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ON THE COMPLEXITY OF THE STABLE MARRIAGE PROBLEM FOR RESTRICTED INSTANCE CLASSES

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Abstract The stable marriage problem is a well studied problem on bipartite graphs, and its special class, MAX SMTI (the problem finding a maximum cardinality stable matching with ties and incomplete lists) is known to be NP-hard. In this paper, we show that MAX SMTI is NP-hard even for highly restricted instances, where each person's preference list includes at most one strict preference. We also show that there is a polynomial time algorithm for solving the problem for a little more restricted class of instances, where each man has at most one strict preference; and if there exists a man who has one strict preference, then he has the unique first choice.

Keywords: stable marriage problem, MAX SMTI, complexity analysis

1. Introduction

The *stable marriage problem* is a classical matching problem on bipartite graphs. An instance of the stable marriage problem consists of n men, n women, and each person's preference list which represents his/her preference to the opposite sex. A *matching* M is a set of pairs of men and women such that they are acceptable to each other. The *cardinality* of matching is the number of pairs in M. A pair of a man and a woman is called a *blocking pair* for M if both of them prefer others to their current partners. A matching M is *stable* if there is no blocking pair for M. The stable marriage problem is to find a stable matching in a given input instance. Since Gale and Shapley [4] introduced it in 1962, this problem has been studied by numerous researchers because of its mathematical charm and applicability to the real world.

In its original setting, each person's preference list is complete and does not include ties. In this case, every stable matching is a perfect matching, and it can be obtained in polynomial time using a well-known algorithm, the *Gale-Shapley algorithm* [4]. However, if preference lists include ties and only a part of the members of the opposite sex, the problem of finding a maximum cardinality stable matching is known to NP-hard [9]. We call such a class MAX SMTI (MAXimum Stable Marriage with Ties and Incomplete Lists). MAX SMTI is also known to be NP-hard under certain restrictions. For example, the one-sided-ties problem is such a problem, where ties can appear only in women's preference lists [13]. A number of researchers have studied such special cases as well as the general MAX SMTI (see e.g. [12]).

On the other hand, it is known that some special cases of MAX SMTI with harder restrictions can be solved in polynomial time, e.g., classes where each person's preference list includes no tie [5], and where each man's preference list includes at most two women [8]. Therefore, a natural question comes to mind: what is the hardest restriction for which the problem remains NP-hard? In other words, what is the boundary between polynomial-time solvability and NP-hardness for this problem? We refer to it as "the boundary of complexity" for MAX SMTI.

In this paper, we show that MAX SMTI is still NP-hard even for highly restricted instances where each person's preference list includes at most one strict preference. We also show that there exists a polynomial time algorithm for a little more restricted instance, where each man's preference list includes at most one strict preference; and, if someone's list includes a strict preference, then it has a unique first choice. (In Section 3, we define this precisely.) We refer to the former class of instance as *Class 1*, and the latter *Class 2*. Since these two variants are so similar, it can be considered as a kind of the boundary of complexity.

The paper is organized as follow. In Section 2, we review some related works. In Section 3, we give some definitions concerning our problems. In Section 4, we show that MAX SMTI for Class 1 is NP-hard. In contrast, we show in Section 5 that MAX SMTI for Class 2 is polynomial time solvable.

2. Related works

The NP-hardness of MAX SMTI is shown by Iwama *et al.* in 1999 [9]. A few years later, Manlove *et al.* established some valuable results for the problem [13]. One is that MAX SMTI remains NP-hard under some hard restrictions (this variant is called MAX SSMTI in [15]), another is that MAX SMTI is approximable within a factor of 2 by breaking ties arbitrary and finding any stable matching. Since then, researchers have focused on a variety of special instance classes of MAX SMTI.

