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Properties of the solutions of certain differential equations

Hitoshi SAITOH
Department of Mathematics, National Institute of Technology, Gunma College, Maebashi, Gunma 371-8530, Japan
e-mail: saitoh@nat.gunma-ct.ac.jp

Abstract

The main object of this paper is to investigate several geometric properties of the solutions of second order ordinary differential equations.

1. Introduction

Let A denote the class of functions $f(z)$ normalized by

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n, \qquad (1.1)$$

which are analytic in the open unit disk

$$U=\{z: z\in\mathbb{C} \text{ and } |z|<1\}.$$

Also, let S, S^* and $S^*(\alpha)$ denote the subclasses of A consisting of functions which are, respectively, univalent, starlike with respect to the origin, and starlike of order α in U $(0 \leqq \alpha < 1)$. Thus, by definition, we have (see for detail [1, 4]),

$$S^*(\alpha) := \left\{ f : f \in A \text{ and } \operatorname{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \ (z \in U; 0 \leqq \alpha < 1 \right\} \quad (1.2)$$

and

$$S^* := S^*(0). \quad (1.3)$$

Furthermore, S_P denote the subclasses of A with the property

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < \operatorname{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} \quad (z \in U), \quad (1.4)$$

and UCV denote the subclasses of A with the property

$$\left| \frac{zf''(z)}{f'(z)} \right| < \operatorname{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \quad (z \in U). \quad (1.5)$$

Remark 1. (1) $f(z) \in UCV \Leftrightarrow zf'(z) \in S_P$.

(2) $S_P \subset S^*(\frac{1}{2})$

2. A class of bounded functions and earlier results

Let B_J denote the class of bounded functions

$$w(z) = \sum_{n=1}^{\infty} c_n z^n, \quad (2.1)$$

analytic in U, for which

$$|w(z)| < J \quad (z \in U;\ J > 0). \tag{2.2}$$

<u>Definition 1</u> Let H_J be the class of complex functions $h(u,v)$ satisfying each of the following conditions:

(i) $h(u,v)$ is continuous in a domain $D \subset \mathbb{C} \times \mathbb{C}$;

(ii) $(0,0) \in D$ and $|h(0,0)| < J$ $(J > 0)$;

(iii) $|h(Je^{i\theta}, Ke^{i\theta})| < J$ whenever $(Je^{i\theta}, Ke^{i\theta}) \in D$

$(\theta \in \mathbb{R},\ K \geqq J > 0)$.

<u>Example 1.</u> It is easily seen that the functions

(1) $h(u,v) = \gamma u + v \in H_J$, $\gamma \in \mathbb{C}$ $(\mathrm{Re}\,\gamma \geqq 0)$, $D = \mathbb{C} \times \mathbb{C}$.

(2) $h(r,s) = r^2 + r + s \in H_J$, $D = \mathbb{C} \times \mathbb{C}$

<u>Definition 2</u> Let $h \in H_J$ with corresponding domain D. We denote by $B_J(h)$ the class of functions $w(z)$ given by (2.1), which are analytic in U and satisfy each of following conditions:

(i) $(w(z), zw'(z)) \in D$ $(z \in U)$,

(ii) $|h(w(z), zw'(z))| < J$ $(z \in U;\ J > 0)$.

The function class $B_J(h)$ is not empty. Indeed, for any $h \in H_J$, we have

$$w(z) = c_1 z \in B_J(h), \tag{2.3}$$

for sufficiently small $|c_1|$ depending on h.

We need the following lemmas to prove our results.

Lemma 1 (see [6]) For any $h \in H_J$,

$$B_J(h) \subset B_J \quad (h \in H_J;\ 0 < J \leqq 1).$$

Lemma 1 leads us immediately to the following result, which also given by [6].

Lemma 2 ([6]) Let $h \in H_J$ and let the function $b(z)$ be analytic in U with

$$|b(z)| < J \quad (z \in U;\ 0 < J \leqq 1).$$

If the following initial-value problem:

$$h(w(z), zw'(z)) = b(z) \quad (w(0) = 0) \tag{2.4}$$

has a solution $w(z)$ analytic in U, then

$$|w(z)| < J \quad (z \in U;\ 0 < J \leqq 1). \tag{2.5}$$

Using Lemma 2, we proved several results.

