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Rate of convergence of an algorithm for curvature-dependent motions of hypersurfaces

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1 Introduction

This is a brief report of my joint work [6] with Professor Masato Kimura (Kanazawa University).

Let $\{\Gamma(t)\}_{t\geq 0}$ be a family of compact hypersurfaces in $\mathbb{R}^{N}$. We say this family is a curvature-dependent motion (CDM for short) if $\Gamma(t)$ moves by the following equation:

$$V=\kappa+\langle\mathbf{b},\mathbf{n}\rangle+g \quad \text{on } \Gamma(t),\ t\in(0,T). \tag{1.1}$$

Here $T>0$, $\mathbf{n}=\mathbf{n}(t,x)$ is the inner unit normal vector field on $\Gamma(t)$, $V=V(t,x)$ is the velocity of $\Gamma(t)$ in the direction of $\mathbf{n}$, $\kappa=\kappa(t,x)(:=-\mathrm{div}\,\mathbf{n}(t,x))$ is the $((N-1)$-times) mean curvature of $\Gamma(t)$, $\mathbf{b}=\mathbf{b}(t,x)=(b^{1}(t,x),\cdots,b^{N}(t,x))$ denotes a given vector field in $\mathbb{R}^{N}$, $g=g(t,x)$ is a forcing term and $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^{N}$. As well known, the case of $\mathbf{b}\equiv \mathbf{0}$ and $g\equiv 0$ is the mean curvature flow (MCF for short). The CDM arises in various fields such as two-phase Stefan problems, phase transitions, image processing, two-phase fluid flows and so on.

From the viewpoints of the above applications, many people have studied numerical methods for CDM. Among them, we treat the following algorithm: Let C_{0} be a compact set in $\mathbb{R}^{N}$ and fix a time step $h>0$. For $k=0,1,2,\ldots$, set $\mathbf{b}_{k}(t,x):=\mathbf{b}(t+kh,x)$ and $g_{k}(t,x):=g(t+kh,x)$. Let $w_{0}=w_{0}(t,x)$ be a unique solution of the initial value problem for the linear parabolic equation with $k=0$:

$$w_{t}-\Delta w+\langle\mathbf{b}_{k},Dw\rangle+g_{k}=0 \quad \text{in } (0,h]\times\mathbb{R}^{N}, \tag{1.2}$$

$$w(0,x)=d(x,C_{k}) \quad \text{for } x\in\mathbb{R}^{N}. \tag{1.3}$$

Here $d(x,D)$ is the signed distance function to ∂D defined by

$$d(x,D):=\begin{cases}\mathrm{dist}\,(x,\partial D) & \text{for } x\in D,\\ -\mathrm{dist}\,(x,\partial D) & \text{for } x\notin D,\end{cases} \tag{1.4}$$

for each closed subset $D(\neq\emptyset)$ of $\mathbb{R}^{N}$. We then set

$$C_{1}:=\{w_{0}(h,\cdot)\geq 0\}. \tag{1.5}$$

Let w_{1} be a unique solution of (1.2) - (1.3) with $k=1$. Again we define C_{2} as the set in (1.5) with w_{1} replacing w_{0}. Repeating this process, we have a sequence $\{C_{k}\}_{k=0}^{+\infty}$ of compact subsets of $\mathbb{R}^{N}$. We set

$$C^{h}(t):=C_{k} \quad \text{for } t\in[kh,(k+1)h),\ k=0,1,2,\ldots \tag{1.6}$$

Letting $h \to 0$, we formally obtain a limit flow $\{C(t)\}_{t\geq 0}$ of compact sets in $\mathbb{R}^N$ and observe that $\partial C(t)$ moves by (1.1) with the initial data ∂C_0.

The above algorithm was numerically studied by Kimura - Notsu [7] and Esedoḡlu - Ruuth - Tsai [3]. In [7] Kimura and Notsu proposed a fully discrete finite element scheme based on the above level set method of the signed distance function. In [7, Section 4] they gave some numerical examples for MCF with a forcing term. In [3] Esedoḡlu, Ruuth and Tsai considered various geometric motions with using the signed distance function, including CDM, MCF with triple junctions and the motion by surface diffusion. The extension of the signed distance approach to vector setting for numerical computation of multiphase problems was addressed in Mohammand - Švadlenka [9]. Our algorithm is also regarded as a variant of the Bence - Merriman - Osher (BMO for short) algorithm to MCF (cf. Bence - Merriman - Osher [1]), which utilizes the solutions of the usual heat equation, continually reinitialized after short time steps. The BMO algorithm and its generalizations are studied by many people. Among them Vivier [10] and Leoni [8] generalized the BMO algorithm with using the linear/semilinear parabolic equations and proved the convergence of their scheme to the anisotropic CDM's associated with these equations. Our algorithm is quite similar to theirs on the point that we use the linear parabolic equation (1.2) to construct the approximate sequence for CDM. However, the choice of the initial data is the main difference between the (generalized) BMO algorithm and ours. In the (generalized) BMO algorithm they choose the initial data

$$w(0,x) = \begin{cases} 1 & \text{for } x \in C_k, \\ -1 & \text{for } x \notin C_k, \end{cases} \quad (= \mathrm{sgn}^*(d(x,C_k)))$$

instead of (1.3), where $\mathrm{sgn}^*(r) := 1$ for $r \geq 0$, $:= -1$ for $r < 0$.

The main purpose of this article is to present the optimal rate of convergence of this algorithm to the smooth and compact CDM.

The strategy is direct calculations for the distance between CDM and the approximate motion. For this purpose the estimate of Dw_k plays an important role. Then we obtain that for any $\varepsilon > 0$, there are constants L_1, $h_0 > 0$ such that

$$\sup_{t\in[0,T-\varepsilon]} d_H(C^h(t), C(t)) \leq L_1 h \quad \text{for all } h \in (0,h_0). \tag{1.7}$$

The optimality of this estimate is obtained by precise calculations in the case of a circle evolving by curvature.

