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A Binomial Model for Portfolio Insurance with transaction costs

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1 Introduction

We consider discrete hedging strategies with transaction costs. In order to hedge the downside risk of risky asset, we compare 3 strategies (i) Option Based Portfolio Insurance(0BPI), (ii) Super hedging (iii) Constant Proportion Portfolio Insurance (CPPI). The risky asset price is assumed to follow Binomial process, however the numbers of strategies of hedging portfolio increases by 2^n due to transaction costs. As pointed in Bensaid[1], the hedging strategy of OBPI is not cost minimal strategy because the transaction cost increases the initial cost.

Bensaid [1] and Ruskovsky [4] demonstrate numerical examples where the super hedging has lower initial cost to attains the same terminal payoff. Let look at their examples; The price process follows as Figure 1. The price change rates are $u=1.3, d=0.9$ and the initial stock price and the strike price are $S_0=100, K=100$. The transaction cost rate is set as $k=0.20$.

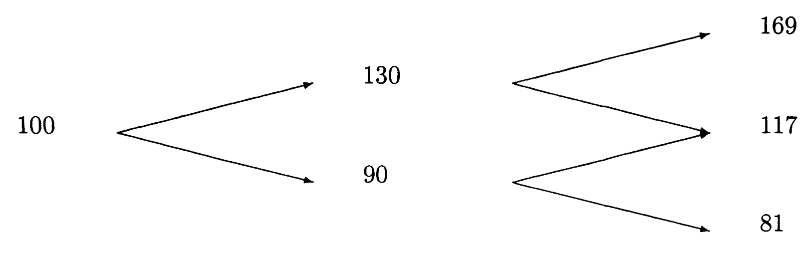

Figure 1: Stock prices tree

We assume that terminal payoff is assumed to be deliver the stock itself without paying any transaction cost at the maturity;
$100(1+0.3)^2\times 1-100=69, 100(1+0.3)(1-0.1)\times 1-100=17, 100(1-0.1)^2\times 0-0=0$
The initial cost of OBPI with transaction cost is 15.33 where $x_0=0.7263, B_0=-57.31$ in Figure 2, where (x_i, B_i) is denoted (the stock number, value of risk free asset).

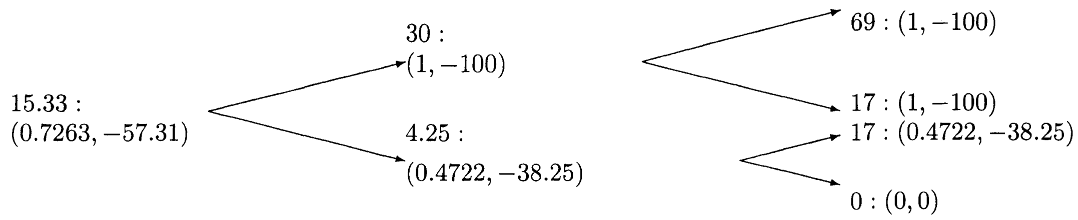

Figure 2: OBPI with transaction cost

The initial cost of Super hedging is 14.19 where $x_0 = 0.7467, B_0 = -60.48$, see in [4]. The following payoffs are at maturity time;
$100(1+0.3)^2 \times 1 - 100 = 69$
$100(1+0.3)(1-0.1) \times 1 - 100 = 17$
$100(1-0.1)(1+0.3) \times 0.7467 - 60.48 = 26.88$
$100(1-0.1)^2 \times 0.7467 - 60.48 = 0$
The terminal values are bigger the OBPI and also the initial cost is smaller than the OBPI.

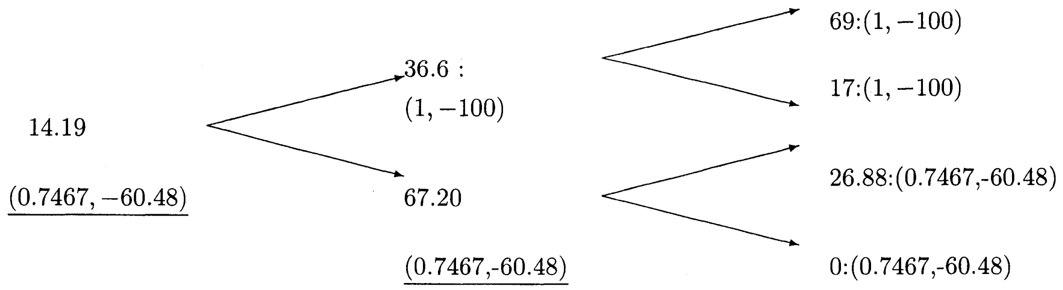

Figure 3: Super hedging Portfolio

The strategy is not changed when the price goes down in Figure 3, where it is shown by the underline. Because no trading saves the transaction cost, it decreases the initial investment. Previous works [1, 4] focussed on the necessary condition that OBPI coincides with Super hedging. We consider the computing algorithm to hedging strategies with transaction cost. By using new probabilities of martingale method as [3, 4], we construct an algorithm to calculate minimum cost hedging for long European call option in Binomial price model.

