数理解析研究所講究録
第 1983 巻 2016 年 22-26

A Note on an Extension of Asset Pricing Models*

Hiroshi ISHIJIMA
Graduate School of International Accounting, Chuo University
Akira MAEDA
Graduate School of Arts and Sciences, University of Tokyo

1. Introduction

In this note, we review the recent advances of asset pricing models by Ishijima and Maeda (2015) and discuss the direction of extending these models.

As in Ishijima and Maeda (2015), we consider an economy that comprises three types of exchange market; i.e. financial market, real estate market and its leasing market. We then start our discussion from the conventional dynamic portfolio choice problem with a representative agent. Besides the utility of nondurable-goods consumption, we focus on the utility that stems from the bundle of real estate attributes which benefit the agent activities. Examples of those characteristics include square footage, year built, walking distance from the nearest subway / railway station and so on. Given an occupancy rate with plausible market clearing conditions at any point in time, we endogenously provide a competitive pricing system for financial assets, real estate and its rent. We then we discuss the direction of extending the pricing system. The rest of this note is organized as follows: Section 2 reviews and discusses the asset pricing models of Ishijima and Maeda (2015) and in Section 3, we conclude.

2. Review of asset pricing models for real estate and financial assets

In addition to financial security markets, we assume that there are real estate property markets and real estate lease markets within the economy. We then introduce the following notation for real estate $i = 1, \cdots, N^H$ and financial asset $j = 1, \cdots, N^P$.

$t = 0$ to ∞: Discrete timing of market trades.

$\boldsymbol{P}_t = \left(P_{j,t}\right)_{1 \le j \le N^P}$: Financial security price vector at time t.

$\boldsymbol{H}_t = \left(H_{i,t}\right)_{1 \le i \le N^H}$: Real estate price vector at time t.

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* This paper is presented at RIMS Workshop on Financial Modeling and Analysis 2015. We are grateful to Professor Toshikazu Kimura (organizer) and the participants for their comments. We remark that we are planning to submit the extended version of this paper in which some parts of this paper might be included.

$\boldsymbol{D}_t^P = \left(D_{j,t}^P\right)_{1\le j\le N^P}$: Vector of dividends yielded by financial securities at time t.

$\boldsymbol{D}_t^H = \left(D_{i,t}^H\right)_{1\le i\le N^H}$: Vector of rents paid by lessees to lessors at time t.

$\boldsymbol{\theta}_t = \left(\theta_{j,t}\right)_{1\le j\le N^P}$: Portfolio vector of financial security holdings at time t.

$\boldsymbol{\varphi}_t = \left(\varphi_{i,t}\right)_{1\le i\le N^H}$: Portfolio vector of real estate property right holdings at time t.

$\boldsymbol{\phi}_t = \left(\phi_{i,t}\right)_{1\le i\le N^H}$: Portfolio vector of real estate leasing at time t, i.e., the portfolio of real estate properties currently under lease contracts at time t.

$\mathbf{L}_t = \mathrm{diag}\left(L_{i,t}\right)_{1\le i\le N^H}$: Diagonal matrix of occupancy rates of real estate at time t.

Y_t: Representative agent's income at time t.

C_t: Representative agent's consumption at time t.

V_t^-: Representative agent's portfolio value before portfolio rebalancing at time t.

V_t: Representative agent's portfolio value after portfolio rebalancing at time t.

A self-financing portfolio strategy of the representative agent is described as follows:

$$V_t = V_t^- + Y_t - C_t - \boldsymbol{\phi}_t'\boldsymbol{D}_t^H + \boldsymbol{\varphi}_t'\mathbf{L}_t'\boldsymbol{D}_t^H \tag{1.1}$$

V_t and $V_{t+1}^-$are represented as follows:

$$V_t = \boldsymbol{\theta}_t'\boldsymbol{P}_t + \boldsymbol{\varphi}_t'\boldsymbol{H}_t \tag{1.2}$$

$$V_{t+1}^- = \boldsymbol{\theta}_t'(\boldsymbol{P}_{t+1} + \boldsymbol{D}_{t+1}^P) + \boldsymbol{\varphi}_t'\boldsymbol{H}_{t+1} \tag{1.3}$$

Thus, the representative agent's consumption at *t* is given by:

$$C_t = \boldsymbol{\theta}_{t-1}'(\boldsymbol{P}_t + \boldsymbol{D}_t^P) + \boldsymbol{\varphi}_{t-1}'\boldsymbol{H}_t + Y_t - \boldsymbol{\phi}_t'\boldsymbol{D}_t^H + \boldsymbol{\varphi}_t'\mathbf{L}_t'\boldsymbol{D}_t^H - \boldsymbol{\theta}_t'\boldsymbol{P}_t - \boldsymbol{\varphi}_t'\boldsymbol{H}_t \tag{1.4}$$