In the following of this article, to simplify the description we set $\mathbf{b} \equiv 0$ and $g \equiv 0$, that is, we treat the MCF $\{\Gamma(t)\}_{t\in[0,T)}$:

$$V = \kappa \quad \text{on } \Gamma(t),\ t \in (0,T). \tag{1.8}$$

and instead of (1.2) - (1.3), we solve the initial value problem for the usual heat equation:

$$w_t - \Delta w = 0 \quad \text{in } (0,h] \times \mathbb{R}^N, \tag{1.9}$$

$$w(0,x) = d(x,C_k) \quad \text{for } x \in \mathbb{R}^N. \tag{1.10}$$

This article is organized as follows. In section 2 we state our assumptions and briefly explain the notions of the generalized MCF. Section 3 is devoted to some estimates on solutions $\{w_k\}_{k=0}^{[T/h]}$ of (1.2) - (1.3) and $\{C^h(t)\}_{t\in[0,T),h>0}$. In section 4 we obtain (1.7) in the case of the smooth and compact MCF and show its optimality.

We use the following notations: For $m \in \mathbb{N}\cup\{0\}$, $\alpha \in (0,1)$, $Q \subset [0,T)\times\mathbb{R}^N$, $f : Q \longrightarrow \mathbb{R}$,

$$\begin{aligned}
&Df = D_x f := (\partial f/\partial x_1, \cdots, \partial f/\partial x_N),\ D_t f = f_t := \partial_t f,\\
&D_x^l f := \partial^{|l|} f/\partial x_1^{l_1}\cdots\partial x_N^{l_N}, |l| = l_1 + \cdots + l_N \text{ for } l = (l_1,\cdots,l_N) \in (\mathbb{N}\cup\{0\})^N\\
&D^2 f := (\partial^2 f/\partial x_i \partial x_j)_{1\le i,j\le N}.
\end{aligned}$$

For $u : \mathbb{R}^N \longrightarrow \mathbb{R}$, $v : [0,T)\times\mathbb{R}^N \longrightarrow \mathbb{R}$ and $\mu \in \mathbb{R}$,

$$\begin{aligned}
\{u \ge \mu\} &:= \{x \in \mathbb{R}^N \mid u(x) \ge \mu\},\\
\{v \ge \mu\} &:= \{(t,x) \in [0,T)\times\mathbb{R}^N \mid v(t,x) \ge \mu\},\\
\{v(t,\cdot) \ge \mu\} &:= \{x \in \mathbb{R}^N \mid v(t,x) \ge \mu\} \text{ etc.}
\end{aligned}$$

Let $\mathcal{U}$ be a metric space and $\mathcal{V}$ a dense subset of $\mathcal{U}$.

$$UC(\mathcal{U}) := \text{the set of all uniformly continuous functions.}$$

For $Q \subset [0,T)\times\mathbb{R}^N$,

$$\begin{aligned}
f(t,x) = O(g(t,x)) \Longleftrightarrow\ &|f(t,x)| \le Kg(t,x)\\
&\text{for some } K > 0 \text{ independent of } (t,x) \in Q.
\end{aligned}$$

Besides we use the following symbols.

$$\begin{aligned}
&\langle p,q\rangle = \text{the inner product between } p,q \in \mathbb{R}^N,\\
&\mathrm{cl}\,A = \text{the closure of } A,\\
&P(x,\delta) := \prod_{i=1}^{N}(x_i - \delta, x_i + \delta) \text{ for } x = (x_1,\cdots,x_N) \in \mathbb{R}^N \text{ and } \delta > 0\\
&\qquad\quad = N\text{-dimensional open cube centered at } x,\\
&[r] = \text{Gauss symbol for } r \in \mathbb{R},\\
&\mathbb{S}^N = \text{the set of all } N\times N\text{-real symmetric matrices},\\
&\mathrm{tr}\,X = \text{the trace of } X \in \mathbb{S}^N,\\
&d_H(A,B) := \max\left\{\sup_{x\in A}\mathrm{dist}\,(x,B), \sup_{x\in B}\mathrm{dist}\,(x,A)\right\} \quad \text{for } A,B \subset \mathbb{R}^N\\
&\qquad\quad = \text{Hausdorff distance between the sets } A \text{ and } B.
\end{aligned}$$

2 Preliminaries

2.1 Assumption

For a given compact hypersurface $\Gamma_0 \subset \mathbb{R}^N$, assume that

(2.1) $$\Gamma_0 \in C^{5+\alpha} \quad \text{for some } \alpha \in (0,1).$$

Then there uniquely exists a smooth and compact MCF $\{\Gamma(t)\}_{t\in[0,T_0)}$ with $\Gamma(0)=\Gamma_0$ for some $T_0>0$. Define the signed distance function $\rho(t,x)$ to $\Gamma(t)$ by

$$\rho(t,x):=d(x,D(t)) \tag{2.2}$$

where $D(t)$ denotes the compact set such that $\partial D(t)=\Gamma(t)$ and $d(x,D(t))$ is defined by (1.4) with $D=D(t)$ for each $t\in[0,T_0)$. Then for each $\varepsilon>0$ there exists $\delta>0$ such that

$$\rho\in C^{(5+\alpha)/2,(5+\alpha)}(\mathcal{N}_{\varepsilon,10\delta}),\ \mathcal{N}_{\varepsilon,10\delta}:=\{(t,x)\in[0,T_0-\varepsilon]\times\mathbb{R}^N\mid|\rho(t,x)|\leq 10\delta\}. \tag{2.3}$$

and the derivatives $D_t^mD_x^l\rho$ $(2m+|l|\leq 5)$ are bounded on $\mathcal{N}_{\varepsilon,10\delta}$. See Evans - Spruck [4].