2 Model

A risky asset price is assumed to follow binomial process as, for the time interval Δt and $u > 1 > d$.

$$S = \begin{cases} Su & \text{up case} \\ Sd & \text{down case} \end{cases} \tag{2.1}$$

There is another risk-less asset which has a constant rate of return r, for the interval. The price process is depicted in figure 4

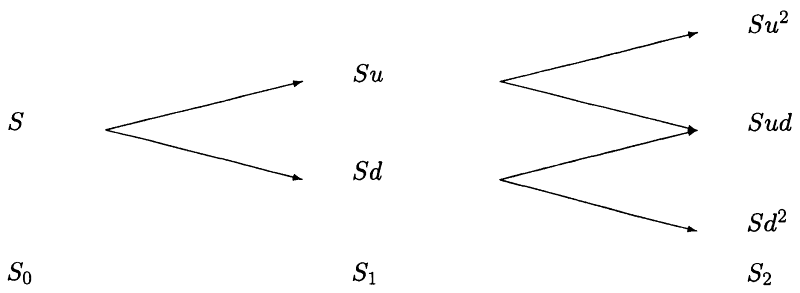

Figure 4: risky asset price

The transaction cost is proportional to the amount of the transaction amount; Let k_1, k_2 be the transaction cost per trade amount for purchase and sale of stock respectively. Let $R = 1 + r$,

$$S_t x_t + RB_t = S_t x_{t+1} + B_{t+1} + k|S_t x_{t+1} - S_t x_t|,$$

Where $k = k_1$ for $x_{t+1} > x_t$ and $k = k_2$ for $x_{t+1} \leq x_t$. The replication strategy for the long position's derivative in Binomial model satisfies $x_1^u > x_0 > x_1^d$ from theorem 1 in [3],

$$\begin{cases} S_t(1+k_1)x_t + RB_t = S_t(1+k_1)x_{t+1} + B_{t+1}, \text{ if } S_t = uS_{t-1} \\ S_t(1-k_2)x_t + RB_t = S_t(1-k_2)x_{t+1} + B_{t+1}, \text{ if } S_t = dS_{t-1} \end{cases} \tag{2.2}$$

The initial portfolio is assumed to be endowment of portfolios without transaction cost. In the following section the martingale method is introduced to calculate the portfolio value.

2.1 Dynamic hedging of long call option and martingale probabilities

Let start the option hedging by recursive way; The initial cost is for the first step by solving (2.2).

$$\begin{pmatrix} x_0 \\ B_0 \end{pmatrix} = \begin{pmatrix} Su^* & R \\ Sd^* & R \end{pmatrix}^{-1} \begin{pmatrix} Su^* x_1^u + B_1^u \\ Sd^* x_1^d + B_1^d \end{pmatrix}$$

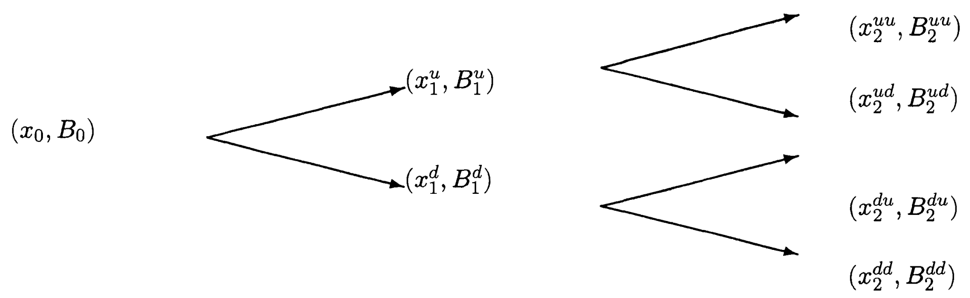

Figure 5: Portfolio tree

For single period model,the initial cost is $Sx_0 + B_0$ then

$$V_0 = (S, 1)\begin{pmatrix} x_0 \\ B_0 \end{pmatrix} = (S, 1)\begin{pmatrix} Su^* & R \\ Sd^* & R \end{pmatrix}^{-1}\begin{pmatrix} Su^* x_1^u + B_1^u \\ Sd^* x_1^d + B_1^d \end{pmatrix}.$$

Because $u(1+k_1) > d(1-k_2)$, define

$$P = \frac{R - d^*}{u^* - d^*} \tag{2.3}$$

and $u^* = (1+k_1)u, d^* = (1-k_2)d$ then,

$$(S, 1)\begin{pmatrix} Su^* & R \\ Sd^* & R \end{pmatrix}^{-1} = \frac{1}{R}(P, (1-P)).$$

We can use P as a probability from the $0 < P < 1$.

$$\begin{aligned} V_0 &= \{P((1+k_1)x_1^u Su + B_1^u) + (1-P)((1-k_2)x_1^d Sd + B_1^d)\}/R \\ &= \frac{1}{R}E^*[V_1] \end{aligned}$$

where $V_1^u := Su^* x_1^u + B_1^u$ and $V_1^d := Sd^* x_1^d + B_1^d$.