For example,

<u>Theorem A</u> (See [6]) Let $a(z)$ and $b(z)$ be analytic in U with

$$\left|z\left(b(z)-\frac{1}{2}a'(z)-\frac{1}{4}[a(z)]^2\right)\right|<\frac{1}{2} \tag{2.6}$$

and

$$|a(z)|<1. \tag{2.7}$$

Let $w(z)$ denote the solution of the initial-value problem:

$$w''(z)+a(z)w'(z)+b(z)w(z)=0 \quad . \tag{2.8}$$
$$(z\in U,\ w(0)=w'(0)-1=0)$$

Then $w(z)$ is starlike in U.

<u>Example 2</u> Let $a(z)=-z$, $b(z)=\frac{z^2}{4}$ in Theorem A, then a solution of

$$w''(z)-zw'(z)+\frac{z^2}{4}w(z)=0 \tag{2.9}$$

is $w(z)=\sqrt{2}\,e^{\frac{z^2}{4}}\sin\frac{z}{\sqrt{2}}$. This function $w(z)$ is starlike function.

3 Main results and their consequences

<u>Theorem 1</u> Let $a(z)$ and $b(z)$ be analytic in U with

$$\left| z\left(b(z)-\frac{1}{2}a'(z)-\frac{1}{4}[a(z)]^2\right)\right| < J \quad (z\in U;\ 0<J<1) \qquad (3.1)$$

and

$$|a(z)| \leqq K \quad (0<K\leqq 2-2J). \qquad (3.2)$$

Let $w(z)$ $(z\in U)$ be the solution of the initial-value problem (2.8). Then, $w(z)$ is starlike of order $1-J-\frac{K}{2}$.

Proof. The transformation

$$w(z)=\exp\left(-\frac{1}{2}\int_0^z a(\zeta)d\zeta\right)v(z) \qquad (3.3)$$

leads to the normal form

$$v''(z)+\left(b(z)-\frac{1}{2}a'(z)-\frac{1}{4}[a(z)]^2\right)v(z)=0 \qquad (3.4)$$

and $v(0)=v'(0)-1=0$. If we put

$$u(z)=\frac{zv'(z)}{v(z)}-1 \quad (z\in U), \qquad (3.5)$$

then $u(z)$ is analytic in U, $u(0)=0$ and (3.4) becomes

$$[u(z)]^2+u(z)+zu'(z)=-z^2\left(b(z)-\frac{1}{2}a'(z)-\frac{1}{4}[a(z)]^2\right), \qquad (3.6)$$

or equivalently

$$h(u(z), zu'(z)) = -z^2\left(b(z) - \frac{1}{2}a'(z) - \frac{1}{4}[a(z)]^2\right), \qquad (3.7)$$

where $h(r,s) = r^2 + r + s$. It is easy to check

$h(r,s) \in H_J$, i.e.,

(i) $h(r,s)$ is continuous in $\mathbb{C} \times \mathbb{C}$;

(ii) $(0,0) \in \mathbb{C} \times \mathbb{C}$, $|h(0,0)| = 0 < J$;

(iii) $|h(Je^{i\theta}, Ke^{i\theta})| \geqq J$ $(K \geqq J)$.

From (3.1), we have

$$\left|-z^2\left(b(z) - \frac{1}{2}a'(z) - \frac{1}{4}[a(z)]^2\right)\right| < J \quad (z \in U).$$

By using Lemma 2, we obtain $|u(z)| < J \ (z \in U)$.

Therefore, we have

$$\left|\frac{zv'(z)}{v(z)} - 1\right| < J \quad (z \in U).$$

This implies

$$1 - J < \mathrm{Re}\left\{\frac{zv'(z)}{v(z)}\right\} < 1 + J \quad (z \in U). \qquad (3.8)$$

From (3.3), we have

$$\exp\left(\frac{1}{2}\int_0^z a(\xi)d\xi\right) \cdot w(z) = v(z). \qquad (3.9)$$

Logarithmically differentiating of (3.9) leads to

$$\frac{zw'(z)}{w(z)} = \frac{zv'(z)}{v(z)} - \frac{z}{2}a(z). \qquad (3.10)$$

Combing (3.8), (3.10) and (3.2), we have

$$\operatorname{Re}\left\{\frac{zw'(z)}{w(z)}\right\} \geqq \operatorname{Re}\left\{\frac{zv'(z)}{v(z)}\right\} - \left|\frac{z}{2}a(z)\right|$$

$$> 1 - J - \frac{K}{2} \quad (z \in U), \tag{3.11}$$

where $0 < 2J + K \leqq 2$, and thus $w(z)$ is starlike of order $1 - J - \frac{K}{2}$. Q.E.D.

