mathcal{L}\mathbb{Z}) \cap W^{1,\infty}(\mathbb{R}/\mathcal{L}\mathbb{Z})$. Then it holds that*

$$\begin{aligned}
\mathcal{E}(\boldsymbol{f}) &= \mathcal{E}_1(\boldsymbol{f}) + \mathcal{E}_2(\boldsymbol{f}) + 4, \\
\mathscr{M}_1(\boldsymbol{f}) &= \frac{\|\boldsymbol{\tau}(s_1) - \boldsymbol{\tau}(s_2)\|_{\mathbb{R}^n}^2}{2\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2}, \\
\mathscr{M}_2(\boldsymbol{f}) &= \frac{2}{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^4} \\
&\quad \times \det \begin{pmatrix} \boldsymbol{\tau}(s_1) \cdot \boldsymbol{\tau}(s_2) & (\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)) \cdot \boldsymbol{\tau}(s_1) \\ (\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)) \cdot \boldsymbol{\tau}(s_2) & \|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2 \end{pmatrix}, \\
\mathcal{E}_i(\boldsymbol{f}) &= \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \mathscr{M}_i(\boldsymbol{f}) ds_1 ds_2.
\end{aligned}$$

The first energy $\mathcal{E}_1$ characterizes the proper domain of $\mathcal{E}$. Indeed

$$\mathcal{E}_1(\boldsymbol{f}) \approx \frac{1}{2} \iint \frac{\|\boldsymbol{\tau}(s_1) - \boldsymbol{\tau}(s_2)\|^2}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))^2} ds_1 ds_2 = \frac{1}{2} [\boldsymbol{f}']_{H^{1/2}}^2.$$

The second energy $\mathcal{E}_2$ has the determinant structure, this implies some cancellation of singularities. O'Hara kindly informed the author that the second energy is a constant multiple of the O'Hara-Solanes energy ([18]). Using the decomposition, we can derive the explicit expressions of variational formulae and reasonable estimates in several function spaces including the absolute integrability ([11, 13]).

Here we present the absolute integrability of $\mathscr{M}_i$ only. Since $\mathscr{M}_1(\boldsymbol{f})$ is non-negative, the energy $\mathcal{E}_1$ is defined as the absolute integration. The sum of two energy densities is non-negative.

Theorem 7.3 ([10]) *It holds that*

$$\mathscr{M}_1(\boldsymbol{f})(s_1, s_2) + \mathscr{M}_2(\boldsymbol{f})(s_1, s_2) \geqq 0.$$

From this theorem we find that the second energy $\mathcal{E}_2$ is also defined as the absolute integration. The above theorem was firstly proved in [10], but the author gives here an alternative proof which is simpler than the original one.

Proof. We use the notation Δ to mean the difference between values at $s = s_1$ and s_2:

$$\Delta s = s_1 - s_2, \quad \Delta \boldsymbol{f} = \boldsymbol{f}(s_1) - \boldsymbol{f}(s_2), \quad \text{etc.}$$

By using the Lagrange formula, we have

$$\mathscr{M}_2(\boldsymbol{f}) = \frac{2}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} \left\langle \left(\boldsymbol{\tau}(s_1) \wedge \frac{\Delta \boldsymbol{f}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}}\right), \left(\boldsymbol{\tau}(s_2) \wedge \frac{\Delta \boldsymbol{f}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}}\right) \right\rangle,$$

where $\wedge$ is the wedge product of vectors, and $\langle \cdot, \cdot \rangle$ is the inner product on $\bigwedge^2 \mathbb{R}^n$. It is easy to see for and any two vectors $\boldsymbol{x}$, $\boldsymbol{y} \in \mathbb{R}^n$ and any unit vector $\boldsymbol{e} \in \mathbb{R}^n$ that

$$\langle (\boldsymbol{x} \wedge \boldsymbol{e}), (\boldsymbol{y} \wedge \boldsymbol{e}) \rangle = (P_e^\perp \boldsymbol{x}) \cdot (P_e^\perp \boldsymbol{y}),$$

where

$$P_e \boldsymbol{x} = (\boldsymbol{x} \cdot \boldsymbol{e})\boldsymbol{e}, \quad P_e^\perp \boldsymbol{x} = \boldsymbol{x} - P_e \boldsymbol{x}.$$