In particular, MAX SMOTI (MAXimum Stable Marriage with One-sided Ties and Incomplete lists) [15] and MAX SSMTI (Special SMTI) are two well-studied variants of MAX SMTI. Instances of MAX SMOTI have a common property that ties can appear only in women's preference lists (i.e., men's preference lists are strictly ordered and does not include ties). MAX SSMTI is even more restricted variant of MAX SMOTI, where ties are only allowed at the ends of the women's preference lists. As mentioned above, MAX SSMTI is NP-hard, and so is MAX SMOTI. Currently, the best approximation ratios for these subclasses are 19/13 (<1.4616) [2] and 5/4 (=1.25) [7], respectively (both are achieved by an algorithm using LP-reduction, called GSA-LP [11]).

In connection with "the boundary of complexity", Irving *et al.* showed some key findings [8]. That is, MAX SMTI is solvable in polynomial time if each man's list is of length at most two (even when women's lists are of unbounded length), but if each man's list is of length at most three, the problem turns NP-hard (even if each women's list is of length at most three). They referred to the former class as $(2, \infty)$-MAX SMTI and the latter as $(3, 3)$-MAX SMTI. More generally, (p, q)-MAX SMTI is a restriction of MAX SMTI in terms of the length of preference lists where each man's preference list includes at most p women and each woman's preference list includes at most q men [10]. There are some other works about NP-hardness and inapproximability of similar variants. Halldósson *et al.* [6] showed that $(4, 7)$-MAX SMTI is NP-hard and cannot be approximable by a constant unless P=NP.

3. Preliminaries

In this section, we prepare some notation and terminology used in this paper.

Definition 3.1. An instance of the stable marriage problem is a 4-tuple $I = (U, Y, pri_U, pri_Y)$, where

$$\begin{aligned} |U| = |Y| = n \in \mathbb{N}_+; \quad U \cap Y = \emptyset; \\ pri_U : U \times Y \to \{0, 1, \cdots, n\}; \\ pri_Y : Y \times U \to \{0, 1, \cdots, n\}. \end{aligned}$$

The sets U and Y consist of the members of men and women in I, respectively, and pri_U and pri_Y are priority functions which represent each person's preference to the opposite sex. Let $m \in U$, and $\{w, w'\} \subset Y$. We say that m prefers w to w' if $pri_U(m, w) > pri_U(m, w')$. If $pri_U(m, w) = pri_U(m, w')$, then we say that w and w' are in a tie with respect to m (or m equally prefers w and w'). We sometimes denote an instance I as $(U_I, Y_I, pri_{U_I}, pri_{Y_I})$ to avoid a confusion.

Definition 3.2. A pair (m, w) of a man m and a woman w with a property $pri_U(m, w) \neq 0$ and $pri_Y(w, m) \neq 0$ is called an *acceptable pair*. Let $\mathbb{A}$ be the set of all acceptable pairs in the input instance. If $(m, w) \in \mathbb{A}$, then we say that m is acceptable to w, and vice versa.

Definition 3.3. Let $m \in U$, and $k \in \mathbb{N}_+$ be the number of women who are acceptable to m. An ordered list $w_1, w_2, \cdots, w_k$ with a property that $pri_U(m, w_i) \geq pri_U(m, w_j) > 0$ $(i = 1, \cdots, k)$ if $i < j$, is called a *preference list* of m. Women's preference lists are defined similarly.

We say that a man m has k *strict preferences* if women in m's preference list can be partitioned into $k+1$ groups with respect to $pri_U(m, \cdot)$ (i.e., $|\{pri_U(m, w) | w \in Y\} \setminus \{0\}| = k + 1$). Similarly, it is defined for women.

