数理解析研究所講究録
第 1982 巻 2016 年 50-62

Fractional calculus of analytic functions concerned with Möbius transformations

Nicoleta Breaz, Daniel Breaz and Shigeyoshi Owa

1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_{k}z^{k} \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$.
If $f(z)\in \mathcal{A}$ satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)>\alpha \qquad (z\in \mathbb{U}) \tag{1.2}$$

for some real $\alpha\,(0\leqq\alpha<1)$, then $f(z)$ is said to be starlike of order α in $\mathbb{U}$. We denote by $\mathcal{S}^{*}(\alpha)$ the class of all starlike functions $f(z)$ of order α in $\mathbb{U}$ and $\mathcal{S}^{*}(0)\equiv \mathcal{S}^{*}$. Furthermore, if $f(z)\in \mathcal{A}$ satisfies

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha \qquad (z\in \mathbb{U}) \tag{1.3}$$

for some real $\alpha\,(0\leqq\alpha<0)$, then we say that $f(z)$ is convex of order α in $\mathbb{U}$. We also denote by $\mathcal{K}(\alpha)$ the class of all such functions $f(z)$ and $\mathcal{K}(0)\equiv \mathcal{K}$. In view of definitions for the classes $\mathcal{S}^{*}(\alpha)$ and $\mathcal{K}(\alpha)$, we know that
(i) $f(z)\in \mathcal{K}(\alpha)$ if and only if $zf'(z)\in \mathcal{S}^{*}(\alpha)$
and
(ii) $f(z)\in \mathcal{S}^{*}(\alpha)$ if and only if $\int_{0}^{z}\frac{f(t)}{t}dt\in \mathcal{K}(\alpha)$. Further, MacGregor [4] and Wilken and Feng [14] have the sharp inclusion relation that $\mathcal{K}(\alpha)\subset \mathcal{S}^{*}(\beta)$ for each $\alpha(0\leqq\alpha<1)$ with

$$\beta=\begin{cases}\dfrac{1-2\alpha}{2^{2(1-\alpha)}(1-2^{2\alpha-1})} & (\alpha\neq\dfrac{1}{2})\\[2ex] \dfrac{1}{2\log 2} & (\alpha=\dfrac{1}{2}).\end{cases} \tag{1.4}$$

For $\alpha=0$, Marx [5] and Strohhäcker [13] showed that $\mathcal{K}\subset \mathcal{S}^{*}(\frac{1}{2})$. Also, by Robertson [12], we know that the extremal function $f(z)$ for the class $\mathcal{S}^{*}(\alpha)$ is

$$f(z)=\frac{z}{(1-z)^{2(1-\alpha)}}=z+\sum_{k=2}^{\infty}\frac{\prod_{j=2}^{k}(j-2\alpha)}{(k-1)!}z^{k} \tag{1.5}$$

and the extremal function $f(z)$ for the class $\mathcal{K}(\alpha)$ is

(1.6) $$f(z) = \begin{cases} \dfrac{1-(1-z)^{2\alpha-1}}{2\alpha-1} = z + \displaystyle\sum_{k=2}^{\infty} \dfrac{\prod_{j=2}^{k}(j-2\alpha)}{k!} z^k & (\alpha \neq \frac{1}{2}) \\ -\log(1-z) = z + \displaystyle\sum_{k=2}^{\infty} \dfrac{1}{k} z^k & (\alpha = \frac{1}{2}). \end{cases}$$

For $f(z) \in \mathcal{A}$, we apply the following Möbius transformation

(1.7) $$w(\zeta) = \frac{z+\zeta}{1+\bar{z}\zeta} \qquad (\zeta \in \mathbb{U})$$

for a fixed $z \in \mathbb{U}$. This Möbius transformation $w(\zeta)$ maps $\mathbb{U}$ onto itself and $\zeta = 0$ to $w(0) = z$.

2 Fractional calculus

From among the various definitions for fractional calculus (that is, fractional derivatives and fractional integrals) given in the literature, we have to recall here the following definitions for fractional calculus which are used by Owa [8], [9] and by Owa and Srivastava [10].

Definition 2.1 *The fractional integral of order λ is defined, for $f(z) \in \mathcal{A}$, by*

(2.1) $$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta,$$

where $\lambda > 0$ and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z - \zeta > 0$.