Therefore

$$\mathscr{M}_2(\boldsymbol{f}) = \frac{2}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} (P_{Rf}^\perp \boldsymbol{\tau}(s_1)) \cdot (P_{Rf}^\perp \boldsymbol{\tau}(s_2)),$$

where

$$R\boldsymbol{f} = \frac{\Delta \boldsymbol{f}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}}.$$

On the other hand, the first density is

$$\mathscr{M}_1(\boldsymbol{f}) = \frac{1}{2} \frac{\|\Delta \boldsymbol{\tau}\|_{\mathbb{R}^n}^2}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2}.$$

Consequently we obtain

$$\begin{aligned}
&2\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2 (\mathscr{M}_1(\boldsymbol{f}) + \mathscr{M}_2(\boldsymbol{f})) \\
&\quad = \|\Delta \boldsymbol{\tau}\|_{\mathbb{R}^n}^2 + 4(P_{Rf}^\perp \boldsymbol{\tau}(s_1)) \cdot (P_{Rf}^\perp \boldsymbol{\tau}(s_2)) \\
&\quad = \|P_{Rf} \Delta \boldsymbol{\tau}\|_{\mathbb{R}^n}^2 + \|P_{Rf}^\perp \Delta \boldsymbol{\tau}\|_{\mathbb{R}^n}^2 + 4(P_{Rf}^\perp \boldsymbol{\tau}(s_1)) \cdot (P_{Rf}^\perp \boldsymbol{\tau}(s_2)) \\
&\quad = \|P_{Rf} \Delta \boldsymbol{\tau}\|_{\mathbb{R}^n}^2 + \|P_{Rf}^\perp (\boldsymbol{\tau}(s_1) + \boldsymbol{\tau}(s_2))\|_{\mathbb{R}^n}^2,
\end{aligned}$$

which is non-negative. □

As a consequence, we can give a proof of Theorem 2.1.

Proof of Theorem 2.1. Theorems 7.2–7.3 implies $\mathcal{E}(\boldsymbol{f}) \geqq 4$. Furthermore $\mathcal{E}(\boldsymbol{f}) = 4$ holds if and only of $\mathscr{M}_1(\boldsymbol{f}) + \mathscr{M}_2(\boldsymbol{f}) \equiv 0$. This is also equivalent to

$$P_{Rf} \Delta \boldsymbol{\tau} \equiv \boldsymbol{o}, \quad P_{Rf}^\perp (\boldsymbol{\tau}(s_1) + \boldsymbol{\tau}(s_2)) \equiv \boldsymbol{o}.$$

In particular from the second relation we find a function μ such that

$$\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2) = \mu(s_1, s_2)(\boldsymbol{\tau}(s_1) + \boldsymbol{\tau}(s_2)). \tag{1}$$

Since a minimizer is smooth, so is μ. We differentiate the above relation three times with respecto to s_1 to obtain

$$\text{(2)} \quad \boldsymbol{\tau}(s_1) = \frac{\partial \mu}{\partial s_1}(s_1, s_2)\left(\boldsymbol{\tau}(s_1) + \boldsymbol{\tau}(s_2)\right) + \mu(s_1, s_2)\boldsymbol{\kappa}(s_1),$$

$$\text{(3)} \quad \boldsymbol{\kappa}(s_1) = \frac{\partial^2 \mu}{\partial s_1^2}(s_1, s_2)\left(\boldsymbol{\tau}(s_1) + \boldsymbol{\tau}(s_2)\right) + 2\frac{\partial \mu}{\partial s_1}(s_1, s_2)\boldsymbol{\kappa}(s_1) + \mu(s_1, s_2)\boldsymbol{\kappa}'(s_1),$$

$$\text{(4)} \quad \boldsymbol{\kappa}'(s_1) = \frac{\partial^3 \mu}{\partial s_1^3}(s_1, s_2)\left(\boldsymbol{\tau}(s_1) + \boldsymbol{\tau}(s_2)\right) + 3\frac{\partial^2 \mu}{\partial s_1^2}(s_1, s_2)\boldsymbol{\kappa}(s_1) + 3\frac{\partial \mu}{\partial s_1}(s_1, s_2)\boldsymbol{\kappa}'(s_1) + \mu(s_1, s_2)\boldsymbol{\kappa}''(s_1).$$