The initial cost for the second step is obtained similarly. At the state Su, the 2 period model are follows;

$$\begin{cases} Suux_1^u + B_1^u R = Suux_2^{uu} + B_2^{uu} + Suu(x_2^{uu} - x_1^u)k_1 \\ Sudx_1^u + B_1^u R = Sudx_2^{ud} + B_2^{ud} + Sud(x_1^u - x_2^{ud})k_2 \end{cases}$$

Then the up state value of portfolio becomes as

$$\begin{aligned} V_1^u &= Su^* x_1^u + B_1^u \\ &= (Su^*, 1)\begin{pmatrix} Suu^* & R \\ Sud^* & R \end{pmatrix}^{-1}\begin{pmatrix} Suu^* x_2^{uu} + B_2^{uu} \\ Sud^* x_2^{ud} + B_2^{ud} \end{pmatrix} \end{aligned}$$

Define

$$P_u = \frac{R(1+k_1) - d^*}{u^* - d^*}, 1 - P_u = \frac{u^* - R(1+k_1)}{u^* - d^*} \tag{2.4}$$

$$\frac{1}{R}(P_u, 1-P_u) = (Su^*, 1)\begin{pmatrix} Suu^* & R \\ Sud^* & R \end{pmatrix}^{-1}$$

Similarly for the price down,

$$V_1^d = Sd^* x_1^d + B_1^d = \frac{1}{R}(P_d, 1-P_d)\begin{pmatrix} Sdu^* x_2^{du} + B_2^{du} \\ Sdd^* x_2^{dd} + B_2^{dd} \end{pmatrix}$$

where

$$\begin{cases} P_d = \frac{R(1-k_2)-d^*}{u^*-d^*} \\ 1-P_d = \frac{u^*-R(1-k_2)}{u^*-d^*} \end{cases} \tag{2.5}$$

The initial value of 2 period model

$$\begin{aligned} V_0 &= \frac{1}{R}(PV_1^u + (1-P)V_1^d) \\ &= \frac{1}{R^2}\{PP_uV_2^{uu} + P(1-P_u)V_2^{ud} + (1-P)P_dV_2^{du} + (1-P)(1-P_d)V_2^{dd}\} \\ &= \frac{1}{R^2}E^*[V_2] \end{aligned}$$

$$V_2 = \begin{cases} V_2^{uu} = Suu^* x_2^{uu} + B_2^{uu} = Suu(1+k_1)x_2^{uu} + B_2^{uu} \\ V_2^{ud} = Sud^* x_2^{ud} + B_2^{ud} = Sud(1-k_2)x_2^{ud} + B_2^{ud} \\ V_2^{du} = Sdu^* x_2^{du} + B_2^{du} = Sdu(1+k_1)x_2^{du} + B_2^{du} \\ V_2^{dd} = Sdd^* x_2^{dd} + B_2^{dd} = Sdd(1-k_2)x_2^{dd} + B_2^{dd} \end{cases}$$

The price vector at time 2 is (Su^2, Sud, Sd^2), where $(PP_u, P(1-P_u)+(1-P)P_d, (1-P)(1-P_d))$ is the distribution as seen in Figure 6.

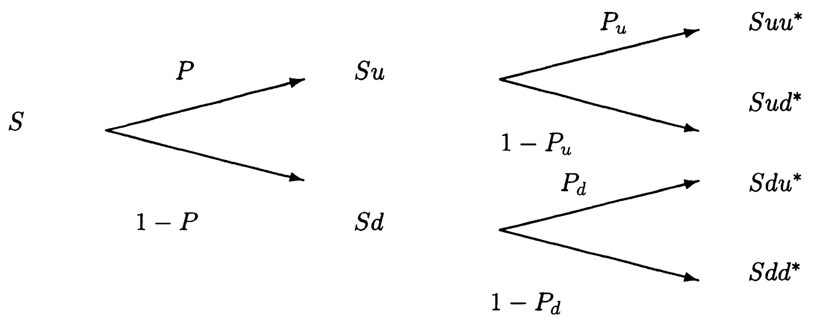

Figure 6: Risky price tree with new probabilities

The option value of maturity n is calculated using the expectation of these probabilities;

$$V_0 = \frac{1}{R^n}E^*[((1+k)Se^Y - K)\mathbf{1}_{Se^Y \geq K}]$$

where

$$Y = \sum_{i=1}^{n} X_i, \text{ where } X_i = \begin{cases} \log u \\ \log d \end{cases}$$

and

$$\begin{cases} k = k_1, & \text{if } X_n = \log u \\ k = -k_2, & \text{if } X_n = \log d \end{cases}$$

2.2 The distribution of Martingale probability

The vector of stock prices in binomial model for time 3 is $(Su^3, Su^2d.Sud^2, Sd^3)$. The probability vector is $(PP_u^2, PP_u(1-P_u) + P(1-P_u)P_d + (1-P)P_dP_u, P(1-P_u)(1-P_d) + (1-P)P_d(1-P_d) + (1-P)(1-P_d)P_d, (1-P)(1-P_d)^2)$ as seen in Figure 7.