Definition 2.2 *The fractional derivative of order λ is defined, for $f(z) \in \mathcal{A}$, by*

(2.2) $$D_z^{\lambda} f(z) = \frac{d}{dz}\left(D_z^{\lambda-1} f(z)\right) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta,$$

where $0 \leqq \lambda < 1$ and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed as in Definition 2.1 above.

Definition 2.3 *Under the hypotheses of Definition 2.2, the fractional derivative of order $n + \lambda$ is defined by*

(2.3) $$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n}\left(D_z^{\lambda} f(z)\right),$$

where $0 \leqq \lambda < 1$ and $n \in \mathbb{N}_0 = 0, 1, 2, \cdots$.

Remark 2.1 In view of definitions for the fractional calculus of $f(z) \in \mathcal{A}$, we see that

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(2+\lambda)} z^{1+\lambda} + \frac{2!}{\Gamma(3+\lambda)} a_2 z^{2+\lambda} + \cdots + \frac{k!}{\Gamma(k+1+\lambda)} a_k z^{k+\lambda} + \cdots \tag{2.4}$$

$$= \frac{1}{\Gamma(2+\lambda)} z^{1+\lambda} + \sum_{k=2}^{\infty} \frac{k!}{\Gamma(k+1+\lambda)} a_k z^{k+\lambda} \quad (\lambda > 0),$$

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(2-\lambda)} z^{1-\lambda} + \frac{2!}{\Gamma(3-\lambda)} a_2 z^{2-\lambda} + \cdots + \frac{k!}{\Gamma(k+1-\lambda)} a_k z^{k-\lambda} + \cdots \tag{2.5}$$

$$= \frac{1}{\Gamma(2-\lambda)} z^{1-\lambda} + \sum_{k=2}^{\infty} \frac{k!}{\Gamma(k+1-\lambda)} a_k z^{k-\lambda} \quad (0 \leqq \lambda < 1),$$

and

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} \left(\frac{1}{\Gamma(2-\lambda)} z^{1-\lambda} + \sum_{k=2}^{\infty} \frac{k!}{\Gamma(k+1-\lambda)} a_k z^{k-\lambda} \right) \tag{2.6}$$

$$= \frac{1}{\Gamma(2-n-\lambda)} z^{1-n-\lambda} + \sum_{k=2}^{\infty} \frac{k!}{\Gamma(k+1-n-\lambda)} a_k z^{k-n-\lambda}$$

for $0 \leqq \lambda < 1$ and $n \in \mathbb{N}_0$.

Therefore, we can write that

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} \left(D_z^{\lambda} f(z) \right) = D_z^{\lambda} \left(\frac{d^n}{dz^n} f(z) \right) \tag{2.7}$$

and

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(2-\lambda)} z^{1-\lambda} + \sum_{k=2}^{\infty} \frac{k!}{\Gamma(k+1-\lambda)} a_k z^{k-\lambda} \tag{2.8}$$

for any real number λ.

Using the fractional calculus (2.8), we define

$$F(z) = \Gamma(2-\lambda) z^{\lambda} D_z^{\lambda} f(z) = z + \sum_{k=2}^{\infty} \frac{k!\Gamma(2-\lambda)}{\Gamma(k+1-\lambda)} a_k z^k \quad (\lambda \in \mathbb{R}, \lambda \neq 2). \tag{2.9}$$

If we take $\lambda = -1$ in (2.9), then

$$F(z) = \Gamma(3) z^{-1} D_z^{-1} f(z) = \frac{2}{z} \int_0^z f(t) dt = z + \sum_{k=2}^{\infty} \frac{2}{k+1} z^k$$

implies the Libera integral operator defined by Libera [3]. Therefore, $F(z)$ given by (2.9) is the generalization operator of Libera integral operator.

Let us give two examples for the fractional operator $F(z)$ defined in (2.9).