Putting $s_1 = s_2 = s$ in (1)–(4), we have

$$\mu(s, s) = 0, \quad \frac{\partial \mu}{\partial s_1}(s, s) = \frac{1}{2}, \quad \frac{\partial^2 \mu}{\partial s_1^2}(s, s) = 0,$$

and

$$\text{(5)} \qquad \boldsymbol{\kappa}'(s) = -4\frac{\partial^3 \mu}{\partial s_1^3}(s, s)\boldsymbol{\tau}(s).$$

Taking the inner product between each side of (5) and $\boldsymbol{\kappa}(s)$, we know $(\|\boldsymbol{\kappa}(s)\|_{\mathbb{R}^n}^2)' = 0$. Therefore $\|\boldsymbol{\kappa}(s)\|_{\mathbb{R}^n}$ is independent of s, and we write it κ. If $\kappa = 0$, then $\boldsymbol{\tau}'(s) = \boldsymbol{\kappa}(s) = \boldsymbol{o}$, and therefore $\boldsymbol{\tau}(s)$ is a constant vector. It is impossible because $\boldsymbol{f}$ is a closed curve. Consequently $\kappa > 0$. Taking the inner product between each sides of (5) and $\boldsymbol{\tau}(s)$, we know

$$-4\frac{\partial^3 \mu}{\partial s_1^3}(s, s) = \boldsymbol{\kappa}'(s) \cdot \boldsymbol{\tau}(s) = -\boldsymbol{\kappa}(s) \cdot \boldsymbol{\kappa}(s) = -\kappa^2.$$

Inserting this into (5), we obtain

$$\boldsymbol{\kappa}'(s) + \kappa^2 \boldsymbol{\tau}(s) = \boldsymbol{o}$$

for every $s \in \mathbb{R}/\mathcal{L}\mathbb{Z}$. Since $\boldsymbol{\tau}(s) = \boldsymbol{f}'(s)$, there exists a constant vector $\boldsymbol{c}$ such that

$$\text{(6)} \qquad \boldsymbol{\kappa}(s) + \kappa^2\left(\boldsymbol{f}(s) - \boldsymbol{c}\right) = \boldsymbol{o}.$$

Integrating with respect to s on $\mathbb{R}/\mathcal{L}\mathbb{Z}$, and deviding by $\mathcal{L}$, we find

$$\boldsymbol{c} = \frac{1}{\mathcal{L}} \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \boldsymbol{f}(s)\, ds.$$

We can rewrite (6) as the second order differential equation

$$\left(\boldsymbol{f}(s) - \boldsymbol{c}\right)'' + \kappa^2\left(\boldsymbol{f}(s) - \boldsymbol{c}\right) = \boldsymbol{o}.$$

The solution is

$$\boldsymbol{f}(s) - \boldsymbol{c} = (\boldsymbol{f}(0) - \boldsymbol{c})\cos\kappa s + \frac{\sin\kappa s}{\kappa}\boldsymbol{\tau}(0),$$

that is, $\boldsymbol{f}$ is a right circle with center $\boldsymbol{c}$ and radius κ^{-1}. Since the total length is $\mathcal{L}$, the radius is $\kappa^{-1} = \dfrac{\mathcal{L}}{2\pi}$. □

The Möbius invariance of each decomposed energy $\mathcal{E}_i$ is discussed in [10, 12].

Theorem 7.4 ([10]) *Each $\mathcal{E}_i$ is Möbius invariant in the following sense.*

1. *It is invariant under the dilation.*
2. *Let*

$$\boldsymbol{f} \mapsto \boldsymbol{p} = \boldsymbol{c} + \frac{r^2(\boldsymbol{f} - \boldsymbol{c})}{\|\boldsymbol{f} - \boldsymbol{c}\|_{\mathbb{R}^n}^2}$$

be the inversion with respect to sphere with center $\boldsymbol{c}$ and radius r.