2.2 Level set equation and generalized MCF

The level set equation to (1.1) is given by

$$\begin{aligned} &u_t+F(Du,D^2u)=0 \quad \text{in } (0,T)\times\mathbb{R}^N,\\ &F(p,X):=-\mathrm{tr}X+\frac{\langle Xp,p\rangle}{|p|^2} \quad \text{for } (p,X)\in(\mathbb{R}^N\backslash\{0\})\times\mathbb{S}^N. \end{aligned} \tag{2.4}$$

Since (2.4) has a singularity at $p=0$, we adopt the notion of viscosity solutions to consider weak solutions of (2.4). Here we only give the definition and the well-definedness of the generalized MCF. See [2] and [5] for the detail.

Definition 2.1. *Let $u\in UC([0,T)\times\mathbb{R}^N)$ be a viscosity solution of (2.4). Set*

$$\Gamma_L(t):=\{u(t,\cdot)=0\},\ \Omega_L^+(t):=\{u(t,\cdot)>0\},\ \Omega_L^-(t):=\{u(t,\cdot)<0\} \tag{2.5}$$

for each $t\in[0,T)$. We call the family $(\Gamma_L(t),\Omega_L^+(t),\Omega_L^-(t))_{t\in[0,T)}$ a generalized MCF.

Theorem 2.1. *Let $(\Gamma_L(t),\Omega_L^+(t),\Omega_L^-(t))_{t\in[0,T)}$ be defined by (2.5). Here $u\in UC([0,T)\times\mathbb{R}^N)$ is a unique viscosity solution of (2.4) with the initial data $u_0\in UC(\mathbb{R}^N)$. Then this family is determined independently of the choice of $u_0\in UC(\mathbb{R}^N)$ satisfying $\Gamma_L(0)=\{u_0=0\}$, $\Omega_L^+(0)=\{u_0>0\}$ and $\Omega_L^-(0)=\{u_0<0\}$.*

3 Estimates on $\{w_k\}_{k=0}^{[T/h]}$ and $\{C^h(t)\}_{t\in[0,T),h>0}$

Let $\{w_k\}_{k=0}^{[T/h]}$ be the sequence of classical solutions of (1.9) - (1.10) and let $C^h(t)$ be given by (1.6). In this section we derive some estimates for $\{w_k\}_{k=0}^{[T/h]}$ and $\{C^h(t)\}_{t\in[0,T),h>0}$.

3.1 Basic estimates

First, we show the uniform boundedness of $\{C^h(t)\}_{t\in[0,T),h>0}$.

Proposition 3.1. *Let $C_0\subset\mathbb{R}^N$ be compact and take $R_0>0$ so that $C_0\subset\mathrm{cl}B(0,R_0)$. Then $C^h(t)\subset\mathrm{cl}B(0,R_0)$ for all $t\in[0,T)$ and $h>0$.*

Proof. For any $x_0 \in \partial B(0, R_0)$ set $D_0(x_0) := \{x \in \mathbb{R}^N \mid \langle x - x_0, x_0 \rangle \leq 0\}$. Let $d(\cdot, D_0(x_0))$ be the signed distance function given by (1.4) with $D = D_0(x_0)$ and $\overline{w}_0 = \overline{w}_0(t,x) := d(x, D(x_0))$. Noting that $\Delta\overline{w}_0 = \Delta d(\cdot, D_0(x_0)) = 0$ in $\mathbb{R}^N$ since $\partial D_0(x_0)$ is a hyperplane, we easily see that $\overline{w}_0$ is a classical supersolution of (1.9) satisfying $d(\cdot, C_0) \leq \overline{w}_0(0, \cdot)$ in $\mathbb{R}^N$. Hence we use the maximum principle to have $w_0(t,x) \leq \overline{w}_0(t,x)$ for $(t,x) \in [0,h] \times \mathbb{R}^N$. Thus $C_1 \subset D_0(x_0)$.

Repeating the above argument, we get $C_k \subset D_0(x_0)$ for $k = 0, 1, 2, \dots, [T/h]$. As $x_0 \in \partial B(0, R_0)$ is arbitrary, we have the desired result. □

We have some global bounds of $\{w_k\}_{k=0}^{[T/h]}$ uniformly in $h > 0$.

Proposition 3.2. *We get* $-\sqrt{|x|^2 + 2Nt} - R_0 \leq w_k(t,x) \leq -|x| + R_0$ *for all* $(t,x) \in [0,h] \times \mathbb{R}^N$, $k = 0, 1, 2, \dots, [T/h]$ *and* $h > 0$, *where* R_0 *is given in Proposition 3.1.*

Proof. Fix $h > 0$ and $k = 0, 1, 2, \dots, [T/h]$. As for the upper estimate, we see from the proof of Proposition 3.1 that for all $h > 0$, $k = 0, 1, 2, \dots, [T/h]$ and $(t,x) \in [0,h] \times \mathbb{R}^N$,

$$w_k(t,x) \leq d(x, \mathrm{cl}\, B(x_0, R_0)) \leq -|x| + R_0.$$

Next we show the lower estimate. Set $k = 0$ for simplicity. Define $\underline{w} = \underline{w}(t,x) := -\sqrt{|x|^2 + 2Nt} - R_0$. Then we easily observe that $\underline{w}$ is a classical subsolution of (1.9) with $k = 0$ and that $\underline{w}(0, \cdot) \leq d(\cdot, C_0)$ in $\mathbb{R}^N$. We obtain the lower estimate by the maximum principle. □

Proposition 3.3. $|Dw_k(t,x)| \leq 1$ *for all* $(t,x) \in [0,h] \times \mathbb{R}^N$, $k = 0, 1, 2, \dots, [T/h]$ *and* $h > 0$.