Example 2.1 Let us define $f(z)$ by

$$f(z) = z + \frac{2-\lambda}{6} z^2 \in \mathcal{A} \qquad (-1 \leqq \lambda < 2). \tag{2.10}$$

Then, we have that

$$\begin{aligned} \text{Re}\left(\frac{zf'(z)}{f(z)}\right) &= \text{Re}\left(2 - \frac{1}{1+Mz}\right) \\ &= 2 - \frac{1 + M\cos\theta}{1 + M^2 + 2M\cos\theta} \qquad (z = e^{i\theta}), \end{aligned} \tag{2.11}$$

where $M = \dfrac{2-\lambda}{6} > 0$. If we define

$$h(t) = \frac{1 + Mt}{1 + M^2 + 2Mt} \qquad (t = \cos\theta), \tag{2.12}$$

then

$$h'(t) = \frac{M(M+1)(M-1)}{(1 + M^2 + 2Mt)^2} < 0 \qquad (0 < M \leqq \frac{1}{2}). \tag{2.13}$$

This shows us that

$$h(t) \leqq h(-1) = \frac{1}{1-M}, \tag{2.14}$$

that is, that

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > 2 - \frac{1}{1-M} = \frac{2+2\lambda}{4+\lambda} > 0 \qquad (z \in \mathbb{U}). \tag{2.15}$$

Therefore, $f(z) \in \mathcal{S}^*\left(\dfrac{2+2\lambda}{4+\lambda}\right)$.

Let us define $F(z)$ by

$$F(z) = \Gamma(2-\lambda) z^\lambda D_z^\lambda f(z) = z + \frac{1}{3} z^2 \qquad (-1 \leqq \lambda < 2). \tag{2.16}$$

Then, we see that $F(z) \in \mathcal{S}^*\left(\dfrac{1}{2}\right)$.

Next, let us consider the function $g(\zeta)$ given by

$$g(\zeta) = \frac{(F \circ w)(\zeta) - F(z)}{(1 - |z|^2) F'(z)} \qquad (\zeta \in \mathbb{U}) \tag{2.17}$$

for a fixed $z \in \mathbb{U}$, where $w(\zeta)$ is given by (1.7). Then, it is easy to see that $g(\zeta) \in \mathcal{A}$. Taking $z = \dfrac{1}{2}$ in (2.17), we have that

$$g(\zeta) = \frac{\zeta(11\zeta + 16)}{4(\zeta + 2)^2} \qquad (\zeta \in \mathbb{U}) \tag{2.18}$$

and

$$\text{Re}\left(\frac{\zeta g'(\zeta)}{g(\zeta)}\right) = \text{Re}\left(1 - \frac{\zeta(11\zeta+10)}{(\zeta+2)(11\zeta+16)}\right) \tag{2.19}$$

$$= 1 - \frac{704\cos^2\theta + 848\cos\theta + 149}{1408\cos^2\theta + 3268\cos\theta + 1885} \quad (\theta = e^{i\theta}).$$

Letting

$$H(t) = \frac{704t^2 + 848t + 149}{1408t^2 + 3268t + 1885} \qquad (t = \cos\theta), \tag{2.20}$$

we obtain that

$$H'(t) = \frac{12(9224t^2 + 186208t + 92629)}{(1408t^2 + 3268t + 1885)^2}. \tag{2.21}$$

This shows that $H'(-1) < 0, H'(0) > 0$, and $H'(1) > 0$. Therefore, there exists some t_0 such that $H'(t_0) = 0$ for $-1 < t_0 < 0$. It follows that

$$\text{Max}_{-1 \leqq t \leqq 1} H(t) = \text{Max}\{H(-1), H(1)\} = H(1) = \frac{7}{27}. \tag{2.22}$$

Thus, we say that

$$\text{Re}\left(\frac{\zeta g'(\zeta)}{g(\zeta)}\right) > 1 - \frac{7}{27} = \frac{20}{27} \qquad (\zeta \in \mathbb{U}). \tag{2.23}$$

Consequently, we say that $F(z) \in \mathcal{S}^*\left(\frac{1}{2}\right), g(\zeta) \in \mathcal{S}^*\left(\frac{20}{27}\right)$ for $f(z) \in \mathcal{S}^*\left(\frac{2+2\lambda}{4+\lambda}\right)$ given by (2.10).

If $\lambda = -\frac{1}{2}$, then

$$f(z) = z + \frac{5}{12}z^2 \in \mathcal{S}^*\left(\frac{2}{7}\right).$$

The open unit disk $\mathbb{U}$ is mapped on the starlike domain of order $\frac{2}{7}$.

If $\lambda = \frac{1}{3}$, then

$$f(z) = z + \frac{5}{18}z^2 \in \mathcal{S}^*\left(\frac{8}{13}\right).$$

Thus, $f(z)$ maps $\mathbb{U}$ on to the starlike domain of order $\frac{8}{13}$.