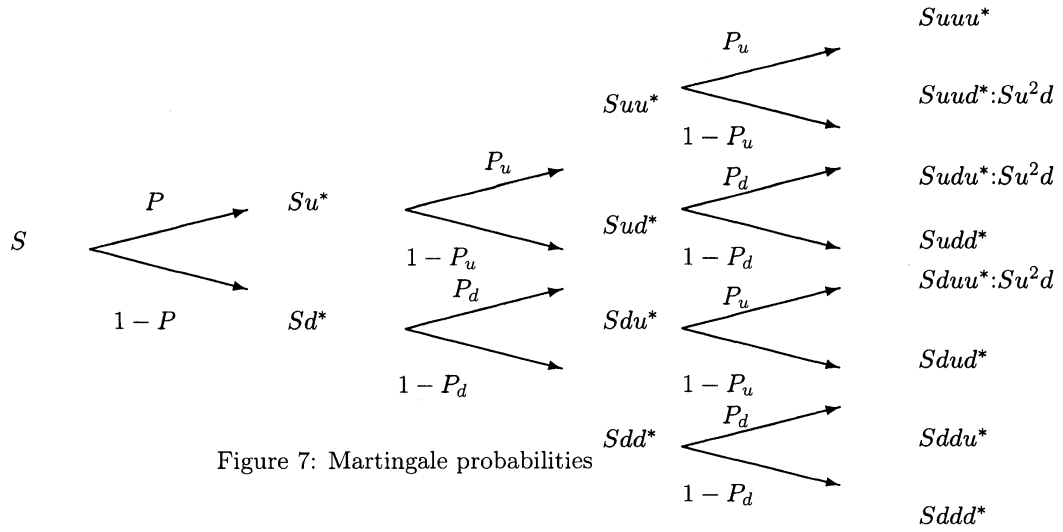

Figure 7: Martingale probabilities

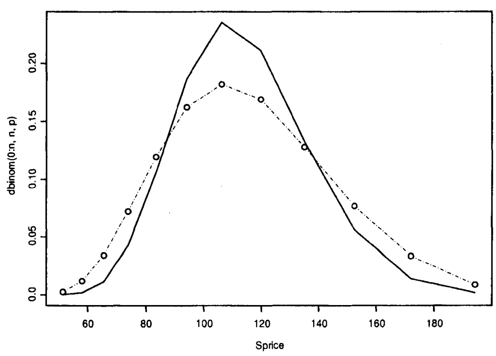

Figure 8: Binomial vs Bushy tree distribution

In the Figure 8 we compare the Binomial distribution drawn with the solid line and the

Martingale probability distribution is with dotted line in the case of $n = 11$, where we sum up 2^{11} probability points. The shape of Martingale probability is heavier than the Binomial distribution in the tail part.

To converge the Binomial model to Black Scholes formulas, it is assumed that $u = e^{\sigma\Delta_t}, ud = 1, R = exp(log(r)\Delta_t), \Delta_t = t_{i+1} - t_i = T/n, i \in (0, n-1)$ like [3].

We consider the convergence of price to replicate the long position of call option. For large n and small cost rate in case of $k_1 = k_2$, V_0 coverges to Black-Schloes value with the following volatility,

$$\sigma^2(1 + \frac{2k\sqrt{n}}{\sigma\sqrt{T}}).$$

For the short call option, the transaction cost of hedging affects in the reverse way. If the price is up, then hedging is to sell the stock. It implies to change the up rate to $u^* = u(1 - k_2)$. If the price is down, then the hedging is to buy and $d^* = d(1 + k_1)$. In order to exists the hedging solution, the following conditions should be satisfies from the existence of martingale probabilities of (2.4) and (2.5).

$$u(1 - k_2) \geq R(1 + k_1), R(1 - k_2) \geq d(1 + k_1). \tag{2.6}$$

The following variance for Black Scholes option formula is existed as

$$\sigma^2(1 - \frac{2k\sqrt{n}}{\sigma\sqrt{T}}) \geq 0 \text{ if } u(1 - k_2) > d(1 + k_1),$$

because $u(1 - k_2) > d(1 + k_1)$ is obvious from (2.6).