(1) If $\boldsymbol{f} \in W^{1,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})$, then $\mathcal{E}_1(\boldsymbol{f}) + \mathcal{E}_2(\boldsymbol{f}) = \mathcal{E}_1(\boldsymbol{p}) + \mathcal{E}_2(\boldsymbol{p})$.

(2) If $\boldsymbol{c} \notin \operatorname{Im}\boldsymbol{f}$ and if $\mathcal{E}(\boldsymbol{f}) < \infty$, then $\mathcal{E}_1(\boldsymbol{f}) = \mathcal{E}_1(\boldsymbol{p})$, $\mathcal{E}_2(\boldsymbol{f}) = \mathcal{E}_2(\boldsymbol{p})$.

Remark 7.1 In the assertion (1) we *assume neither $\boldsymbol{c} \notin \operatorname{Im}\boldsymbol{f}$ nor $\mathcal{E}(\boldsymbol{f}) < \infty$.*

For proof, see [10].

Assume that $\operatorname{Im}\boldsymbol{f} = S^1$ and $\boldsymbol{c} \in \operatorname{Im}\boldsymbol{f}$, and then $\boldsymbol{p}$ is a straight line. Therefore $\mathcal{E}_1(\boldsymbol{f}) = 2\pi$, and $\mathcal{E}_1(\boldsymbol{p}) = 0$. This shows $\mathcal{E}_1(\boldsymbol{f}) = \mathcal{E}_1(\boldsymbol{p})$ does not hold without $\boldsymbol{c} \notin \operatorname{Im}\boldsymbol{f}$. In fact,

Theorem 7.5 ([12]) *Let $\boldsymbol{f} \in C^{1,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})$. Then for inversion with center $\boldsymbol{c}$ on $\operatorname{Im}\boldsymbol{f}$, we have*

$$\mathcal{E}_1(\boldsymbol{f}) = \mathcal{E}_1(\boldsymbol{p}) + 2\pi^2, \quad \mathcal{E}_2(\boldsymbol{f}) = \mathcal{E}_2(\boldsymbol{p}) - 2\pi^2.$$

Sketch of proof. Let θ be a general parameter. Under our assumption, $\boldsymbol{f}$ has no self-intersections and therefore $\boldsymbol{f}$ passes through the center $\boldsymbol{c}$ only once at most per a period. Thus we may assume that $\boldsymbol{c} = \boldsymbol{f}(0)$. Then if $\theta \neq 0 \pmod{2\pi}$, it holds that $\boldsymbol{f}(\theta) \neq \boldsymbol{c}$. As shown in [10], we have

$$\mathscr{M}_1(\boldsymbol{f})\|\dot{\boldsymbol{f}}(\theta_1)\|_{\mathbb{R}^n}\|\dot{\boldsymbol{f}}(\theta_2)\|_{\mathbb{R}^n} - \mathscr{M}_1(\boldsymbol{p})\|\dot{\boldsymbol{p}}(\theta_1)\|_{\mathbb{R}^n}\|\dot{\boldsymbol{p}}(\theta_2)\|_{\mathbb{R}^n} = \frac{1}{2}J(\theta_1, \theta_2), \tag{7}$$

where

$$\frac{1}{2}J(\theta_1,\theta_2) = \left(\frac{\partial}{\partial\theta_1}\log\|\boldsymbol{f}(\theta_1)-\boldsymbol{f}(0)\|^2_{\mathbb{R}^n}\right)\left(\frac{\partial}{\partial\theta_2}\log\|\boldsymbol{f}(\theta_1)-\boldsymbol{f}(\theta_2)\|^2_{\mathbb{R}^n}\right)$$
$$+\left(\frac{\partial}{\partial\theta_1}\log\|\boldsymbol{f}(\theta_1)-\boldsymbol{f}(\theta_2)\|^2_{\mathbb{R}^n}\right)\left(\frac{\partial}{\partial\theta_2}\log\|\boldsymbol{f}(\theta_2)-\boldsymbol{f}(0)\|^2_{\mathbb{R}^n}\right)$$
$$-\left(\frac{\partial}{\partial\theta_1}\log\|\boldsymbol{f}(\theta_1)-\boldsymbol{f}(0)\|^2_{\mathbb{R}^n}\right)\left(\frac{\partial}{\partial\theta_2}\log\|\boldsymbol{f}(\theta_2)-\boldsymbol{f}(0)\|^2_{\mathbb{R}^n}\right).$$