Definition 3.4. Let M be a matching on the bipartite graph $(U, Y; \mathbb{A})$ and SU and SY denote the sets of single men and women (i.e., person, who do not match in M). Let $M(p)$ denote the partner of a person p (i.e., $M(m) = w \Leftrightarrow M(w) = m \Leftrightarrow (m, w) \in M$). A *blocking pair* for M is a pair $(m, w) \in \mathbb{A}$ of a man and a woman which satisfies one of the following conditions:

$$pri_U(m, w) > pri_U(m, M(m)) \wedge pri_Y(w, m) > pri_Y(w, M(w)); \tag{3.1}$$
$$pri_U(m, w) > pri_U(m, M(m)) \wedge w \in SY; \tag{3.2}$$
$$pri_Y(w, m) > pri_Y(w, M(w)) \wedge m \in SU; \tag{3.3}$$
$$w \in SY \wedge m \in SU. \tag{3.4}$$

Definition 3.5. A set $M \subset \mathbb{A}$ of acceptable pairs is called a *stable matching* if every $(m, w) \in \mathbb{A}$ is not a blocking pair for M.

Definition 3.6. The problem of finding the maximum cardinality of a stable matching in the input instance (where cardinality of a matching is the number of its components) is called *MAX SMTI* (MAXimum Stable Marriage with Ties and Incomplete lists).

Definition 3.7. *PER SMTI (PERfect SMTI)* is the problem of checking whether a *perfect* stable matching M exists or not (i.e., M is a stable matching, and there is no single person with respect to M) for a given instance of MAX SMTI.

Definition 3.8. Let V be the set of variables of a satisfiability problem. *(2,2)E3-SAT* is the problem of checking whether B is satisfiable, given an boolean formula B in conjunctive normal form in which each clause contains exactly three literals and, for each $v \in V$, each of literals v and $\bar{v}$ appears exactly twice in B.

The problem $(2,2)$E3-SAT is known to be NP-complete [1]. Irving *et al.* showed the NP-hardness of $(3,3)$-MAX SMTI by using a reduction from $(2,2)$E3-SAT [8].

Definition 3.9. Let $m \in U$ and w be a woman in the m's preference list. We say that w is the *unique first choice* of m if $pri_U(m, w) > pri_U(m, w')$ for any $w' \in Y \setminus \{w\}$.

Definition 3.10. Let p be a person in an instance (i.e, $p \in U \cup Y$). Consider the following conditions for p:

(S0) p has no strict preference;

(S1) p has one strict preference;

(S2) p has two strict preferences;

(UFC) p has the unique first choice;

(L2) p's preference list is length of two;

(L3) p's preference list is length of three.

Class 1, *Class 2*, and *Class 3* are the sets of instances of MAX SMTI satisfying some of the above conditions. The Class 1 is composed of instances of MAX SMTI in which each person satisfies

$$(S0) \text{ or } (S1).$$

The Class 2 consists of instances of MAX SMTI in which each man satisfies

$$(S0) \text{ or } ((S1) \text{ and } (UFC)),$$

and the Class 3 consists of instances of MAX SMTI in which each person satisfies

$$((S0) \text{ and } (L2)) \text{ or } ((S2) \text{ and } (L3)).$$

We give an example of instances of Classes 1, 2, and 3 in Tables 1, 2, and 3, respectively. Note that a man m's preference list includes a woman w if and only if w's preference includes m. The set of men is $\{a, b, c, d\}$ and the set of women is $\{w, x, y, z\}$ for each instance. The symbol " = " denotes a tie, and " > " denotes a strict preference. In Table 1, each person's preference list has at most one strict preference, i.e., the symbol " > " does not appear twice in each person's preference list. In Table 2, each man m has the same property, and if m has a strict preference, then he has the unique first choice (see man a). On the other hand, the strict preference can appear arbitrary number of times on women's preference lists in Table 2 (woman x). In Table 3, the length of each person's preference list is either two or three, and if the preference list is of length two, then it has no strict preference (man b), and if the preference list is of length three, then it has two strict preferences (man a).