It is worthwhile to note that the timing of cash-in and cash-out expressed as Eqs. (1.1) – (1.4) are two folds; the timing of rent payout is different from that of financial securities' dividend payment $\boldsymbol{\theta}_t'\boldsymbol{D}_{t+1}^P$. It is because lease (or rent) contracts are usually cash-in-advance contracts: the lessee must pay the rent $\boldsymbol{\phi}_t'\boldsymbol{D}_t^H$ at the beginning of each period, which brings the rent of "dividends" $\boldsymbol{\varphi}_t'\mathbf{L}_t'\boldsymbol{D}_t^H$ to the owner of the property. This fact contrasts to financial security investment, in which dividends are brought to the security holder at the end of each period. Also we remark that the representative agent has a choice between consumption goods C_t and housing goods $\boldsymbol{\phi}_t'\boldsymbol{D}_t^H$.

As a source of benefits from real estate, each piece of real estate is a representation of a bundle of attributes. That is, with the following notation:

$b_{ik,t}$: Unit content of attribute k that is included in real estate i at time $(k = 1, \dots, K;\ i = 1, \dots, N^H)$. Lancaster (1966, 1971) referred this variable as *consumption technology*,

$Z_{k,t}$: Amount of attribute $k\ (= 1, \cdots, K)$ that is included in the entire real estate portfolio $\boldsymbol{\varphi}_t$ at time t, the bundle of attributes can be represented as:

$$Z_{k,t} = \sum_{i=1}^{N^H} b_{ik,t}\,\varphi_{i,t} \;\; or \;\; \boldsymbol{Z}_t = \mathbf{B}_t'\boldsymbol{\varphi}_t \tag{1.5}$$

The representative agent is to maximize the sum of the instantaneous utility derived during each period from the present into the infinite future. As the instantaneous utility at time t is time-additive and assumed to be a function of consumption at the time and the bundle of attributes, the objective to be maximized is defined as follows:

$$U(\{C_t, \boldsymbol{Z}_t\}, \ldots, \{C_{t+\tau}, \boldsymbol{Z}_{t+\tau}\}, \ldots) = E_t\left[\sum_{\tau=0}^{\infty} \delta^\tau u(C_{t+\tau}, \boldsymbol{Z}_{t+\tau})\right] \tag{1.6}$$

The agent's problem is described as follows:

$$\left|\begin{array}{ll} \underset{\{\boldsymbol{\theta}_t, \boldsymbol{\varphi}, \boldsymbol{\phi}_t\}}{\text{maximize}} & E_t\left[\sum_{\tau=0}^{\infty} \delta^\tau u(C_{t+\tau}, \boldsymbol{Z}_{t+\tau})\right] \\ \text{subject to} & C_{t+\tau} = \boldsymbol{\theta}_{t+\tau-1}'(\boldsymbol{P}_{t+\tau} + \boldsymbol{D}_{t+\tau}^P) + \boldsymbol{\varphi}_{t+\tau-1}'\boldsymbol{H}_{t+\tau} + Y_{t+\tau} - \boldsymbol{\phi}_{t+\tau}'\boldsymbol{D}_{t+\tau}^H \\ & \quad +\boldsymbol{\varphi}_{t+\tau}'\mathbf{L}_{t+\tau}\boldsymbol{D}_{t+\tau}^H - \boldsymbol{\theta}_{t+\tau}'\boldsymbol{P}_{t+\tau} - \boldsymbol{\varphi}_{t+\tau}'\boldsymbol{H}_{t+\tau} \\ & \boldsymbol{Z}_{t+\tau} = \mathbf{B}_{t+\tau}'\boldsymbol{\phi}_{t+\tau} \\ & \tau = 0,1,\ldots \end{array}\right. \tag{1.7}$$

It can clearly be observed that the problem is merely an extension of typical dynamic portfolio selection problems. The first-order necessary conditions for Eq. (1.7) and the plausible market clearing conditions expressed constitute a competitive equilibrium as stated in Proposition 1.