Example 2.1 means that there is some function $f(z) \in \mathcal{S}^*(\alpha)$ such that $F(z) \in \mathcal{S}^*(\beta)$ and $g(\zeta) \in \mathcal{S}^*(\gamma)$.

Next, we consider

Example 2.2 Let a function $f(z)$ be given by

$$f(z) = z + \frac{2-\lambda}{12}z^2 \in \mathcal{A} \qquad (-1 \leqq \lambda < 2). \tag{2.24}$$

Then, we have that

$$\begin{aligned} \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) &= \operatorname{Re}\left(2 - \frac{1}{1+2Mz}\right) \\ &= 2 - \frac{1+2M\cos\theta}{1+4M^2+4M\cos\theta} \qquad (z = e^{i\theta}), \end{aligned} \tag{2.25}$$

where $M = \dfrac{2-\lambda}{12} > 0$. Defining $h(t)$ by

$$h(t) = \frac{1+2Mt}{1+4M^2+4Mt} \qquad (t = \cos\theta), \tag{2.26}$$

we have that

$$h'(t) = \frac{2M(2M+1)(2M-1)}{(1+4M^2+4Mt)^2} < 0 \qquad (0 < M \leqq \frac{1}{4}) \tag{2.27}$$

which shows us that

$$h(t) \leqq h(-1) = \frac{1}{1-2M}. \tag{2.28}$$

Thus, we obtain that

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 2 - \frac{1}{1-2M} = \frac{2+2\lambda}{4+\lambda} > 0 \qquad (z \in \mathbb{U}). \tag{2.29}$$

This gives us that $f(z) \in \mathcal{K}\left(\dfrac{2+2\lambda}{4+\lambda}\right)$.

For such $f(z)$, we define

$$F(z) = \Gamma(2-\lambda)z^{\lambda}D_z^{\lambda}f(z) = z + \frac{1}{6}z^2 \qquad (-1 \leqq \lambda < 2). \tag{2.30}$$

Then, it is easy to see that $F(z) \in \mathcal{K}\left(\dfrac{1}{2}\right)$.

For this $F(z)$, we consider $g(\zeta)$ defined by (2.17). If we take $z = \dfrac{1}{2}$ for $g(\zeta)$, we have that

$$g(\zeta) = \frac{\zeta(17\zeta+28)}{7(\zeta+2)^2} \qquad (\zeta \in \mathbb{U}) \tag{2.31}$$

and

$$\operatorname{Re}\left(1 + \frac{\zeta g''(\zeta)}{g'(\zeta)}\right) = \operatorname{Re}\left(1 - \frac{\zeta(10\zeta+11)}{(5\zeta+7)(\zeta+2)}\right) \tag{2.32}$$

$$= 1 - \frac{280\cos^2\theta + 379\cos\theta + 97}{280\cos^2\theta + 646\cos\theta + 370} \quad (\zeta = e^{i\theta}).$$

If we write that

$$H(t) = \frac{280t^2 + 379t + 97}{280t^2 + 646t + 370} \quad (t = \cos\theta), \tag{2.33}$$

then

$$H'(t) = \frac{24(3115t^2 + 6370t + 3232)}{(280t^2 + 646t + 370)^2}. \tag{2.34}$$

Since $H'(-1) < 0, H'(0) > 0$, and $H'(1) > 0$, there exists some t_0 such that $H'(t_0) = 0$ for $-1 < t_0 < 0$. This gives us that

$$\text{Max}_{-1 \leqq t \leqq 1} H(t) = \text{Max}\{H(-1), H(1)\} = H(1) = \frac{7}{12}. \tag{2.35}$$

It follows that

$$\text{Re}\left(1 + \frac{\zeta g''(\zeta)}{g'(\zeta)}\right) > 1 - \frac{7}{12} = \frac{5}{12} \quad (\zeta \in \mathbb{U}). \tag{2.36}$$

Therefore, we say that $F(z) \in \mathcal{K}\left(\frac{1}{2}\right), g(\zeta) \in \mathcal{K}\left(\frac{5}{12}\right)$ for $f(z) \in \mathcal{K}\left(\frac{2+2\lambda}{4+\lambda}\right)$.