3 Super Hedging for European call option

The option payoff at maturity is $(S_T - K)^+$ which implies $x_T = 1, B_T = -K$ for $S_T > K$ otherwise $x_T = 0, B_T = 0$. In the definition of super hedging, the portfolio value at $T-1$ is set to be equal or larger than the value of OBPI. Let $S_{T-1}x_{T-1} + B_{T-1}$ be a portfolio at $T-1$. If the price is up, then the rebalance is as follows,

$$S_{T-1}ux_{T-1} + B^u_{T-1}R \geq S_{T-1}ux_T + B_T + k|S_{T-1}u(x_T - x_{T-1})|.$$

For $x_T > x_{T-1}$, the cost to increase the stock is $k_1S_{T-1}u(x_T - x_{T-1})$ then

$$S_{T-1}ux_{T-1} + B^u_{T-1}R \geq S_{T-1}ux_T + B^u_T + k_1S_{T-1}u(x_T - x_{T-1}),$$

which is equivalent to the following,

$$B^u_{T-1} \geq \frac{1}{R}\left(B^u_T + S_{T-1}u(x_T - x_{T-1})(1 + k_1)\right). \tag{3.1}$$

For $x_T < x_{T-1}$, the cost to decrease the stock is $k_2S_{T-1}u(x_{T-1} - x_T)$ then

$$S_{T-1}ux_{T-1} + B^u_{T-1}R \geq S_{T-1}ux_T + B_T + k_2S_{T-1}u(x_{T-1} - x_T).$$

Equivalently,

$$B^u_{T-1} \geq \frac{1}{R}\left(B^u_T + S_{T-1}u(x_{T-1} - x_T)(1-k_2)\right). \tag{3.2}$$

Define the cost function as:

$$\psi(y) = \begin{cases} (1+k_1)y & ; y \geq 0 \\ (1-k_2)y & ; y < 0. \end{cases}$$

Using the cost function, two equations (3.1) and (3.2) are merged into one equation as

$$B^u_{T-1} \geq \frac{1}{R}\left(B^u_T + S_{T-1}u\psi(x_T - x_{T-1})\right). \tag{3.3}$$

For the portfolio value $S_{T-1}x_{T-1} + B_{T-1}$, if the price is down, then $S_{T-1}dx_{T-1} + B^d_{T-1}R$ is re-balanced similarly to the up case;

$$B^d_{T-1} \geq \frac{1}{R}\left(B^d_T + S_{T-1}d\psi(x_T - x_{T-1})\right). \tag{3.4}$$

We describe two inequality (3.3) and (3.4) together to the following inequality using random variables $S_T(\omega), B_T(\omega)$;

$$B_{T-1}(\omega) \geq \frac{1}{R}\left(B_T(\omega) + S_T(\omega)\psi(x_T - x_{T-1})\right). \tag{3.5}$$

The minimum cost at time 0 is obtained by solving the following problem from T to $t = 0$.

$$\begin{aligned} Q_t(x_{t-1}, \omega) &= \min_{(x_t, B_t)} \frac{1}{R}\left(B_t + S_t(\omega)\psi(x_t - x_{t-1})\right) && (3.6) \\ & s.t. \quad B_t \geq Q_{t+1}(x_t, u) && (3.7) \\ & \qquad\;\; B_t \geq Q_{t+1}(x_t, d). && (3.8) \end{aligned}$$

The terminal conditions for the long European call option are given as

$$Q_T(x_{T-1}, \omega) = \frac{1}{R}\left(B_T + S_T(\omega)\psi(x_T - x_{T-1})\right),$$

and

$$RB_T + S_T(\omega)x_T = (S_T(\omega) - K)^+.$$

We find the optimal strategy in Theorem 1 in [1];

Theorem 1 *Fore all $t = 0, \cdots, T-1$,there exists two numbers denoted $x^{min}_t(\omega)$ and $x^{max}_t(\omega)$ such that a solution for $Q_t(x_{t-1}, \omega)$ is $x^*_t(x_{t-1}, \omega)$ with*

$$x^*_t(x_{t-1}, \omega) = \begin{cases} x^{min}_t(\omega) & x_{t-1} \leq x^{min}_t(\omega) \\ x_{t-1} & x_{t-1} \leq x_t \in [x^{min}_t(\omega), x^{max}_t(\omega)] \\ x^{max}_t(\omega) & x_{t-1} \geq x^{max}_t(\omega). \end{cases}$$

From this theorem,there exists non trading interval in the optimal solution of problem (3.6).

What is the condition to exist the non trading interval in optimal solution. The following theorem provides the condition to the rate of transaction cost.