Proof. Fix $h > 0$, $k = 0, 1, 2, \dots, [T/h]$. Since $v_k := |Dw_k|^2$ is a classical subsolution of (1.9) satisfying $v_k(0,x) = 1$ for a.e. $x \in \mathbb{R}^N$, the result follows from the maximum principle. □

3.2 Local estimates for $\{w_k\}_{k=0}^{[T/h]}$

Let $\rho = \rho(t,x)$ be the signed distance function to a smooth and compact CDM $\{\Gamma(t)\}_{t \in [0,T)}$ given by (2.2). This subsection is devoted to some local estimates for $\{w_k\}_{k=0}^{[T/h]}$ under (2.3).

The solution w_k of (1.9) - (1.10) is given by

$$w_k(t,x) = \int_{\mathbb{R}^N} E(t, x-y)\rho(kh, y)dy, \tag{3.1}$$

where $E = E(t,x)$ is the heat kernel. We use this formula and (2.3) to get the following.

Proposition 3.4. *The solution* w_k *of (1.9) - (1.10) with* $C_k := \{\rho(kh, \cdot) \geq 0\}$ *satisfies*

$$\sup_{\substack{k=0,1,2,\dots,[T/h] \\ h>0,\, 2m+|l| \leq 5}} \|D_t^m D_x^l w_k\|_{C([0,h] \times \{|\rho(kh,\cdot)| \leq 5\delta\})} =: K_1 < +\infty. \tag{3.2}$$

We need an estimate for $\{Dw_k\}_{k=0}^{[T/h]}$ to obtain the rate of convergence of our algorithm to a smooth and compact MCF.

Proposition 3.5. *For each $k = 0, 1, 2, \dots, [T/h]$, let w_k be a solution of (1.2) - (1.3) with $C_k = \{\rho(kh, \cdot) \geq 0\}$. There are constants $K_2 > 0$ and $t_1 > 0$ such that*

(3.3) $$\langle Dw_k, Dd(kh, \cdot)\rangle \geq 1 - K_2 t (> 0) \quad on\ [0, h] \times \{|\rho(kh, \cdot)| \leq 5\delta\}$$

for all $k = 0, 1, 2, \dots, [T/h]$ and $h \in (0, t_1)$.

Proof. We consider only the case $k = 0$ since the other ones are similarly proved. Recall that $\rho(0, \cdot) \in C^{5+\alpha}(\{|\rho(0, \cdot)| \leq 10\delta\})$ by (2.3). By (3.1) and the smoothness of $\rho(0, \cdot)$, we get

$$\begin{aligned} w_{0,x_i}(t, x) &= \int_{\mathbb{R}^N} E_{x_i}(t, y - x)\rho(0, y)dy = \int_{P(x,\delta')} E(t, y - x)\rho_{x_i}(0, y)dy + O(e^{-(\delta')^2/8t}) \\ &=: I_1 + O(e^{-(\delta')^2/8t}). \end{aligned}$$

We estimate I_1. It is observed by the change of variables $y - x \mapsto y$ and Taylor's theorem that for some $\theta \in (0, 1)$ and small $t > 0$,

$$\begin{aligned} I_1 = \int_{P(0,\delta')} E(t, y)\Bigg\{ &\rho_{x_i}(0, x) + \langle D\rho_{x_i}(0, x), y\rangle + \frac{1}{2}\langle D^2\rho_{x_i}(0, x)y, y\rangle \\ &+ \frac{1}{3!}\left(\sum_{i=1}^N y_i \frac{\partial}{\partial x_i}\right)^3 \rho_{x_i}(0, x + \theta y)\Bigg\}dy. \end{aligned}$$

By virtue of

$$\int_{P(0,\delta')} E(t, y)y_i dy = \int_{P(0,\delta')} E(t, y)y_i y_j dy = 0, \quad \int_{P(0,\delta')} E(t, y)y_i^2 dy = 2t + O(e^{-(\delta')^2/8t})$$

for all $i, j = 1, 2, \dots, N$ $(i \neq j)$, we get

$$|I_1 - \{\rho_{x_i}(0, x) + t\Delta\rho_{x_i}(0, x)\}| \leq K_{2,1} t^{3/2}.$$

for all $(t, x) \in [0, t_{1,1}] \times \{|\rho| \leq 5\delta\}$ and some $K_{2,1}$, $t_{1,1} > 0$. Hence Choosing $K_2 \geq K_{2,1}$ and $t_1 \leq t_{1,1}$, we obtain the desired result. □

Remark 3.1. It follows from Propositions 3.4 and 3.5 that

$$\langle Dw_k, Dd\rangle \geq 1 - K_3 t \quad \text{on } [0, h] \times \{|\rho(kh, \cdot)| \leq 5\delta\}$$

for all $k = 0, 1, 2, \dots, [T/h]$ and $h \in (0, t_1)$ and some $K_3 > 0$.

4 Convergence

4.1 Convergence to generalized MCF

The convergence of our algorithm can be obtained by the estimates in Propositions 3.1 - 3.3 and the method due to [8].