If $\lambda = -\frac{2}{3}$, then

$$f(z) = z + \frac{2}{9}z^2 \in \mathcal{K}\left(\frac{1}{5}\right)$$

maps $\mathbb{U}$ on to the convex domain of order $\frac{1}{5}$.

If $\lambda = \frac{3}{2}$, then

$$f(z) = z + \frac{1}{24}z^2 \in \mathcal{K}\left(\frac{10}{11}\right).$$

This function $f(z)$ maps $\mathbb{U}$ on to the convex domain of order $\frac{10}{11}$.

Example 2.2 say that there exists some function $f(z) \in \mathcal{K}(\alpha)$ such that $F(z) \in \mathcal{K}(\beta)$ and $g(\zeta) \in \mathcal{K}(\gamma)$.

In view of Example 2.1 and Example 2.2, we introduce

Definition 2.4 *Let $f(z) \in \mathcal{A}, F(z) = \Gamma(2-\lambda)z^\lambda D_z^\lambda f(z)$ with $-1 \leqq \lambda < 2$ and let $g(\zeta)$ be defined by (2.17) for a fixed $z \in \mathbb{U}$. Then we say that*
(i) $f(z) \in \mathcal{S}_0$ if $g(\zeta)$ is univalent in $\mathbb{U}$,
(ii) $f(z) \in \mathcal{S}_0^*(\alpha)$ if $g(\zeta) \in \mathcal{S}^*(\alpha)$
and
(iii) $f(z) \in \mathcal{K}_0(\alpha)$ if $g(\zeta) \in \mathcal{K}(\alpha)$.

Also, we write that $\mathcal{S}_0^*(0) \equiv \mathcal{S}_0^*$ *and* $\mathcal{K}_0(0) \equiv \mathcal{K}_0$ *when* $\alpha = 0$.

In order to discuss our classes $\mathcal{S}_0, \mathcal{S}_0^*(\alpha)$ and $\mathcal{K}_0(\alpha)$, we need the following lemma due to Robertson [12] (also see Duren [1]).

Lemma 2.1 *If* $f(z) \in \mathcal{S}^*(\alpha)$, *then*

$$|a_k| \leqq \frac{\prod_{j=2}^{k}(j-2\alpha)}{(k-1)!} \qquad (k = 2, 3, 4, \cdots) \tag{2.37}$$

with the equality in (2.37) with $f(z)$ *given by (1.5). If* $f(z) \in \mathcal{K}(\alpha)$, *then*

$$|a_k| \leqq \frac{\prod_{j=2}^{k}(j-2\alpha)}{k!} \qquad (k = 2, 3, 4, \cdots) \tag{2.38}$$

with the equality in (2.38) with $f(z)$ *given by (1.6).*

Lemma 2.2 *If* $g(\zeta)$ *is defined by*

$$g(\zeta) = \frac{(f \circ w)(\zeta) - f(z)}{(1-|z|^2)f'(z)} \qquad (\zeta \in \mathbb{U}) \tag{2.39}$$

for a fixed $z \in \mathbb{U}$ *for* $f(z) \in \mathcal{A}$, *then*

$$\frac{\dfrac{d^n}{d\zeta^n}(f \circ w)(\zeta)}{f'(z)} \tag{2.40}$$

$$= \frac{n!(n-1)!(1+\overline{z}\zeta)^{2n}}{(1-|z|^2)^{n-1}} \left(\sum_{j=0}^{n-1} \frac{g^{(n-j)}(\zeta)\overline{z}^j}{(n-j)!(n-j-1)!j!(1+\overline{z}\zeta)^j} \right)$$

for $n = 1, 2, 3, \cdots$, *where* $w(\zeta)$ *is given by (1.7).*

Taking $\zeta = 0$ in Lemma 2.2, we have

Corollary 2.1 *If* $g(\zeta)$ *is defined by (2.39) for* $f(z) \in \mathcal{A}$, *then we have*

$$\left| \frac{f^{(n)}(z)}{f'(z)} \right| \leqq \frac{n!(n-1)!}{(1-|z|^2)^{n-1}} \left(\sum_{j=0}^{n-1} \frac{|g^{(n-j)}(0)||z|^j}{(n-j)!(n-j-1)!j!} \right) \tag{2.41}$$

for $z \in \mathbb{U}$. *Furthermore, we have*

$$\left| \frac{f''(z)}{f'(z)} \right| \leqq \frac{|g''(0)| + 2|g'(0)||z|}{1-|z|^2} \qquad (z \in \mathbb{U}). \tag{2.42}$$