Since

$$\frac{\partial}{\partial\theta_i}(\cdots)\,d\theta_i = \frac{\partial}{\partial s_i}(\cdots)\,ds_i,$$

it is easily known that

$$\iint_{(\mathbb{R}/2\pi\mathbb{Z})^2} J(\theta_1,\theta_2)d\theta_1 d\theta_2 = \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \mathscr{J}(s_1,s_2)\,ds_1 ds_2, \tag{8}$$

where

$$\mathscr{J}(s_1,s_2) = \left(\frac{\partial}{\partial s_1}\log\|\boldsymbol{f}(s_1)-\boldsymbol{f}(0)\|^2_{\mathbb{R}^n}\right)\left(\frac{\partial}{\partial s_2}\log\|\boldsymbol{f}(s_1)-\boldsymbol{f}(s_2)\|^2_{\mathbb{R}^n}\right)$$
$$+\left(\frac{\partial}{\partial s_1}\log\|\boldsymbol{f}(s_1)-\boldsymbol{f}(s_2)\|^2_{\mathbb{R}^n}\right)\left(\frac{\partial}{\partial s_2}\log\|\boldsymbol{f}(s_2)-\boldsymbol{f}(0)\|^2_{\mathbb{R}^n}\right)$$
$$-\left(\frac{\partial}{\partial s_1}\log\|\boldsymbol{f}(s_1)-\boldsymbol{f}(0)\|^2_{\mathbb{R}^n}\right)\left(\frac{\partial}{\partial s_2}\log\|\boldsymbol{f}(s_2)-\boldsymbol{f}(0)\|^2_{\mathbb{R}^n}\right)$$

Here we put $s = 0$ at $\theta = 0$. We introduce a function $\mathscr{F}$ replacing the Euclidean distance $\|\cdot\|_{\mathbb{R}^n}$ in $\mathscr{J}$ with the intrinsic distance $\mathscr{D}$:

$$\mathscr{F}(s_1,s_2) = \left(\frac{\partial}{\partial s_1}\log\mathscr{D}(\boldsymbol{f}(s_1),\boldsymbol{f}(0))^2\right)\left(\frac{\partial}{\partial s_2}\log\mathscr{D}(\boldsymbol{f}(s_1),\boldsymbol{f}(s_2))^2\right)$$
$$+\left(\frac{\partial}{\partial s_1}\log\mathscr{D}(\boldsymbol{f}(s_1),\boldsymbol{f}(s_2))^2\right)\left(\frac{\partial}{\partial s_2}\log\mathscr{D}(\boldsymbol{f}(s_2),\boldsymbol{f}(0))^2\right)$$
$$-\left(\frac{\partial}{\partial s_1}\log\mathscr{D}(\boldsymbol{f}(s_1),\boldsymbol{f}(0))^2\right)\left(\frac{\partial}{\partial s_2}\log\mathscr{D}(\boldsymbol{f}(s_2),\boldsymbol{f}(0))^2\right).$$

We can show

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \mathscr{F}(s_1,s_2)\,ds_1 ds_2 = 4\pi^2 \tag{9}$$

and

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} (\mathscr{J}(s_1,s_2)-\mathscr{F}(s_1,s_2))\,ds_1 ds_2 = 0. \tag{10}$$

The assertion of Theorem easily follows from (7)–(10). We use there the existence of bounded $\boldsymbol{\kappa}$ almost everywhere, so we need assume $\boldsymbol{f} \in C^{1,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})$. See [12] for details. □

Corollary 7.1 *Right circles are the only global minimizers of* $\mathcal{E}_1$ *in* $C^{1,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})$.