Theorem 2 *Suppose*

$$\frac{1+k_1}{1-k_2} \geq \frac{R}{d}, \tag{3.9}$$

then non trading interval exists in the optimal solution.
Proof:

In order to obtain hedging strategies, it is necessary to exist the martingale probability. For the first step $0 < P = \frac{R-d^*}{u^*-d^*} < 1$. After the second step $0 < P_u = \frac{R(1+k_1)-d^*}{u^*-d^*} < 1$ and $0 < P_d = \frac{R(1-k_2)-d^*}{u^*-d^*} < 1$. The linear programing problem of (3.6) has the following 4 cases of pivot value;

(i) The price is up and buying stock and the price is down and selling stock
$u^* = u(1+k_1), d^* = d(1-k_2)$ then all martingale probabilities exist and the trading strategy is determined, which coincides that of OBPI. There does not exist non-trading strategy.

(ii) Selling stock irrelevantly the price change
$u^* = u(1-k_2), d^* = d(1-k_2)$ then all martingale probabilities exist and the trading strategy is also determined.

(iii) Buying stock irrelevantly the price change
$u^* = u(1+k_1), d^* = d(1+k_1)$ then it requires the following conditions for the strategy;

$$R > d(1+k_1), R(1-k_2) > d(1+k_1)$$

Then suppose

$$\frac{1+k_1}{1-k_2} \geq \frac{R}{d},$$

then there exist non-trading strategy.

(iv) The price is up and selling stock and the price is down and buying stock
$u^* = u(1-k_2), d^* = d(1+k_1)$. But this case of pivot does not exist, for the long call option hedging, the terminal payoff $(S_T - K)^+$ is increasing function of $Su^id^{n-i}, i = 0, \cdots, n$. □

3.1 Algorithm of super hedging strategy

Let us construct the super hedging strategy if (3.9) holds. The following useful lemma for the strategy is available in [1].

Lemma 1 *The following minimizing problem have a same optimal solution as (3.6);*

$$H_0 = \min_{(x_0,B_0)} \frac{1}{R}\left(B_0 + S_t(\omega)\psi(x_1 - x_0)\right) \tag{3.10}$$

$$s.t. \quad B_{T-1}(\omega) \geq \frac{1}{R}\left(B_T + S_T(\omega)\psi(x_T - x_{T-1})\right) \tag{3.11}$$

$$B_{t-1}(\omega) = \frac{1}{R}\left(B_t + S_t(\omega)\psi(x_t - x_{t-1})\right), \forall t \in (1, T-1). \tag{3.12}$$

In order to obtain the super hedging strategy, it is necessary to check the possibility of inequality of (3.11). If the condition (3.9) holds, there exists the buy and hold strategy which means no trading for x_T and x_{T-1}.

Let consider the example of super hedging in Figure 3. The condition (3.9)holds for $R = 1, k_1 = 0.2, k_2 = k/(1+k) = 0.2/1.2, d = 0.8$. The terminal payoff are $(Su^2 - K)^+ > (Sud - K)^+ > (Sd^2 - K)^+$. Suppose $(Sd^2 - K)^+ = 0$, and let no trading strategy (x_0, B_0), then $(x_1^d, B_1^d) = (x_0, B_0)$. It is necessary to minimize the initial cost under the following constraints;

$$\min_{x_0, B_0} Sx_0 + B_0 \tag{3.13}$$

$$\begin{aligned} x_1^d Sd^2 + B_1^d R &\geq 0 \\ x_1^d Sud + B_1^d R &\geq (Sud - K) \\ x_1^u Sud + B_1^u R &\geq (Sud - K) \\ x_1^u Su^2 + B_1^u R &\geq (Su^2 - K) \\ x_0 Su(1+k_1) + B_0 R &= x_1 Su(1+k_1) + B_1^u \end{aligned}$$

3.2 Martingale algorithm for super hedging

Let consider the super hedging of general case for long call option of maturity T, seen the way in Figure 9.

(i) The terminal values are for no transaction cost because of assumption of stock delivery which use the risk neutral probability of $P_0 = \frac{R-d}{u-d}$. They are calculated by the expectation;

$$V(S_T(\omega)) = E^{P_0}[(S_T(\omega) - K)^+]$$

The strategies of stock are $x_T = 1$ or $x_T = 0$ except the boundary price of stock: $S_T(\omega_i), \min(\omega_i : S_T(\omega) > K)$,which is denoted $Su^{(T-i)}d^i$.

(ii) Solve the super hedge problem for 2 periods of (3.13) for $Su^{(T-i)}d^i$ and $Su^{(T-i+1)}d^{i-1}$. It determines the non trading nodes.

(iii) Calculate $V(S_i(\omega)) = E^{P_*}[V_{i+1}(S_{i+1}(\omega))], i \in (1, T-1)$, where the probabilities are (P_u, P_d) of (2.4) and (2.5).

(iv) The super hedging cost is using the probability P of (2.3) as

$$V(S) = E^P[V_1(S_i(\omega))]$$

(v) The strategies are calculated (x_0, B_0) from $V(S_1)$ and (2.2). All strategies are similarly calcucalated.