Table 1: Preference lists of an instance of Class 1

a : $w = z > y = x$		w : $a > c = d$
b : $x > z$		x : $d = a = b$
c : $z = y = w$		y : $c = d > a$
d : $w > z = x = y$		z : $a = c = b = d$

Table 2: Preference lists of an instance of Class 2

a : $w > z = y = x$		w : $a > c = d$
b : $x > z$		x : $d > a > b$
c : $z = y = w$		y : $c = d > a$
d : $w > z = x = y$		z : $a = c = b = d$

Table 3: Preference lists of an instance of Class 3

a : $w > z > y$		w : $a > c > d$
b : $x = z$		x : $d = b$
c : $z > y > w$		y : $c = d$
d : $w = z$		z : $a > c > b$

4. NP-hardness of MAX SMTI for Class 1

In this section, we show the NP-hardness of MAX SMTI for Class 1. The result holds even if each person's preference list is of length at most three.

We can show the NP-completeness of PER SMTI for Class 3 by reduction from $(2,2)$E3-SAT. We can also show the NP-completeness of PER SMTI with Class 1 by using a reduction from PER SMTI for Class 3. Accordingly, MAX SMTI for Class 1 is NP-hard.

Transformation from (2,2)E3-SAT to PER SMTI for Class 3

Let us show the NP-completeness of PER SMTI for Class 3 by using a reduction from $(3,3)$E3-SAT. First, we introduce a transformation from an instance of $(2,2)$E3-SAT to an instance of $(3,3)$-MAX SMTI, which Irving *et al.* used to show the NP-hardness of $(3,3)$-MAX SMTI in [8].

Let $B = (V, C)$ be an instance of $(2,2)$E3-SAT, where $V = \{v_0, v_1, ..., v_{n-1}\}$ and $C = \{c_1, c_2, ..., c_m\}$ are the sets of variables and clauses, respectively. An instance I of $(3,3)$-MAX SMTI is constructed as follows. The set of men in I is $X \cup P \cup Q$, where

$$X = \bigcup_{i=0}^{n-1} X_i, \quad X_i = \{x_{4i+r} \mid 0 \le r \le 3\} \quad (0 \le i \le n-1),$$

$$P = \bigcup_{j=1}^{m} P_j, \quad P_j = \{p_j^1, p_j^2, p_j^2\} \quad (1 \le j \le m),$$

$$Q = \{q_j \mid c_j \in C\}.$$

The set of women in I is $Y' \cup C' \cup Z$, where

$$
\begin{aligned}
Y' &= \bigcup_{i=0}^{n-1} Y_i', \quad Y_i' = \{y_{4i+r} \mid 0 \le r \le 3\} \quad (0 \le i \le n-1),\\
C' &= \{c_j^s \mid 1 \le j \le m \ , \ 1 \le s \le 3\},\\
Z &= \{z_j \mid 1 \le j \le m\}.
\end{aligned}
$$

The preference lists of I are given by Table 4.

In the preference list of a man $x_{4i+r} \in X$ $(0 \le i \le n-1,\ r \in \{0,1\})$, the symbol $c(x_{4i+r})$ denotes the woman $c_j^s \in C'$ such that the $(r+1)$th occurrence of literal v_i appears at position s of c_j. Similarly if $r \in \{2,3\}$ then the symbol $c(x_{4i+r})$ denotes the woman $c_j^s \in C'$ such that the $(r-1)$th occurrence of literal $\bar{v}_i$ appears at position s of c_j. Also in the preference list of a woman $c_j^s \in C'$, if literal v_i appears at position s of clause $c_j \in C$, the symbol $x(c_j^s)$ denotes the man x_{4i+r-1}, where $r \in \{1,2\}$ in accordance with the first or the second occurrence of literal v_i in B. Otherwise, if literal $\bar{v}_i$ appears at position s of clause $c_j \in C$, the symbol $x(c_j^s)$ denotes the man x_{4i+r+1}, where $r \in \{1,2\}$ in accordance with the first or the second occurrence of literal $\bar{v}_i$ in B. Then the following proposition holds.

Proposition 4.1 (Irving *et al.* [8]). An instance B of (2, 2)E3-SAT is satisfiable if and only if I admits a perfect stable matching.