Proof. Since $\mathscr{M}_1(\boldsymbol{p}) \geqq 0$, it is clear $\mathcal{E}_1(\boldsymbol{p}) \geqq 0$, hence $\mathcal{E}_1(\boldsymbol{f}) = \mathcal{E}_1(\boldsymbol{p}) + 2\pi^2 \geqq 2\pi^2$. Furthermore $\mathcal{E}_1(\boldsymbol{f}) = 2\pi^2$ holds if and only if $\mathcal{E}_1(\boldsymbol{p}) = 0$, which is equivalent to $\mathscr{M}_1(\boldsymbol{p}) \equiv 0$. This implies $\boldsymbol{p}'(s)$ is a constant vector, that is, $\boldsymbol{p}$ is a straight line. Consequently the pre-itmage $\boldsymbol{f}$ is a right circle. □

As said above, the lower bound of $\mathcal{E}_1$ in $C^{1,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})$ is $2\pi^2$. This estimate seems to be related to the first eigenvalue of $(-\Delta_s)^{3/2}$, which is the principal part of L^2-gradient, is $\left(\frac{2\pi}{\mathcal{L}}\right)^3$. Indeed by the bi-Lipschitz continuity of $\boldsymbol{f}$ we have

$$\begin{aligned}
\mathcal{E}_1(\boldsymbol{f}) &= \frac{1}{2}\iint_{(\mathcal{R}/\mathcal{Z})^2} \frac{\|\Delta\boldsymbol{\tau}\|_{\mathbb{R}^n}^2}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}\, ds_1 ds_2 \simeq \frac{1}{2}\iint (\mathcal{R}/\mathcal{Z})^2 \frac{\|\Delta\boldsymbol{f}'\|_{\mathbb{R}^n}^2}{|\Delta s|^2}\, ds_1 ds_2 \\
&\simeq \pi \int_{\mathcal{R}/\mathcal{Z}} \|(-\Delta_s)^{1/4}\boldsymbol{f}'\|_{\mathbb{R}^n}^2 ds = \pi \int_{\mathcal{R}/\mathcal{Z}} \|(-\Delta_s)^{3/4}\boldsymbol{f}\|_{\mathbb{R}^n}^2 ds \\
&= \pi \int_{\mathcal{R}/\mathcal{Z}} \boldsymbol{f}\cdot(-\Delta_s)^{3/2}\boldsymbol{f}\, ds \geqq \pi\left(\frac{2\pi}{\mathcal{L}}\right)^3 \int_{\mathcal{R}/\mathcal{Z}} \|\boldsymbol{f}\|_{\mathbb{R}^n}^2 ds.
\end{aligned}$$

The equality in the above estimate implies that each component of $\boldsymbol{f}$ is the first eigenfunctions. That means $\boldsymbol{f}$ is a right circle with the center origin and the perimeter $\mathcal{L}$. Therefore

$$\|\boldsymbol{f}\|_{\mathbb{R}^n} \equiv \frac{\mathcal{L}}{2\pi}.$$

Consequently we ontain

$$\pi\left(\frac{2\pi}{\mathcal{L}}\right)^3 \int_{\mathcal{R}/\mathcal{Z}} \|\boldsymbol{f}\|_{\mathbb{R}^n}^2 ds = \pi\left(\frac{2\pi}{\mathcal{L}}\right)^3 \cdot \mathcal{L} \cdot \left(\frac{\mathcal{L}}{2\pi}\right)^2 = 2\pi^2$$

This is not rigorous argument, however it is interesting for the author.

It is still open that Theorem 7.5 holds or not for $\boldsymbol{f} \in H^{3/2}(\mathbb{R}/\mathcal{L}\mathbb{Z}) \cap W^{1,\infty}(\mathbb{R}/\mathcal{L}\mathbb{Z})$. What is show for the curve in $H^{3/2}(\mathbb{R}/\mathcal{L}\mathbb{Z}) \cap W^{1,\infty}(\mathbb{R}/\mathcal{L}\mathbb{Z})$ is a (non-sharp) lower estimate on $\mathcal{E}_1$.

Proposition 7.1 *We have*

$$\mathcal{E}_1(\boldsymbol{f}) \geqq 2\pi \int_{-\pi}^{\pi} \frac{1-\cos u}{u^2}\, du$$

for any $\boldsymbol{f} \in H^{3/2}(\mathbb{R}/\mathcal{L}\mathbb{Z}) \cap W^{1,\infty}(\mathbb{R}/\mathcal{L}\mathbb{Z})$.