Applying Corollary 2.1, we have

Theorem 2.1 *Let $F(z)$ be defined by (2.9) for $f(z) \in \mathcal{A}$ with $-1 \leqq \lambda < 2$.*
(i) If $f(z) \in \mathcal{S}_0$, then

$$\left|\frac{F^{(n)}(z)}{F'(z)}\right| \leqq \frac{n!(n+|z|)}{(1-|z|)^{n-1}(1+|z|)} \qquad (n = 1, 2, 3, \cdots) \tag{2.43}$$

with the equality for $g(\zeta)$ given by

$$g(\zeta) = \frac{\zeta}{(1+e^{i\theta}\zeta)^2} \qquad (\theta \in \mathbb{R}). \tag{2.44}$$

(ii) If $f(z) \in \mathcal{S}_0^*(\alpha)$, then

$$\left|\frac{F^{(n)}(z)}{F'(z)}\right| \leqq \frac{n!(n-1)!}{(1-|z|^2)^{n-1}} \left(\sum_{j=0}^{n-1} \frac{\prod_{k=2}^{n-j}(k-2\alpha)}{j!((n-j-1)!)^2}|z|^j\right) \tag{2.45}$$

with the equality for $g(\zeta)$ given by

$$g(\zeta) = \frac{\zeta}{(1+e^{i\theta}\zeta)^{2(1-\alpha)}} \qquad (\theta \in \mathbb{R}). \tag{2.46}$$

(iii) If $f(z) \in \mathcal{K}_0(\alpha)$, then

$$\left|\frac{F^{(n)}(z)}{F'(z)}\right| \leqq \frac{n!(n-1)!}{(1-|z|^2)^{n-1}} \left(\sum_{j=0}^{n-1} \frac{\prod_{k=2}^{n-j}(k-2\alpha)}{j!(n-j)!(n-j-1)!}|z|^j\right) \qquad (n = 1, 2, 3, \cdots) \tag{2.47}$$

with the equality for $g(\zeta)$ given by

$$g(\zeta) = \begin{cases} \dfrac{1-(1-\zeta)^{2\alpha-1}}{2\alpha-1} & \left(\alpha \neq \dfrac{1}{2}\right) \\ -\log(1-\zeta) & \left(\alpha = \frac{1}{2}\right). \end{cases} \tag{2.48}$$

Letting $n = 2$ in Theorem 2.1, we have

Corollary 2.2 *Let $F(z)$ be defined by (2.9) for $f(z) \in \mathcal{A}$ with $-1 \leqq \lambda < 2$.*
(i) If $f(z) \in \mathcal{S}_0$, then

$$\left|\frac{\lambda(\lambda-1)D_z^{\lambda}f(z) + 2\lambda z D_z^{\lambda+1}f(z) + z^2 D_z^{\lambda}f(z)}{z(\lambda D_z^{\lambda}f(z) + zD_z^{\lambda+1}f(z)}\right| \leqq \frac{2(2+|z|)}{1-|z|^2} \tag{2.49}$$

for $z \in \mathbb{U}$.
(ii) If $f(z) \in \mathcal{S}_0^*(\alpha)$, then

$$\left|\frac{\lambda(\lambda-1)D_z^{\lambda}f(z) + 2\lambda z D_z^{\lambda+1}f(z) + z^2 D_z^{\lambda}f(z)}{z(\lambda D_z^{\lambda}f(z) + zD_z^{\lambda+1}f(z)}\right| \leqq \frac{2(2(1-\alpha)+|z|)}{1-|z|^2} \tag{2.50}$$

for $z \in \mathbb{U}$.
(iii) If $f(z) \in \mathcal{K}_0(\alpha)$, then

$$\left|\frac{\lambda(\lambda-1)D_z^{\lambda}f(z) + 2\lambda z D_z^{\lambda+1}f(z) + z^2 D_z^{\lambda}f(z)}{z(\lambda D_z^{\lambda}f(z) + zD_z^{\lambda+1}f(z)}\right| \leqq \frac{2(1-\alpha+|z|)}{1-|z|^2} \tag{2.51}$$

for $z \in \mathbb{U}$.