Table 4: Preference lists in the constructed instance

x_{4i}	:	y_{4i}	$>$	$c(x_{4i})$	$>$	y_{4i+1}	$(0 \le i \le n-1)$
x_{4i+1}	:	y_{4i+1}	$>$	$c(x_{4i+1})$	$>$	y_{4i+2}	$(0 \le i \le n-1)$
x_{4i+2}	:	y_{4i+3}	$>$	$c(x_{4i+2})$	$>$	y_{4i+2}	$(0 \le i \le n-1)$
x_{4i+3}	:	y_{4i}	$>$	$c(x_{4i+3})$	$>$	y_{4i+3}	$(0 \le i \le n-1)$
p_j^s	:	z_j	$>$	c_j^s			$(1 \le j \le m,\ 1 \le s \le 3)$
q_j	:	c_j^1	$>$	c_j^2	$>$	c_j^3	$(1 \le j \le m)$
y_{4i}	:	x_{4i}	$=$	x_{4i+3}			$(0 \le i \le n-1)$
y_{4i+1}	:	x_{4i}	$=$	x_{4i+1}			$(0 \le i \le n-1)$
y_{4i+2}	:	x_{4i+1}	$=$	x_{4i+2}			$(0 \le i \le n-1)$
y_{4i+3}	:	x_{4i+2}	$=$	x_{4i+3}			$(0 \le i \le n-1)$
c_j^s	:	p_j^s	$>$	$x(c_j^s)$	$>$	q_j^s	$(1 \le j \le m,\ 1 \le s \le 3)$
z_j	:	p_j^1	$=$	p_j^2	$=$	p_j^3	$(1 \le j \le m)$

Next, we consider a transformation from B to an instance of Class 3. Let I' be the instance obtained from I by the replacing preference lists of P and Z, as follows:

$$
\begin{array}{lllllll}
p_j^s & : & z_j & = & c_j^s & & (1 \le j \le m,\ 1 \le s \le 3);\\
z_j & : & p_j^1 & > & p_j^2 \ > \ p_j^3 & & (1 \le j \le m).
\end{array}
$$

Then I' is in Class 3. The following lemma holds.

Lemma 4.1. An instance of (3, 3)-MAX SMTI I admits a perfect stable matching if and only if the instance I' admits a perfect stable matching.

Proof. Let M be a perfect matching on $(U, Y; \mathbb{A})$.

($\Rightarrow$): Suppose that M is stable in I and not stable in I'. Let (m, w) be a blocking pair for M in I'. If $m \notin P$ and $w \notin Z$, then (m, w) is a blocking pair for M in I, and so we obtain $m \in P$ or $w \in Z$. Since each man in P has no strict preference in I', he cannot be in blocking pairs in I'. Note that w's partner must belong to P if $w \in Z$. Thus there is no blocking pair for M in I'.

($\Leftarrow$): Suppose that M is stable in I' and is not stable in I. Let (m, w) be a blocking pair for M in I. We obtain $m \in P$ by similar arguments as above. If $w \in C'$, then $pri_U(m, w) > pri_U(m, M(m))$ does not hold for any woman $M(m)$ because w is the worst woman in m's preference list. On the other hand, each woman in Z has no strict preference in I, and so she cannot be in blocking pairs in I'. Thus there is no blocking pair for M in I. □

Corollary 4.1. PER SMTI for Class 3 is NP-complete.

Transformation from PER SMTI for Class 3 to PER SMTI for Class 1

Let us prove the NP-completeness of PER SMTI for Class 1 by using a reduction from PER SMTI for Class 3.

Algorithm 1 shows a pseudo-code of the algorithm for transforming an instance in Class 3 to an instance in Class 1. The algorithm consists of the two phases. In Phase 1, we remove the pairs which are contained in all stable matchings, in the input instance. To remove a pair (m, w), we replace U_I and Y_I by $U_I \setminus \{m\}$ and $Y_I \setminus \{w\}$, respectively. Let P_U be the set of pairs (m, w) such that m is the unique first choice of w, and w is the unique first choice of m in the input instance. Then, we delete all pairs in P_U.