Remark 7.2 Note that estimate for curves of $C^{1,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})$ class can be written as

$$\mathcal{E}_1(\boldsymbol{f}) \geqq 2\pi^2 = 2\pi \int_{-\infty}^{\infty} \frac{1-\cos u}{u^2}\, du.$$

Proof. Since the Euclidean distance is not longer than the intrinsic distance, we have

$$\mathcal{E}_1(\boldsymbol{f}) = \iint_{(\mathcal{R}/\mathcal{Z})^2} \frac{\|\Delta \boldsymbol{\tau}\|^2_{\mathbb{R}^n}}{2\|\Delta \boldsymbol{f}\|^2_{\mathbb{R}^n}}\, ds_1 ds_2 \geqq \iint_{(\mathcal{R}/\mathcal{Z})^2} \frac{\|\Delta \boldsymbol{\tau}\|^2_{\mathbb{R}^n}}{2\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))^2}\, ds_1 ds_2.$$

We insert the Fourier series

$$\boldsymbol{f} = \sum_{k\in\mathbb{Z}} \varphi_k \boldsymbol{a}_k$$

into the above. In a similar manner to derive the L^2-gradient (see [13]), we can obtain

$$\iint_{(\mathcal{R}/\mathcal{Z})^2} \frac{\|\Delta \boldsymbol{\tau}\|^2_{\mathbb{R}^n}}{2\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))^2}\, ds_1 ds_2 = \sum_{k\in\mathbb{Z}} \left|\frac{2\pi k}{\mathcal{L}}\right|^3 \|\boldsymbol{a}_k\|^2_{\mathbb{C}^n} \int_{-\pi|k|}^{\pi|k|} \frac{1-\cos u}{u^2}\, du.$$

Applying Parseval's identity to

$$\boldsymbol{\tau} = \sum_{k\in\mathbb{Z}} \frac{2\pi i k}{\mathcal{L}} \varphi_k \boldsymbol{a}_k,$$

we have

$$\|\boldsymbol{\tau}\|^2_{L^2(\mathbb{R}/\mathcal{L}\mathbb{Z})} = \sum_{k\in\mathbb{Z}} \left|\frac{2\pi k}{\mathcal{L}}\right|^2 \|\boldsymbol{a}_k\|^2_{\mathbb{C}^n}.$$

On the other hand, since $\boldsymbol{\tau}$ is a unit vector,

$$\|\boldsymbol{\tau}\|^2_{L^2(\mathbb{R}/\mathcal{L}\mathbb{Z})} = \mathcal{L}$$

holds. Therefore it holds that

$$\sum_{k\in\mathbb{Z}} \left|\frac{2\pi k}{\mathcal{L}}\right|^2 \|\boldsymbol{a}_k\|^2_{\mathbb{C}^n} = \mathcal{L}.$$

Consequently we arrive at

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \left|\frac{2\pi k}{\mathcal{L}}\right|^3 \|\boldsymbol{a}_k\|^2_{\mathbb{C}^n} \int_{-\pi|k|}^{\pi|k|} \frac{1-\cos u}{u^2}\, du &\geqq \frac{2\pi}{\mathcal{L}} \int_{-\pi}^{\pi} \frac{1-\cos u}{u^2}\, du \sum_{k\in\mathbb{Z}} \left|\frac{2\pi k}{\mathcal{L}}\right|^2 \|\boldsymbol{a}_k\|^2_{\mathbb{C}^n} \\ &= 2\pi \int_{-\pi}^{\pi} \frac{1-\cos u}{u^2}\, du.\end{aligned}$$

□

From the proof we can see that the above estimate is not sharp. The author expects that the lower bound of $\mathcal{E}_1$ in $H^{3/2}(\mathbb{R}/\mathcal{L}\mathbb{Z}) \cap W^{1,\infty}(\mathbb{R}/\mathcal{L}\mathbb{Z})$ is also $2\pi^2$ and that right circles are only minimizers.

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