Taking $\lambda = 0$ in Corollary 2.2, we have

Corollary 2.3 *If* $f(z) \in \mathcal{S}_0$, *then*

(2.52) $$\left|\frac{f(z)}{f'(z)}\right| \leqq \frac{2(2+|z|)}{1-|z|^2} \qquad (z \in \mathbb{U}),$$

if $f(z) \in \mathcal{S}_0^*(\alpha)$, *then*

(2.53) $$\left|\frac{f(z)}{f'(z)}\right| \leqq \frac{2(2(1-\alpha)+|z|)}{1-|z|^2} \qquad (z \in \mathbb{U}),$$

and if $f(z) \in \mathcal{K}_0(\alpha)$, *then*

(2.54) $$\left|\frac{f(z)}{f'(z)}\right| \leqq \frac{2(1-\alpha+|z|)}{1-|z|^2} \qquad (z \in \mathbb{U}),$$

3 Univalency of fractional calculus

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. Then $f(z)$ is said to be subordinate to $g(z)$, written $f(z) \prec g(z)$, if there exists a function $w(z)$ analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1\,(z \in \mathbb{U})$, and such that $f(z) = g(w(z))$. Furthermore, if $g(z)$ is univalent in $\mathbb{U}$, then the subordination $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$ (cf. Miller and Mocanu [6]).

To discuss the univalency of fractional calculus $F(z)$ given by (2.9), we need the following lemma due to Miller and Mocanu [7] (or due to Jack [2]).

Lemma 3.1 *Let the function* $w(z)$ *be analytic in* $\mathbb{U}$ *with* $w(0) = 0$. *If there exists a point* $z_0 \in \mathbb{U}$ *such that*

(3.1) $$\mathrm{Max}_{|z| \leqq |z_0|} |w(z)| = |w(z_0)|,$$

then

(3.2) $$\frac{z_0 w'(z_0)}{w(z_0)} = k$$

and

(3.3) $$\mathrm{Re}\left(1 + \frac{z_0 w''(z_0)}{w'(z_0)}\right) \geqq k,$$

where $k \geqq 1$.

Now, we derive

Theorem 3.1 *If $F(z)$ defined by (2.9) for $f(z) \in \mathcal{A}$ satisfies*

$$\left|1+\frac{1}{2}\frac{zF''(z)}{F'(z)}-\frac{zF'(z)}{F(z)}\right|<\frac{2-\alpha}{4\alpha} \quad (z \in \mathbb{U}) \tag{3.4}$$

for some real α which satisfies $2(\sqrt{2}-1) \leqq \alpha < 1$, then

$$\frac{z^2F'(z)}{F(z)^2} \prec \frac{1+(1-\alpha)z}{1-z} \quad (z \in \mathbb{U}). \tag{3.5}$$

Next, we show

Theorem 3.2 *If $F(z)$ defined by (2.9) for $f(z) \in \mathcal{A}$ satisfies*

$$\left|1+\frac{1}{2}\frac{zF''(z)}{F'(z)}-\frac{zF'(z)}{F(z)}\right|<\frac{\alpha}{2(1+\alpha)} \quad (z \in \mathbb{U}) \tag{3.6}$$

for some real $\alpha > 0$, then

$$\left|\frac{z^2F'(z)}{F(z)^2}-1\right|<\alpha \quad (z \in \mathbb{U}). \tag{3.7}$$

Taking $\alpha = 1$ in Theorem 3.2, we have

Corollary 3.1 *If $F(z)$ defined by (2.9) for $f(z) \in \mathcal{A}$ satisfies*

$$\left|1+\frac{1}{2}\frac{zF''(z)}{F'(z)}-\frac{zF'(z)}{F(z)}\right|<\frac{1}{4} \quad (z \in \mathbb{U}), \tag{3.8}$$

then

$$\left|\frac{z^2F'(z)}{F(z)^2}-1\right|<1 \quad (z \in \mathbb{U}). \tag{3.9}$$

Remark 3.1 In view of the result for univalency of analytic functions due to Ozaki and Nunokawa [11], we see that $F(z)$ satisfying the inequality (3.9) is univalent in $\mathbb{U}$.