Proposition 1 (PHD Equations)

Let the occupancy rates $\mathbf{L}_t\ (\forall t)$ *and dividends yielded by financial securities* $\boldsymbol{D}_t^P\ (\forall t)$ *be exogenous. Within the framework and according to the assumptions described above, financial security prices, real estate prices, and real estate rents are determined by the following equations:*

P: Financial asset equilibrium prices (P-equation)

$$\boldsymbol{P}_t = E_t\left[(\boldsymbol{D}_{t+1}^P + \boldsymbol{P}_{t+1})M_{t:t+1}^C\right] \Leftrightarrow \tag{1.8}$$

$$P_{j,t} = E_t\left[(D_{j,t+1}^P + P_{j,t+1})M_{t:t+1}^C\right] \;\; (j = 1, \cdots, N^P) \tag{1.9}$$

H: Real estate equilibrium prices (H-equation)

$$\boldsymbol{H}_t = \boldsymbol{L}_t\boldsymbol{D}_t^H + E_t\left[\boldsymbol{H}_{t+1}M_{t:t+1}^C\right] = \boldsymbol{L}_t\boldsymbol{B}_t\boldsymbol{M}_{t:t}^z + E_t\left[\boldsymbol{H}_{t+1}M_{t:t+1}^C\right] \Leftrightarrow \tag{1.10}$$

$$H_{i,t} = L_{i,t}D_{i,t}^H + E_t\left[H_{i,t+1}M_{t:t+1}^C\right] = L_{i,t}\boldsymbol{b}_{i,t}\boldsymbol{M}_{t:t}^Z + E_t\left[H_{i,t+1}M_{t:t+1}^C\right] \;\; (i = 1, \cdots, N^H) \tag{1.11}$$

D: Real estate equilibrium rent (D-equation)

$$\boldsymbol{D}_t^H = \boldsymbol{B}_t\boldsymbol{M}_{t:t}^z \Leftrightarrow \tag{1.12}$$

$$D_{i,t}^{H} = \boldsymbol{b}_{i,t}\boldsymbol{M}_{t:t}^{Z} = \sum_{k=1}^{K} b_{ik,t}M_{k,t:t}^{Z} \ (i = 1, \cdots, N^{H}) \tag{1.13}$$

where

$$M_{t:t+1}^{C} = \delta \cdot \frac{\partial u(C_{t+1}, \boldsymbol{Z}_{t+1})/\partial C_{t+1}}{\partial u(C_t, \boldsymbol{Z}_t)/\partial C_t} \tag{1.14}$$

$$\boldsymbol{M}_{t:t}^{Z} = \frac{\partial u(C_t, \boldsymbol{Z}_t)/\partial \boldsymbol{Z}_t}{\partial u(C_t, \boldsymbol{Z}_t)/\partial C_t} \tag{1.15}$$

$$C_t = \boldsymbol{1}'\boldsymbol{D}_t^{P} + Y_t \tag{1.16}$$

$$\boldsymbol{Z}_t = \boldsymbol{B}_t'\boldsymbol{L}_t\boldsymbol{1} \tag{1.17}$$

To interpret the pricing system, the financial asset price is given as a stochastically discounted value of future dividends as shown in the financial economics literature since Merton (1969) and Lucas (1978). Similarly, the real estate price is given as a stochastically discounted value of future rents which can be regarded as dividends of financial assets. Moreover, the future rents of real estate can be represented as a linear combination of attribute prices for each of real estate as quoted in the literatures of real estate economics or consumer choice since Lancaster (1966, 1971), Rosen (1974), and Ekeland et al. (2004).

We remark that these attribute prices are the product of two components that can be interpreted as the cash-flow pricing kernel and hedonic pricing kernel, respectively. The first component is a cash-flow pricing kernel (or stochastic discount factor) which is a marginal rate of substitution between the present and future nondurable-goods consumptions along time horizon. The second component is a hedonic pricing kernel which a substitution between the nondurable-goods consumption and the real-estate attributes benefit at any point in time in the future. In this regard, our pricing kernel could be an extension to combine two existing pricing kernels. We might also extend the discussion along discrete-time horizon to provide a stochastic process of real estate prices. On the basis of these theoretical pricing systems, we might provide some statistical models that are ready to implement empirical analyses to explore the determinants of real estate prices. That is, our statistical pricing model allows us to incorporate not only the hedonic variables of real estate attributes but also the exogenous variables as the cash-flow pricing kernel. These model specifications would help understand the pricing mechanism of real estate in detail.

3. Conclusion

In this note, we review the recent advances of asset pricing models by Ishijima and Maeda (2015). We then we discuss the direction of extending the pricing system in order to explore the determinants of real estate prices in conjunction with financial assets.

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Hiroshi ISHIJIMA

Graduate School of International Accounting

Chuo University

Tokyo 162-8478 JAPAN

E-mail address: ishijima@tamacc.chuo-u.ac.jp

中央大学・大学院国際会計研究科

石島博

Akira MAEDA

Graduate School of Arts and Sciences

University of Tokyo

Tokyo 153-8902 JAPAN

E-mail address: maeda@global.c.u-tokyo.ac.jp

東京大学・大学院総合文化研究科

前田章