In Phase 2, we delete the pairs which do not belong to any stable matching in the input instance. To delete a pair (m, w), we replace their priority functions with $pri_{U_I}(m, w) = 0$ and $pri_{Y_I}(m, w) = 0$.

Lemma 4.2. Let M be a perfect stable matching in I_3. Then the following two conditions hold:

(a) $P_U \subseteq M$;

(b) M does not contain pairs which are deleted in Phase 2.

Lemma 4.3. I_3 admits a perfect stable matching if and only if I_1 admits a perfect stable matching.

The outline of our proof for Lemma 4.3 is as follows.

($\Rightarrow$): Let M be a perfect stable matching in I_3, and M' be a matching in I_1 defined as $M' := M \setminus P_U$. We can show that M' is a perfect stable matching in I_1.

($\Leftarrow$): Let M' be a perfect stable matching in I_1, and M a matching in I_3 defined as $M := M' \cup P_U$. Then M is a perfect matching in I_3. Suppose that there exists a blocking pair (m, w) for M in I_3. We can show that if (m, w) is a blocking pair for M in I_3, then (m, w) is also a blocking pair for M' in I_1, which is a contradiction.

Theorem 4.1. MAX SMTI for Class 1 is NP-hard. This result holds even if each person's preference list is of length at most three.

Algorithm 1 A transformation from Class 3 to Class 1

Input: An instance I_3 of Class 3;
Output: An instance $I_1$of Class 1;

$I \leftarrow I_3$;
/* Phase 1 */
$P_U \leftarrow \{(m, w) \mid m$ is the unique first choice of w in I_3, and w is the unique first choice of m in $I_3\}$;
remove all pairs in P_U from I;
/* Phase 2 */
for each man m in I whose preference list contains three women in I **do**
  $w_1 \leftarrow$ the first choice of m's preference list in I;
  $w_2 \leftarrow$ the second choice of m's preference list in I;
  for each man m' in I where $pri_{Y_{I_3}}(w_1, m) > pri_{Y_{I_3}}(w_1, m')$ **do**
    delete the pair (m', w_1) from I;
  $pri_{U_I}(m, w_2) \leftarrow pri_{U_I}(m, w_1)$;
for each woman m in I whose preference list contains three men in I **do**
  $m_1 \leftarrow$ the first choice of w's preference list in I;
  $m_2 \leftarrow$ the second choice on w's preference list in I;
  for each woman w' in I where $pri_{U_{I_3}}(m_1, w) > pri_{U_{I_3}}(m_1, w')$ **do**
    delete the pair (w', m_1) from I;
  $pri_{Y_I}(w, m_2) \leftarrow pri_{Y_I}(w, m_1)$;
$I_1 \leftarrow I$;
return I_1;

5. A polynomial time algorithm for Class 2

In this section, we develop an algorithm to solve MAX SMTI for Class 2 in polynomial time. The algorithm is almost the same as the one proposed in [8]. We can show that it solves a more general instance class of the problem.

Algorithm 2 shows a pseudo-code for the algorithm. It consists of three phases. In Phase 1, we delete the pairs which do not belong to any stable matching. To delete a pair (m, w), we rewrite their priority functions as $pri_U(m, w) = 0$ and $pri_Y(w, m) = 0$. Let $m_i \in U$ and w_j be the unique first choice of m_i. If there exists a strict successor m_k of m_i on w_j's preference list (i.e., $pri_Y(w_j, m_i) > pri_Y(w_j, m_k) \neq 0$), then (m_k, w_j) is deleted. Let $\mathbb{A}'$ denotes the set of pairs of a man and a woman which are still acceptable to each other after Phase 0 and Phase 1. In Phase 2, we find a minimum cost maximum cardinality matching M_G on the bipartite graph $(U, Y; \mathbb{A}')$ where the cost of an edge (m, w) is $-pri_Y(w, m)$. This procedure terminates within polynomial time [3].