Example 3. Let $a(z) = -\frac{2}{3}z$, $b(z) = \frac{z^2}{9}$ in Theorem 1, then the solution of

$$w''(z) - \frac{2}{3}z\,w'(z) + \frac{z^2}{9}w(z) = 0 \tag{3.12}$$

is $w(z) = \sqrt{3}\, e^{\frac{z^2}{6}} \cdot \sin\frac{z}{\sqrt{3}} \in S^*\left(\frac{1}{3}\right)$.

Next, we prove

Theorem 2 Let $a(z)$ and $b(z)$ be analytic in U with

$$\left|z\left(b(z) - \frac{1}{2}a'(z) - \frac{1}{4}[a(z)]^2\right)\right| < J \quad \left(z \in U,\ 0 < J < \frac{1}{2}\right) \tag{3.13}$$

and

$$|a(z)| \leqq K \quad (0 < K \leqq 1 - 2J). \tag{3.14}$$

Let $w(z)$ $(z \in U)$ be the solution of the initial-value problem (2.8). Then, $w(z) \in S_P$.

Proof. From (3.10) in the proof of Theorem 1,

$$\frac{zw'(z)}{w(z)}-1=\frac{zv'(z)}{v(z)}-1-\frac{z}{2}a(z).$$

Then we have

$$\left|\frac{zw'(z)}{w(z)}-1\right| \leqq \left|\frac{zv'(z)}{v(z)}-1\right|+\left|\frac{z}{2}a(z)\right|$$
$$< J+\frac{K}{2} \quad (z\in U). \tag{3.15}$$

From (3.11) and (3.15), we obtain

$$\mathrm{Re}\left\{\frac{zw'(z)}{w(z)}\right\} > \left|\frac{zw'(z)}{w(z)}-1\right| \quad (0<2J+K\leqq 1, \; z\in U), \tag{3.16}$$

that is, $w(z)\in S_P$. Q.E.D.

Example 4 Let $a(z)=-\frac{z}{2}$ and $b(z)=\frac{z^2}{16}$ in Theorem 2, the solution of

$$w''(z)-\frac{z}{2}w'(z)+\frac{z^2}{16}w(z)=0 \tag{3.17}$$

is $w(z)=2e^{\frac{z^2}{8}}\sin\frac{z}{2}\in S_P$.

Also, $2e^{\frac{z^2}{8}}\sin\frac{z}{2}\in S^*(\frac{1}{2})$.

Furthermore, we prove the following theorems.

Theorem 3 Let $zp(z)$ be analytic in U with $|zp(z)|<J$ $(z\in U;\ 0<J\leqq 1)$. Let $w(z)$, $z\in U$, be the

solution of the following differential equation

$$w''(z)+p(z)w(z)=0 \tag{3.18}$$

with $w(0)=0$ and $w'(z)\neq 0$. Then the solution $w(z)$ is starlike of order $1-J$, that is,

$$\mathrm{Re}\left\{\frac{zw'(z)}{w(z)}\right\} > 1-J \quad (z\in U;\ 0<J\leqq 1). \tag{3.19}$$

Proof. We put

$$u(z)=\frac{zw'(z)}{w(z)}-1 \quad (z\in U). \tag{3.20}$$

Then $u(z)$ is analytic in U, $u(0)=0$ and (3.18) becomes

$$[u(z)]^2+u(z)+zu'(z)=-z^2p(z), \tag{3.21}$$

or equivalently

$$h(u(z),zu'(z))=-z^2p(z), \tag{3.22}$$

where $h(r,s)=r^2+r+s$. It is easy to check $h(r,s)\in H_J$. From assumption, we have

$$|z^2p(z)|<J \quad (z\in U;\ 0<J\leqq 1).$$

By using Lemma 2, we obtain

$$|u(z)|<J \quad (z\in U;\ 0<J\leqq 1),$$

which, in view of the relationship (3.20), yields

$$\left| \frac{z w'(z)}{w(z)} - 1 \right| < J \quad (z \in U;\ 0 < J \leqq 1), \tag{3.23}$$

that is,

$$\mathrm{Re}\left\{ \frac{z w'(z)}{w(z)} \right\} > 1 - J .$$

This means $w(z) \in S^*(1-J)$. Q.E.D.

<u>Remark 2</u> In[7], Shams, Kulkarni and Jahangiri introduced the following class $SD(\alpha,\beta)$.

Let $SD(\alpha,\beta)$ be the family of functions $f(z) \in A$ satisfying the inequality

$$\mathrm{Re}\left\{ \frac{z f'(z)}{f(z)} \right\} > \alpha \left| \frac{z f'(z)}{f(z)} - 1 \right| + \beta \tag{3.24}$$

$$(z \in U,\ \alpha \geqq 0,\ 0 \leqq \beta < 1).$$

We can see $SD(1,0) = S_p$.