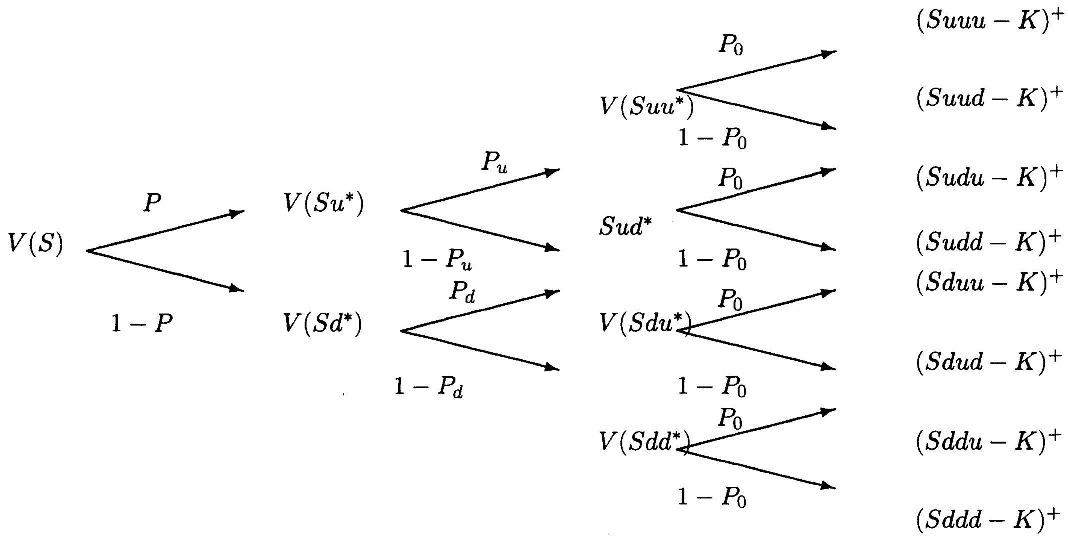

Figure 9: Martingale algorithm for super hedging

4 CPPI in Binomial price tree

We consider the constant proportional portfolio Insurance strategies with transaction costs under the Binomial price process. Let V_0 the initial portfolio value where we keep the guaranteed value F_0 in the present value. The difference $V_0 - F_0$ is called cushion which is denoted C_0. The risky asset price is assumed to follows Binomial process. The risky investment in CPPI is set m times of the cushion. The investment amount to risky asset is called Exposure; $E_i = mC_i$ at any discrete time. The value of CPPI at time 2 satisfies as follows;

$$V_1 = \begin{cases} mC_0u + (V_0 - mC_0)R = C_0(mu + (1-m)R) + F_0R \\ mC_0d + (V_0 - mC_0)R = C_0(md + (1-m)R) + F_0R \end{cases} \tag{4.1}$$

Let define $U = mu + (1-m)R$ and $D = md + (1-m)R$. The cushion values at the time 1 is $C_1 = \{CU, CD\}$. Similarly $C_2 = \{CUU, CUD, CDD\}$ follows the Binomial tree as seen in the following figure;

We consider the transaction cost to change the exposure E_i where the rate is k. The cost at 1 is $k|E_1 - E_0|$.

$$k|E_1 - E_0| = k|m(C_1 - C_0)| = \begin{cases} k_1mC(U-1), \text{if the price up} \\ k_2mC(1-D), \text{if the price down.} \end{cases}$$

Assuming $u > R > 1 > d$ and $1 < m$, it implies that

$$U > u > 1 > d > D.$$

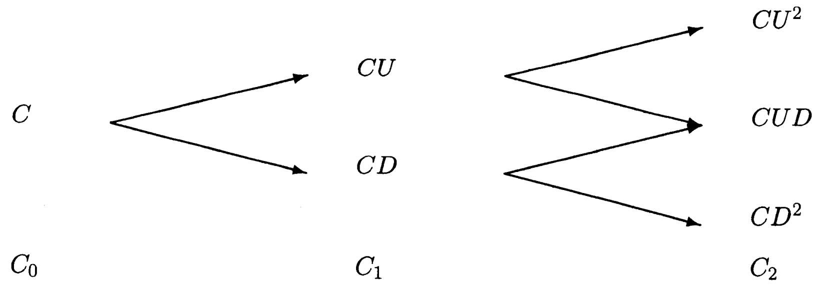

Figure 10: Cushion value tree

The portfolio values payed after transaction costs are

$$V_1' = \begin{cases} CU + F_0R - k_1mC(U-1) = CU(1-k_1m) + k_1mC + F_1 \\ CD + F_0R - k_2mC(1-D) = CD(1+k_2m) - k_2mC + F_1. \end{cases}$$

The cushion values at time 1 with transaction cost are

$$C_1' = \begin{cases} C(U(1-k_1m) + k_1m) \\ C(D(1+k_2m) - k_2m) \end{cases} \tag{4.2}$$