Theorem 4.1. *Let $u \in UC([0,T)\times\mathbb{R}^N)$ be a unique viscosity solution of (2.4) satisfying $u(0,\cdot) = d(\cdot, C_0)$ in $\mathbb{R}^N$. Let $(\Gamma_L(t), \Omega_L^+(t), \Omega_L^-(t))_{t\in[0,T)}$ be a generalized MCF given by (2.5). Let $\{C_k\}_{k=0}^{[T/h]}$ be the discrete evolution by our algorithm. Assume that*

$$\Gamma_L(t) = \partial\Omega_L^+(t) = \partial\Omega_L^-(t) \quad \text{for all } t \in [0,T). \tag{4.1}$$

Then we have

$$\lim_{h\to 0} d_H(C_{[t/h]}, \mathrm{cl}\,\Omega_L^+(t)) = 0 \quad \text{locally uniformly in } [0,T).$$

Remark 4.1. The condition (4.1) roughly means that for each $t \in [0,T)$, $\Gamma(t)$ is a hypersurface in $\mathbb{R}^N$. It is called the non-fattening condition.

4.2 Rate of convergence

Based on Theorem 4.1, we derive the rate of convergence of our algorithm to the smooth and compact MCF. For this purpose we reformulate our algorithm in the following way: Let C_0 be a compact subset of $\mathbb{R}^N$ whose boundary is of class $C^{5+\alpha}$. For each $h > 0$ let $\{w_k\}_{k=0}^{[T_0/h]}$ be a sequence of solutions of (1.2) - (1.3) with setting $C_k := \{w_{k-1}(h,\cdot) \geq 0\}$ ($k = 1, 2, \dots, [T_0/h]$). Define $w^h(t,x) := w_k(t-kh, x)$ for $t \in [kh, (k+1)h)$, $x \in \mathbb{R}^N$, $k = 0, 1, 2, \dots, [T_0/h]$ and $h > 0$ and $C^h(t)$ as

$$C^h(t) := \{w^h(t,\cdot) \geq 0\} \quad \text{for } t \in [0,T_0) \text{ and } h > 0 \tag{4.2}$$

instead of (1.6). Notice that $C^h(kh) = C_k$ for $k = 0, 1, 2, \dots, [T_0/h]$ and $h > 0$. We then obtain the following theorem.

Theorem 4.2. *Assume (2.1). Let $\{\Gamma(t)\}_{t\in[0,T_0)}$ be a smooth and compact MCF with $\Gamma(0) = \partial C_0$ and let $\rho = \rho(t,x)$ be defined by (2.2). Set $C^h(t)$ as (4.2) and $C(t) := \{\rho(t,\cdot) \geq 0\}$ for each $t \in [0,T_0)$ and $h > 0$. For any $\varepsilon > 0$, there exist L_1 and $h_0 > 0$ depending on (2.3) such that*

$$\sup_{t\in[0,T_0-\varepsilon]} d_H(C^h(t), C(t)) \leq L_1 h \quad \text{for all } h \in (0,h_0).$$

Since $\Gamma(t)$ is a hypersurface for every $t \in [0,T_0)$, Theorem 4.1 yields that for any $\varepsilon > 0$, $\eta_0 \in (0, 5\delta)$, there exists $h_{0,1} > 0$ such that

$$\sup_{t\in[0,T_0-\varepsilon]} d_H(C^h(t), C(t)) \leq \eta_0 \quad \text{for all } h \in (0,h_{0,1}). \tag{4.3}$$

Here $\delta > 0$ is the constant in (2.3). Theorem 4.2 is deduced from the following lemma.

Lemma 4.1. *Under the conditions in Theorem 4.2, if $d_H(C^h(kh), C(kh)) \leq \eta$ for small $\eta \in [0,\eta_0)$, then for some K_4, $t_2 > 0$ depending on (2.3),*

$$d_H(C^h(kh+\bar{t}), C(kh+\bar{t})) \leq \frac{\eta + K_4\bar{t}^2/2}{1 - K_4\bar{t}} \quad \text{for all } \bar{t} \in [0,h] \text{ and } h \in (0,t_2).$$

Outline of the proof. Assume that $(0 \leq) d_H(C^h(kh), C(kh)) \leq \eta$. Let W be a solution of (1.9) satisfying $W(0,\cdot) = d(\cdot, C(kh))$ in $\mathbb{R}^N$ and set $D^{\pm}_{\eta}(\bar{t}) := \{W(\bar{t},\cdot) \geq \pm\eta\}$ and $\Omega^{\pm}_{\eta}(kh+\bar{t}) := \{\rho(kh+\bar{t},\cdot) \geq \pm\eta\}$.

We easily get $W - \eta \leq w_k \leq W + \eta$ on $[0,h] \times \mathbb{R}^N$ from the maximum principle since $W(0,\cdot) - \eta \leq w_k(0,\cdot) \leq W(0,\cdot) + \eta$ in $\mathbb{R}^N$. Hence we have $D^+_{\eta}(\bar{t}) \subset C^h(kh+\bar{t}) \subset D^-_{\eta}(\bar{t})$ for all $\bar{t} \in [0,h]$. Since $\Omega^+_{\eta}(kh+\bar{t}) \subset C(kh+\bar{t}) \subset \Omega^+_{\eta}(kh+\bar{t})$, we have

$$\Omega^+_{\eta}(kh+\bar{t}) \cap D^+_{\eta}(\bar{t}) \subset C(kh+\bar{t}), C^h(kh+\bar{t}) \subset \Omega^-_{\eta}(kh+\bar{t}) \cup D^-_{\eta}(\bar{t}) \quad \text{for } \bar{t} \in [0,h].$$

Therefore we observe that for all $\bar{t} \in [0,h]$,

$$\begin{aligned} d_H(C^h(kh+\bar{t}), C^h(kh+\bar{t})) \leq \max\{ & d_H(\Omega^+_{\eta}(kh+\bar{t}) \cap D^+_{\eta}(\bar{t}), C^h(kh+\bar{t})), \\ & d_H((\Omega^-_{\eta}(kh+\bar{t}) \cup D^-_{\eta}(\bar{t}), C^h(kh+\bar{t}))\}. \end{aligned} \tag{4.4}$$