Example 3.1 Let us consider the function $f(z)$ given by

$$f(z) = D_z^{-\lambda}\left(\frac{z^{1-\lambda}}{\Gamma(2-\lambda)}e^{\frac{z}{2}}\right) \quad (-1 \leqq \lambda < 2). \tag{3.10}$$

Then we have that

$$F(z) = \Gamma(2-\lambda)z^{\lambda}D_z^{\lambda}f(z) = ze^{\frac{z}{2}}, \tag{3.11}$$

(3.12)
$$\frac{zF'(z)}{F(z)} = 1 + \frac{1}{2}z,$$

and

(3.13)
$$\frac{zF''(z)}{F'(z)} = \frac{1}{2}z + \frac{z}{2+z}.$$

Therefore, $F(z)$ satisfies

(3.14)
$$\left|1 + \frac{1}{2}\frac{zF''(z)}{F'(z)} - \frac{zF'(z)}{F(z)}\right| = \frac{1}{4}\left|\frac{z^2}{2+z}\right| < \frac{1}{4} \quad (z \in \mathbb{U}).$$

For such a function $F(z)$, we see that

(3.15)
$$\left|\frac{z^2F'(z)}{F(z)^2} - 1\right| = \left|e^{-\frac{z}{2}}\left(1 + \frac{1}{2}z\right) - 1\right| \leqq c \quad (z \in \mathbb{U}).$$

By using the computer, we know that $c < 0.18 < 1$. Indeed, the function $F(z)$ satisfying (3.12) implies that

(3.16)
$$\operatorname{Re}\left(\frac{zF'(z)}{F(z)}\right) > \frac{1}{2} \quad (z \in \mathbb{U}).$$

This shows us that $F(z) \in \mathcal{S}^*\left(\frac{1}{2}\right)$.

References

[1] P. L. Duren, Univalent Functions, Springer-Verlag, New York, Berlin, Heiderberg, Tokyo, 1983

[2] I. S. Jack, *Functions starlike and convex of order alpha*, J. London Math. Soc. **3**(1971), 469 – 474

[3] R. J. Libera, *Some classes of regular univalent functions*, Proc. Amer. Math. Soc. **16**(1965), 755 – 758

[4] T. H. MacGregor, *A subordination for convex fuctions of order α*, J. London Math. Soc. **9**(1975), 530 – 536

[5] A. Marx, *Untersuchunggen uber schlichte Abbildungen*, Math. Ann. **107**(1932/33), 40 – 67

[6] S. S. Miller and P. T. Mocanu, Differential Subordinations, Pure and Applied Mathematics **225**, Marcel Dekker, 2000

[7] S. S. Miller and P. T. Mocanu, *Second order differential inequalities in the complex plane*, J. Math. Anal. Appl. **65**(1978), 289 – 305

[8] S. Owa, *On the distortion theorems. I*, Kyungpook Math. J. **18**(1978), 53 – 59

[9] S. Owa, *On applications of the fractional calculus*, Math. Japon. **25**(1980), 195 - 206

[10] S. Owa and H. M. Srivastava, *Univalent and starlike generalized hypergeometric functions*, Canad. J. Math. **39**(1987), 1057 - 1077

[11] S. Ozaki and M. Nunokawa, *The Schwarzian derivative and univalent functions*, Proc. Amer. Math. Soc. **33**(1972), 392 - 394

[12] M. S. Robertson, *On the theory of univalent functions*, Proc. Amer. Math. Soc. **37**(1936), 374 - 408

[13] E. Strohhäcker, *Beitrage zur Theorie der schlichten Funktionen*, Math. Z. **37**(1933), 356 - 380

[14] D. R. Wilken and J. Feng, *A remark on convex and starlike functions*, J. London Math. Soc. **21**(1980), 287 - 290

Nicoleta Breaz
Department of Mathematics-Informatics
Faculty of Sciences
"1 December 1918" Unidversity of Alba Iulia
510009 Alba Iulia
Romania
E-mail: nbreaz@uab.ro

Daniel Breaz
Department of Mathematics-Informatics
Faculty of Sciences
"1 December 1918" Unidversity of Alba Iulia
510009 Alba Iulia
Romania
E-mail: dbreaz@uab.ro

Shigeyoshi Owa
Department of Mathematics
Faculty of Education
Yamato University
Katayama 2-5-1, Suita
Osaka 564-0082
Japan
E-mail: owa.shigeyoshi@yamato-u.ac.jp