Similarly, the portfolio values at time 2 are

$$V_2' = \begin{cases} C_1'(U(1-k_1m) + k_1m) + F_2 \\ C_1'(D(1+k_2m) - k_2m) + F_2. \end{cases}$$

By substitution of (4.2), the next cushion becomes

$$C_2' = \begin{cases} C(U(1-k_1m) + k_1m)^2 \\ C(U(1-k_1m) + k_1m)(D(1+k_2m) - k_2m) \\ C(D(1+k_2m) - k_2m)^2 \end{cases}$$

Let define $U' = U(1-k_1m) + k_1m$, and $D' = D(1+k_2m) - k_2m$. $CU'^jD'^{(n-j)}$ is the cushion values at time n when the price goes up j times with transaction cost. Define p the observable probability of price up state, the expected value of CPPI with transaction costs is calculated as

$$\begin{aligned} E[C_n'] &= \sum_{j=1}^{n} \binom{n}{j} p^j(1-p)^{n-j}CU'^jD'^{(n-j)} \\ &= C\sum_{j=1}^{n} \binom{n}{j} (U'p)^j(D'(1-p))^{n-j} \\ &= C(U'p + D'(1-p))^n \\ &= CM^n, \end{aligned}$$

where $M = (U'p+D'(1-p))$. The expected value of portfolio at time n is $E[V_n] = C_0M^n + F_0R^n$.

4.1 CPPI parameter and transaction cost rates

We assume that the multiplier m is greater than 1. With transaction cost, the binomial rates of up and down state are defined from (4.1) and (4.2);

$$\begin{aligned} U'(m) &= (mu + (1-m)R)(1-k_1 m) + k_1 m \\ &= -k_1(u-R)m^2 + (u - R - k_1 R + k_1)m + R \\ D'(m) &= (md + (1-m)R)(1 + k_2 m) - k_2 m \\ &= -k_2(R-d)m^2 + (d - R + k_2 R - k_2)m + R \end{aligned}$$

The maximum of $U'(m)$ is achieved at $m^* = \frac{u-R-k_1R+k_1}{2k_1(u-R)}$ as seen in Figure 10.

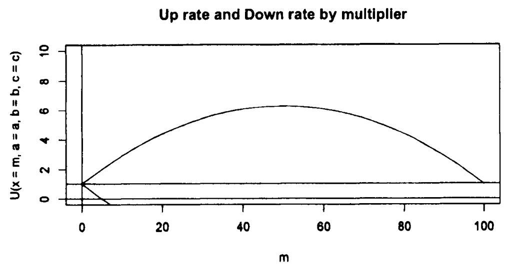

Figure 11: **Cushion change rate by multiplier**

If $m^* > m > 0$, $U'(m)$ is increasing function. If $m = 0$, the CPPI portfolio is just a risk-free asset because of $U'(0) = D'(0) = R$. To satisfy $m^* > 0$ it is necessary to hold

$$k_1 < \frac{u-R}{r}.$$

Because of $D'(0) = R$ and the maximum is achieved at $m_* = \frac{d-R-k_2R+k_2}{2k_2(R-d)} < 0$, Then $D'(m)$ is decreasing function for $m > 0$.

In order to satisfy $D'(m) > 0$, k_2 should satisfies

$$\frac{d - R + k_2 r + \sqrt{(d-R+k_2 r)^2 + 4k_2(R-d)R}}{2k_2(R-d)} > m$$

We sum up in the following theorem:

Theorem 3 *In order tpo protect the down side risk in CPPI, transaction rates k_1 for purchase and k_2 satisfies the following conditions;*

(i) *If $k_1 < \frac{u-R}{r}$, then the up multiplier $U' > R$.*

(ii) *if k_2 satisfies the condition; $\frac{d-R+k_2r+\sqrt{(d-R+k_2r)^2+4k_2(R-d)R}}{2k_2(R-d)} > m$, then the down multiplier satisfies $R > D' > 0$.*

5 Conclusions

Using new probabilities instead of portfolio minimization, the hedging values are calculated as if binary tree process. The comparison between OBPI and Super hedging depends on the cost rate. The high cost of trading induces to stop trading to follow exact replication. It is profitable for the purpose of the down side risk management. In the previous articles in [1] and [4] the necessary condition to have a same value among OBPI and Super hedging is $ud = 1$ and $(1 + k_1)(1 - k_2) < u$. Our study extends the condition of more general parameters and proposes an algorithm of Super hedging if transaction cost rate satisfies (3.9). We also compare the CPPI with transaction costs under the Binomial price assumption. It might be interesting the strategy like non trading time in CPPI in the same frame work. Transaction cost is reduced significantly by using the future product instead to dealing spot stock directly. Even if suitable future contract exists in the market, the transaction cost is depending on the volatility and liquidity, which might be stochastic process.

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