We estimate the right-hand side of (4.4). It is easily seen that

$$\begin{aligned} & d_H(\Omega^+_{\eta}(kh+\bar{t}) \cap D^+_{\eta}(\bar{t}), C^h(kh+\bar{t})) \\ & \quad \leq d_H(D^+_{\eta}(\bar{t}), C^h(kh+\bar{t})) + d_H(\Omega^+_{\eta}(kh+\bar{t}), D^+_{\eta}(\bar{t})), \\ & d_H(\Omega^-_{\eta}(kh+\bar{t}) \cup D^-_{\eta}(\bar{t}), C^h(kh+\bar{t})) \\ & \quad \leq d_H(D^-_{\eta}(\bar{t}), C^h(kh+\bar{t})) + d_H(\Omega^-_{\eta}(kh+\bar{t}), D^-_{\eta}(\bar{t})). \end{aligned}$$

As W satisfies Proposition 3.5, we get from some calculations

$$d_H(D^{\pm}_{\eta}(\bar{t}), C^h(kh+\bar{t})) \leq \frac{\eta}{1-K_1\bar{t}} \quad \text{for all } \bar{t} \in [0,h] \text{ and } h > 0.$$

Step 1. We derive an estimate for $\sup_{x \in D^+_{\eta}(\bar{t})} \operatorname{dist}(x, \Omega^+_{\eta}(kh+\bar{t}))$.

Fix $\bar{t} \in [0,h]$ and $x \in D^+_{\eta}(\bar{t})$. We may assume that $x \in \partial D^+_{\eta}(\bar{t}) \backslash \Omega^+_{\eta}(kh+\bar{t})$. Set $\widetilde{\rho}(\bar{t},x) := \rho(kh+\bar{t},x)$. Notice that for $s \in [0,h]$ the point $z(s,x) := x - \widetilde{\rho}(s,x) D\widetilde{\rho}(s,x) \in \partial\Omega^+_{\eta}(kh+s)$ satisfies $|x - z(s,x)| = |\widetilde{\rho}(s,x)| = \operatorname{dist}(x, \partial\Omega^+_{\eta}(kh+s))$. Tedious calculations yields that

$$\begin{aligned} & \sup_{\substack{s \in [0,h], x \in D^+_{\eta}(\bar{t}), \\ k=0,1,2,\ldots,[T_0/h], h>0}} |W(s, z(s,x)) - \eta| \leq K_{4,1} s^2, \\ & \eta = W(\bar{t},x) = W(\bar{t}, z(\bar{t},x)) + \widetilde{\rho}(\bar{t},x) \langle DW(\bar{t}, z^{\theta}(\bar{t},x)), D\widetilde{\rho}(\bar{t},x) \rangle, \\ & z^{\theta}(\bar{t},x)) := x - \theta\widetilde{\rho}(\bar{t},x) D\widetilde{\rho}(\bar{t},x), \ \theta \in (0,1). \end{aligned}$$

Combining these formulae, we get

$$\sup_{x \in D(\bar{t})} \operatorname{dist}(x, D^+_{\eta}(\bar{t})) = \sup_{x \in D(\bar{t})} |\widetilde{\rho}(\bar{t},x)| \leq \frac{K_{4,1} t^2}{1 - K_3 t}.$$

Here and in the sequel $K_{4,j} > 0$ $(j \in \mathbb{N})$ is a constant depending on (2.3) and (3.2).

Step 2. We estimate $\sup_{x \in \Omega^+_{\eta}(kh+\bar{t})} \operatorname{dist}(x, D^+_{\eta}(\bar{t}))$.

Fix $\bar{t} \in [0,h]$ and $x \in \Omega^+_{\eta}(kh+\bar{t})$. We may assume that $x \in \partial\Omega^+_{\eta}(kh+\bar{t}) \backslash D^+_{\eta}(\bar{t})$. Let $\widehat{\rho}(\bar{t},x)$ be the signed distance function given by (2.2) with $\Gamma(\bar{t}) = \partial D^+_{\eta}(\bar{t})$. For $s \in [0,h]$, the point $\widehat{z}(s,x) := x - \widehat{\rho}(s,x)D\widehat{\rho}(s,x) \in \partial D_\eta^+(s)$ satisfies $|x - \widehat{z}(s,x)| = |\rho(s,x)| = \operatorname{dist}(x, \partial D_\eta^+(s))$. Similar calculations to those in the previous step yield that

$$\sup_{\substack{\bar{t}\in[0,h],x\in\partial\widehat{C}(kh+s)\\ k=0,1,\dots,[T/h],h>0}} |\rho(kh+\bar{t}, \widehat{z}(\bar{t},x)) - \eta| \le K_{4,2}\bar{t}^2,$$

$$\eta = \rho(kh+\bar{t},x) = \rho(kh+\bar{t},\widehat{z}(\bar{t},x)) + \widehat{\rho}(\bar{t},x)\langle D\rho(kh+\bar{t}, x - \theta\widehat{\rho}(\bar{t},x)D\widehat{\rho}(\bar{t},x)), D\widehat{\rho}(\bar{t},x)\rangle.$$

Therefore we have by using Propositions 3.3 and 3.5

$$\sup_{x\in\Omega_\eta^+(kh+\bar{t})} \operatorname{dist}(x, D_\eta^+(\bar{t})) = \sup_{x\in\Omega_\eta^+(kh+\bar{t})} |\widehat{\rho}(\bar{t},x)| \le \frac{K_{4,2}\bar{t}^2}{1-K_3\bar{t}}.$$

Combining the estimates in Step 1, 2 and setting $K_4 := \max\{K_3, K_{4,1}, K_{4,2}\}$ and $t_2 = t_1$, we obtain

$$d_H(\Omega_\eta^+(kh+\bar{t}), D_\eta^+(\bar{t})) \le \frac{K_4\bar{t}^2}{1-K_4\bar{t}} \quad \text{for all } \bar{t}\in[0,h] \text{ and } h\in[0,t_2].$$