The cardinality of M_G is the same as the maximum cardinality stable matching in the input instance, but there can be a pair $(m, w) \in \mathbb{A}'$ which satisfies (3.2) with respect to M_G. To obtain a stable matching, we remove those pairs from the matching in Phase 3. We set $M := M_G$ and iterate the following process. Let m_i be a man who is matched in M and has the unique first choice, say w_j. If w_j is single in M, then we replace M by $(M \cup \{(m_i, w_j)\}) \setminus \{(m_i, M(m_i))\}$. We refer to this procedure as **STABILIZE**. The procedure is repeated until no such a man exists.

Let us start analyzing of the algorithm. First, we discuss the correctness of Phase 1.

Lemma 5.1. There exists no stable matching in the input instance which contains a pair deleted in Phase 1.

Algorithm 2 A polynomial time algorithm for MAX SMTI for Class 2

Input: an instance of Class 2
Output: a matching M

/* Phase 1 */
set all men to be unmarked;
while some man m_i is unmarked **and** m_i has a non-empty preference list **do**
  set m_i to be marked;
  if m_i has the unique first choice **then**
    $w_j \leftarrow$ the unique first choice of m_i;
    for each strict successor m_k of m_i in w_j's preference list **do**
      set m_k unmarked;
      delete the pair (m_k, w_j);
/* Phase 2 */
$G \leftarrow (U, Y, \mathbb{A}')$;
$M_G \leftarrow$ a minimum cost maximum cardinality matching in G;
/* Phase 3 */
$M \leftarrow M_G$;
while there exists a man m_i assigned to his second-choice woman w_k in M **and** the woman w_j who is the unique first choice of m_i is unassigned in M **do**
  $M \leftarrow (M \setminus \{(m_i, w_k)\}) \cup \{(m_i, w_j)\}$;
return M;

Proof. Let M be a stable matching and $(m_k, w_j) \in M$ a pair which is deleted in Phase 1. According to the algorithm, there must be a man $m_i \in U$ whose unique first choice is w_j, and m_k is a strict successor of m_i in w_j's preference list (i.e., $pri_Y(w_j, m_i) > pri_Y(w_j, m_k) \Leftrightarrow pri_Y(w_j, m_i) > pri_Y(w_j, M(w_j))$). If m_i matches someone in M, then $pri_U(m_i, w_j) > pri_U(m_i, M(m_i))$ because w_j is the unique first choice of m_i. Hence, (m_i, w_j) is a blocking pair for M, regardless of whether m_i is matched in M or not. This contradicts the stability of M. □

Next, we discuss the stability of the output matching.

Lemma 5.2. No pair $(m, w) \in \mathbb{A}'$ satisfies any of the three conditions (3.1), (3.3) and (3.4) with respect to M_G.

Lemma 5.3. Let $M \subset \mathbb{A}'$ be a matching which we obtain after some executions of STABILIZE for M_G, and let $M' \subset \mathbb{A}'$ be a matching which we obtain after an execution of STABILIZE for M. If no pair $(m, w) \in \mathbb{A}'$ satisfies any of the three conditions (3.1), (3.3) and (3.4) with respect to M, then M' has the same property.

Note that no pair $(m, w) \in \mathbb{A}'$ satisfies (3.2) with respect to the output matching of Phase 3. Hence, it follows from Lemma 5.2 and Lemma 5.3 that the output matching of the algorithm is stable.

The rest to be discussed is the finiteness of the algorithm. It is easy to see that both Phase 1 and Phase 2 terminate in polynomial time. In each execution of STABILIZE, exactly one man switches his current partner to his unique first choice. Hence, STABILIZE is executed at most $|U|$ times, and thus we obtain the following conclusion.

Theorem 5.1. Given an instance of MAX SMTI in Class 2, Algorithm 2 returns a maximum cardinality stable matching in a polynomial time.

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