Next, we prove the following theorem.

<u>Theorem 4</u> Let $zp(z)$ be analytic in U with $|zp(z)| < J\ (z \in U;\ 0 < J \leqq 1)$. Let $w(z)$, $z \in U$,

be the solution of differential equation (3.18).

Then the solution $w(z)$ is in $SD(\alpha,\beta)$, i.e.,

$$\mathrm{Re}\left\{\frac{zw'(z)}{w(z)}\right\} > \alpha\left|\frac{zw'(z)}{w(z)}-1\right|+\beta \tag{3.25}$$

$$\left(z\in U;\ 0<J\leqq\frac{1-\beta}{1+\alpha}\right),$$

where $\alpha\geqq 0$ and $0\leqq\beta<1$.

Proof. According to the proof of Theorem 3, we have (3.23). Therefore,

$$\mathrm{Re}\left\{\frac{zw'(z)}{w(z)}\right\} > 1-J \geqq \alpha J+\beta \geqq \alpha\left|\frac{zw'(z)}{w(z)}-1\right|+\beta$$

$$\left(0<J\leqq\frac{1-\beta}{1+\alpha},\ \alpha\geqq 0,\ 0\leqq\beta<1\right).$$

That is $w(z)\in SD(\alpha,\beta)$. Q.E.D.

<u>Example 5</u> (1) Let $p(z)=\frac{6}{7}-\frac{4}{49}z^2$ in Theorem 3 and Theorem 4. Then

$$|z\,p(z)| < \frac{46}{49},$$

therefore,

$$\mathrm{Re}\left\{\frac{zw'(z)}{w(z)}\right\} > \frac{3}{49}, \text{ that is, } w(z)\in S^*\left(\frac{3}{49}\right).$$

And the solution $w(z)$ of

$$w''(z)+\left(\frac{6}{7}-\frac{4}{49}z^2\right)w(z)=0$$

is $w(z)=ze^{-\frac{z^2}{7}}$. Also,

$$\mathrm{Re}\left\{\frac{zw'(z)}{w(z)}\right\}>\frac{3}{14}\left|\frac{zw'(z)}{w(z)}-1\right|,$$ i.e., $w(z)\in SD(\frac{3}{14},0)$.

Furthermore, $w(z)\in SD(\frac{1}{7},\frac{1}{49})$.

(2) Let $p(z)=\frac{6}{13}-\frac{4}{169}z^2$ in Theorem 3, Theorem 4.

Then $|zp(z)|<\frac{82}{169}$, therefore

$\mathrm{Re}\left\{\frac{zw'(z)}{w(z)}\right\}>\frac{87}{169}$, that is, $w(z)\in S^*(\frac{87}{169})$.

And the solution $w(z)$ of

$$w''(z)+\left(\frac{6}{13}-\frac{4}{169}z^2\right)w(z)=0$$

is $w(z)=ze^{-\frac{z^2}{13}}$. Furthermore,

$$\mathrm{Re}\left\{\frac{zw'(z)}{w(z)}\right\}>3\left|\frac{zw'(z)}{w(z)}-1\right|+\frac{9}{169},$$ i.e.,

$w(z)\in SD(3,\frac{9}{169})$. Also, $w(z)\in SD(2,\frac{35}{169})$

and $w(z)\in SD(1,\frac{61}{169})$.

Putting $p(z)=\lambda+\frac{1}{2}-\frac{z^2}{4}$ in Theorem 3, Theorem 4, we have

<u>Corollary 1</u> We consider the Weber's differential equation

$$v''(z) + \left(\lambda + \frac{1}{2} - \frac{z^2}{4}\right) v(z) = 0 \qquad (3.26)$$

If $\left|\lambda + \frac{1}{2} - \frac{z^2}{4}\right| < J$ ($z \in U$; $0 < J \leqq 1$),
then the solution $v(z)$ is starlike of order $1-J$,
that is, $v(z) \in S^*(1-J)$.

<u>Corollary 2</u> We consider the Weber's differential equation (3.23).
If $\left|\lambda + \frac{1}{2} - \frac{z^2}{4}\right| < J$ ($z \in U$; $0 < J \leqq \frac{1-\beta}{1+\alpha}$),
where $\alpha \geqq 0$ and $0 \leqq \beta < 1$, then $w(z) \in SD(\alpha, \beta)$.

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