The estimate of $d_H(\Omega_\eta^-(kh+\bar{t}), D_\eta^-(\bar{t}))$ is obtained by the same way. Therefore we get the desired result. □

Proof of Theorem 4.2. In the case $k = 0$, we apply Lemma 4.1 with $\eta := 0$ to have

$$\sup_{\bar{t}\in[0,h]} d_H(C^h(\bar{t}), C(\bar{t})) \le \frac{K_4h^2}{1-K_4h}.$$

In the case $k = 1$, it follows from Lemma 4.1 with $\eta := K_4h^2/\{1-K_4h\}$ to obtain

$$\sup_{\bar{t}\in[0,h]} d_H(C^h(h+\bar{t}), C(h+\bar{t})) \le \frac{K_4h^2}{(1-K_4h)^2} + \frac{K_4h^2}{1-K_4h}.$$

Repeating this process, we see that for $k = 2, 3, \dots, [T_0/h]$

$$\sup_{\bar{t}\in[0,h]} d_H(C^h(kh+\bar{t}), C(kh+\bar{t})) \le \sum_{l=1}^{k+1} \frac{K_4h^2}{(1-K_4h)^l} \le (e^{K_4T_0}-1)h.$$

Letting $L_1 := e^{K_4T_0} - 1$, we get the desired result. □

4.3 Optimality

This subsection is devoted to the optimality of the estimate in Theorem 4.2. For this purpose we consider the radial case. For simplicity, we set $N = 2$, $R(t) := \sqrt{1-2t}$, $T_0 := 1/2$ and $C(t) := \{x \in \mathbb{R}^2 \mid |x| \le R(t)\}$. Since it suffices to consider the radial solution, the initial value problem (1.9) - (1.10) and the definition of $\{C_k\}_{k=0}^{[T/h]}$ turn to

(4.5) $$w_{k,t} = w_{k,rr} + \frac{w_{k,r}}{r},\ w_k = w_k(t,r) \quad \text{in } (0,+\infty)\times(0,+\infty),$$

(4.6) $$w_{k,r}(t,0) = 0 \quad \text{for } t > 0,$$

(4.7) $$w_k(0,r) = R_k - r \quad \text{for } r\in[0,+\infty),$$

$$C_k := \{x\in\mathbb{R}^2 \mid w_k(h,|x|)\ge 0\},\ C_0 := \operatorname{cl} B(0,1),$$

$$R_k := \text{radius of } C_k,\ R_0 := 1.$$

For $t \in [kh, (k+1)h)$, $k = 0, 1, 2, \ldots, [T/h]$ and $h > 0$, set

$$C^h(t) := \{x \in \mathbb{R}^2 \mid w_k(t - kh, |x|) \geq 0\}, \; R^h(t) := \text{radius of } C^h(t).$$

The following proposition says that for each $h > 0$, $C^h(t)$ evolves faster than $C(t)$.

Proposition 4.1. *$C^h(t) \subset C(t)$ for all $t \in [0, T_0)$ and $h > 0$.*

Proof. Let $V_0 = V_0(t, r) := 1 - \sqrt{r^2 + 2t}$. Then $C(t) = \{V_0(t, |\cdot|) \geq 0\}$ for $t \in [0, h]$ and V_0 is a classical supersolution of (4.5) satisfying (4.6) and (4.7). Hence it follows from the maximum principle that $w_0 \leq V_0$ on $[0, h] \times [0, +\infty)$. This inequality yield that $C^h(t) \subset C(t)$ for all $t \in [0, h]$.

Set $V_1 = V_1(t, r) := 1 - \sqrt{r^2 + 2(t + h)}$. Then $C(t + h) = \{V_1(t, |\cdot|) \geq 0\}$ for $t \in [0, h]$ and V_1 is a classical supersolution of (4.5) satisfying (4.6) and $V_1(0, \cdot) \geq w_1(0, \cdot)$ on $[0, +\infty)$. Thus we get $w_1 \leq V_1$ on $[0, h] \times [0, +\infty)$ by the maximum principle. Therefore $C^h(t) \subset C(t)$ for all $t \in [h, 2h]$. We have the result by induction. □

We need an estimate for $w_{k,r}$.

Proposition 4.2. *For any $\delta \in (0, 1/8)$, there are constants $K_5 > 0$ and $h_1 > 0$ depending on δ such that*

$$\left| w_{k,r}(\bar{t}, r) - \left(-1 + \frac{\bar{t}}{r^2} \right) \right| \leq K_5 \bar{t}^2. \tag{4.8}$$

for all $\bar{t} \in [0, h]$, $r \in [\delta, +\infty)$ and $h \in (0, h_1)$.

Proof. Some calculations yield that

$$\left| Dw_k(\bar{t}, |x|) - \left(-\frac{x}{|x|} + \bar{t}\frac{x}{|x|^3} \right) \right| \leq K_5 \bar{t}^2$$

for small $\bar{t} > 0$ and $x \in \mathbb{R}^N \backslash B(0, \delta)$. Noting the formula $w_{k,r} = \langle Dw_k, x/|x| \rangle$, we get the desired result. □

Since we see by Proposition 4.1 and Theorem 4.2 that for any $\varepsilon \in (0, 1/4)$

$$d_H(C^h(t), C(t)) = R(t) - R^h(t) \leq L_1 h, \; R^h(t) \geq \sqrt{\varepsilon} \tag{4.9}$$

for all $t \in [0, 1/2 - \varepsilon]$ and $h \in (0, h_1)$, we consider the lower bound of $R(t) - R^h(t)$ for small $h > 0$ to prove the optimality of Theorem 4.2.

