数理解析研究所講究録
第 1980 巻 2016 年 103-105

Pasting reproducing kernel Hilbert spaces

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Abstract: The aim of this article is to find the necessary and sufficient condition for the mapping $H_K(E) \ni f \mapsto (f|E_1, f|E_2) \in H_{K|E_1\times E_2}(E_1) \oplus H_{K|E_2\times E_2}(E_2)$ to be isomorphic, where K is a positive definite function on $E = E_1 + E_2$.

1 Introduction

Let E be a set and $K : E \times E \to \mathbb{C}$ be a positive definite function. For $f \in H_K(E)$, we can easily check that $f|E_0 \in H_{K|E_0\times E_0}(E_0)$ since

$$f|E_0 \otimes f|E_0 \ll \|f\|_{H_K(E)}{}^2 K|E_0 \times E_0 \otimes K|E_0 \times E_0$$

in the sense that

$$\sum_{j,k=1,2,\ldots,n} \left(\|f\|_{H_K(E)}{}^2 K|E_0 \times E_0(p_j,p_k) - f|E_0 \otimes f|E_0(p_j,p_k)\right) z_j\overline{z_k}$$
$$= \sum_{j,k=1,2,\ldots,n} \left(\|f\|_{H_K(E)}{}^2 K(p_j,p_k) - f(p_j,p_k)\right) z_j\overline{z_k} \geq 0$$

for any finite set $\{p_1, p_2, \ldots, p_k\} \subset E_0$ and $\{z_1, z_2, \ldots, z_k\} \subset \mathbb{C}$. Therefore, when E is partitioned into the sum $E = E_1 + E_2$, the operation the mapping $R : H_K(E) \ni f \mapsto (f|E_1, f|E_2) \in H_{K|E_1\times E_2}(E_1) \oplus H_{K|E_2\times E_2}(E_2)$ makes sense. Note that R is injection, since $f|E_1 = 0$ and $f|E_2 = 0$ imply $f = 0$.

2 Main result

We show the necessary and sufficient condition for R to be isomorphic.

Theorem. *The mapping R is isomorphic if and only if $K|E_1 \times E_2 = 0$.*

Proof. Assume first that $K|E_1 \times E_2 = 0$. Let us first show that R is surjection. To this end, given $g_1 \in H_{K|E_1\times E_1}(E_1)$ and $g_2 \in H_{K|E_2\times E_2}(E_2)$, we define a function f on E by $f(p) = g_1(p)$ on E_1 and $f(p) = g_2(p)$ on E_2. Let us check that $f \in H_K(E)$. To this end, we set $f_1 = \chi_{E_1} f$ and $f_2 = \chi_{E_2} f$. Then for $l = 1, 2$, we have

$$\sum_{j,k=1,2,\ldots,n} \left(\|f_l\|_{H_K(E)}{}^2 K(p_j,p_k) - f_l \otimes f_l(p_j,p_k)\right) z_j\overline{z_k}$$
$$\geq \sum_{j,k=1,2,\ldots,n\,,\,p_j,p_k\in E_l} \left(\|f_l\|_{H_K(E)}{}^2 K(p_j,p_k) - f_l \otimes f_l(p_j,p_k)\right) z_j\overline{z_k}$$

by assumption. Since $\|f_l\|_{H_K(E)} \geq \|g_1\|_{H_{K|E_l\times E_l}(E_l)}$ from a general result on the reproducing kernel Hilbert spaces, we have

$$\sum_{j,k=1,2,\ldots,n} \left(\|f_l\|_{H_K(E)}{}^2 K(p_j,p_k) - f_l \otimes f_l(p_j,p_k)\right) z_j\overline{z_k}$$
$$\geq \sum_{j,k=1,2,\ldots,n\,,\,p_j,p_k\in E_l} \left(\|g_l\|_{H_{K|E_l\times E_l}(E_l)}{}^2 K(p_j,p_k) - g_l \otimes g_l(p_j,p_k)\right) z_j\overline{z_k} \geq 0.$$

Thus, $f_l \in H_{K_l}(E_l)$ as was to be shown.

It remains to show that R is an isomorphism. In fact, $\{K(\cdot,p)\}_{p\in E_l}$ is a dense subspace in $H_{K|E_l\times E_l}(E_l)$, we have only to show that

$$\left(\left\|\sum_{m=1}^{L}(z_1^m K(\cdot,p_1^m) + z_2^m K(\cdot,p_2^m))\right\|_{H_K(E)}\right)^2$$
$$= \left(\left\|\sum_{m=1}^{L} z_1^m K(\cdot,p_1^m)\right\|_{H_{K|E_1\times E_1}(E_1)}\right)^2 + \left(\left\|\sum_{m=1}^{L} z_2^m K(\cdot,p_2^m)\right\|_{H_{K|E_2\times E_2}(E_2)}\right)^2$$

for any $p_1^m \in E_1, p_2^m \in E_2, z_1^m, z_2^m \in \mathbb{C}$ with $m = 1, 2, \ldots, L$.

Conversely, if R is an isomorphism, then

$$\left(\|K(\cdot,p_1)+zK(\cdot,p_2))\|_{H_K(E)}\right)^2$$
$$=\left(\|K(\cdot,p_1)\|_{H_{K|E_1\times E_1}(E_1)}\right)^2+|z|^2\left(\|K(\cdot,p_2)\|_{H_{K|E_2\times E_2}(E_2)}\right)^2$$

for any $p_1\in E_1$ and $p_2\in E_2$ and $z\in\mathbb{C}$. Thus $K|E_1\times E_2=0$. □

This result is essentially based on [1, 2], however, the representation is arranged in the polished version.

References

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