Theorem 4.3. *Set $C(t) := \{|x| \leq R(t)\}$ ($R(t) = \sqrt{1 - 2t}$) and $C^h(t) = \{w_k(t - kh, |x|) \geq 0\}$. Let $R^h(t)$ be the radius of $C^h(t)$. Then for any $\varepsilon \in (0, 1/4)$ there exists $h_2 > 0$ such that for all $h \in (0, h_2)$*

$$R(t) - R^h(t) \geq \begin{cases} \frac{1}{2}t^2 & \text{for } t \in [0, h], \\ \frac{1}{4}th & \text{for } t \in [h, T_0 - \varepsilon]. \end{cases} \tag{4.10}$$

The strategy of the proof of Theorem 4.3 is similar to that of Theorem 4.2.

Lemma 4.2. *Fix $\varepsilon \in (0, 1/4)$. If $R(kh) - R^h(kh) \geq \eta$ for small $\eta \geq 0$, then for some $K_6 = K_6(\varepsilon) > 0$,*

$$R(kh+\bar{t}) - R^h(kh+\bar{t}) \geq \eta + \frac{\bar{t}^2}{(R(kh))^3} - K_6\bar{t}^3 \tag{4.11}$$

for all $\bar{t} \in [0, h]$ and small $h > 0$.

Proof. The argument is quite similar to that in the proof of Theorem 4.2.

Assume that $R(kh) - R^h(kh) \geq \eta$ for small $\eta > 0$. Let w_k be a solution of (4.5) - (4.6) - (4.7). Set $\xi(\bar{t}) := w_k(\bar{t}, R(kh+\bar{t}))$ for $\bar{t} \in [0, h]$. Then we observe by (4.5) and the regularity of w_k near $r = R(kh)$

$$w_k(\bar{t}, R(kh+\bar{t})) \leq -\eta - \frac{3\bar{t}^2}{2(R(kh))^3} + K_6\bar{t}^3 \tag{4.12}$$

for all $\bar{t} \in [0, h]$ and small $h > 0$.

On the other hand, we see by the mean value theorem that

$$\begin{aligned} w_k(\bar{t}, R(kh+\bar{t})) &= w_k(\bar{t}, R^h(kh+\bar{t})) \\ &\quad + w_{k,r}(\bar{t}, R(kh+\bar{t}) + \widetilde{\theta})(R(kh+\bar{t}) - R^h(kh+\bar{t})) \\ &= w_{k,r}(\bar{t}, R(kh+\bar{t}) + \widetilde{\theta})(R(kh+\bar{t}) - R^h(kh+\bar{t})), \end{aligned}$$

where $\widetilde{\theta} := \theta(R^h(kh+\bar{t}) - R(kh+\bar{t}))(< 0)$ and $\theta \in (0, 1)$. Hence we obtain

$$R(kh+\bar{t}) - R^h(kh+\bar{t}) = \frac{-w_k(\bar{t}, R(kh+\bar{t}))}{-w_{k,r}(\bar{t}, R(kh+\bar{t}) + \widetilde{\theta})} \tag{4.13}$$

It follows from (4.8) that $-1 \leq w_{k,r}(\bar{t}, R(kh+\bar{t})+\widetilde{\theta}) \leq -1/2$. Hence $1/2 \leq -w_{k,r}(\bar{t}, R(kh+\bar{t}) + \widetilde{\theta}) \leq 1$. Using (4.12) and this inequality, we obtain (4.11). □

Proof of Theorem 4.3. Take $h_1 > 0$ so small that $1 - K_6\bar{t} \geq 1/2$ for all $\bar{t} \in [0, h]$ and $h \in (0, h_1)$. In the case $k = 0$, as $R(0) = R^h(0) = 1$, we apply Lemma 4.2 with $\eta = 0$ to have

$$R(\bar{t}) - R^h(\bar{t}) \geq \frac{\bar{t}^2}{(R(0))^3} - K_6\bar{t}^3 \geq \frac{\bar{t}^2}{2(R(0))^3}$$

In the case $k = 1$, we use Lemma 4.2 with $\eta = h^2/2(R(0))^2$ to obtain

$$R(h+\bar{t}) - R^h(h+\bar{t}) \geq \eta + \frac{\bar{t}^2}{(R(h))^3} - K_6\bar{t}^3 \geq \frac{1}{2}\left(\frac{h^2}{(R(0))^2} + \frac{\bar{t}^2}{(R(h))^3}\right)$$

for all $\bar{t} \in [0, h]$. Here we have used the fact that $\sqrt{2\varepsilon} \leq R(t) \leq R(0) = 1$ for all $t \in [0, T_0 - \varepsilon]$. Hence we are able to prove by induction that

$$R(kh+\bar{t}) - R^h(kh+\bar{t}) \geq \sum_{l=0}^{k} \frac{h^2}{2(R(lh))^3} + \frac{\bar{t}^2}{2(R(kh))^3}$$

for all $\bar{t} \in [0,h]$, $k = 0,1,2,\dots,[T/h]$ and $h > 0$.

For any $\varepsilon \in (0, T_0/2)$, choosing a small $h_2 > 0$ we get

$$R(kh+\bar{t}) - R^h(kh+\bar{t}) \geq \frac{1}{2}\left\{\sum_{l=0}^{k} \frac{h^2}{(R(lh))^3} + \frac{\bar{t}^2}{(R(kh))^2}\right\} \geq \frac{kh^2+\bar{t}^2}{2} \geq \frac{(kh+\bar{t})h}{4}$$

for all $\bar{t} \in [0,h]$, $k = 1,2,\dots,[T/h]$ and $h \in (0,h_2)$. Hence the proof is completed. □

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