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Solvability of heat equations with hysteresis coupled with Navier-Stokes equations in 2D and 3D

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This is a prompt report of the author [33].

1 Introduction

1.1 Problem and related works

Let $T > 0$ and $\Omega \subset \mathbb{R}^N$ $(N = 2, 3)$ be a bounded domain with smooth boundary Γ. We consider the following problem (P):

$$
\text{(P)} \qquad \begin{cases}
\psi_1(\theta) \leq w \leq \psi_2(\theta) & \text{in } Q := (0,T) \times \Omega, \\
\partial w/\partial t = 0 & \text{in } Q[\psi_1(\theta) < w < \psi_2(\theta)], \\
\partial w/\partial t > 0 & \text{in } Q[w = \psi_1(\theta)], \\
\partial w/\partial t < 0 & \text{in } Q[w = \psi_2(\theta)], \\
\partial\theta/\partial t - \Delta\theta + \boldsymbol{v} \cdot \nabla\theta + w = f & \text{in } Q, \\
\partial \boldsymbol{v}/\partial t - \Delta \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \boldsymbol{g}(\theta) - \nabla\pi & \text{in } Q, \\
\operatorname{div} \boldsymbol{v} = 0 & \text{in } Q, \\
\theta = 0, \quad \boldsymbol{v} = 0 & \text{in } (0,T) \times \Gamma, \\
w(0) = w_0, \quad \theta(0) = \theta_0, \quad \boldsymbol{v}(0) = \boldsymbol{v}_0 & \text{in } Q,
\end{cases}
$$

where $w : Q \to \mathbb{R}$, $\theta : Q \to \mathbb{R}$, $\boldsymbol{v} : Q \to \mathbb{R}^N$ and $\pi : Q \to \mathbb{R}$ stand for the hysteresis term, the temperature, the velocity and the pressure, respectively, and these are unknown functions; $\psi_1, \psi_2 : \mathbb{R} \to \mathbb{R}$, $f : Q \to \mathbb{R}$, $\boldsymbol{g} : \mathbb{R} \to \mathbb{R}^N$, $w_0 : \Omega \to \mathbb{R}$, $\theta_0 : \Omega \to \mathbb{R}$ and $\boldsymbol{v}_0 : \Omega \to \mathbb{R}^N$ are given functions.

From a view point of physics the problem (P) describes the temperature θ, the velocity $\boldsymbol{v}$ and the pressure π of incompressible fluid in a bounded region Ω on a time period $[0, T]$. It is especially peculiar that the temperature will be controlled by the heat source $-w$. which is fluctuated by the present temperature. Such phenomenon comes from the temperature-dependent constraint on w:

$$\psi_1(\theta) \leq w \leq \psi_2(\theta).$$

Typical examples of ψ_1, ψ_2 are non-decreasing functions. Then such model represents e.g., phenomenon by thermostat devices. For more details, if the temperature θ rises (falls), then the heat source $-w$ will fall (rise), influenced by the obstacle functions ψ_1, ψ_2. This means that thermostat devices cool (heat) the fluid, responding to too high (low) temperature.

Mathematically, the problem (P) is the Boussinesq system with hysteresis formulated in a quasi-variational inequality, which represents the phenomenon by thermostat devices. Boussinesq systems are dealt with in many works such as Morimoto [25], Fukao-Kenmochi [8], Kubo [20], Fukao-Kubo [10], [11], Sobajima-the author-Yokota [28], Larios-Lunasin-Titi [21], Li-Xu [22], Miao-Zheng [23], Fukao-Kenmochi [9] and the author [31]. Thermostat models for hysteresis formulated in a quasi-variational inequality are studied in e.g., Kenmochi-Koyama-Meyer [17], and other models for such hysteresis are also studied in, Kubo [19], Colli-Kenmochi-Kubo [4] and so on. Thermostat models with relay hysteresis are studied by many authors such as Glashoff-Sprekels [12], [13], Visintin [34], Kopfová-Kopf [18], Gurevich-Jäger-Skubachevskii [15] and Gurevich-Tikhomirov [16].

Recently, the author [32] showed existence for the problem (P) in the 2D case with the Navier-Stokes equation in a weak sense. That is, (P) has at least one solution $(w, \theta, \boldsymbol{v})$ satisfying

$$\boldsymbol{v} \in H^1(0,T;(\boldsymbol{H}^1_\sigma(\Omega))^*) \cap L^\infty(0,T;\boldsymbol{L}^2_\sigma(\Omega)) \cap L^2(0,T;\boldsymbol{H}^1_\sigma(\Omega)) \tag{1.1}$$

with the condition

$$\boldsymbol{v}_0 \in \boldsymbol{L}^2_\sigma(\Omega) = D(A^0),$$

where $\boldsymbol{L}^2_\sigma(\Omega)$ and $\boldsymbol{H}^1_\sigma(\Omega)$ are roughly sets of Lebesgue and Sobolev functions satisfying divergence freeness, respectively (see Section 1.2), and

$$A : D(A) := \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_\sigma(\Omega) \subset \boldsymbol{L}^2_\sigma(\Omega) \to \boldsymbol{L}^2_\sigma(\Omega)$$

is the Stokes operator, which is defined as roughly $-\Delta$ (see Section 1.2). However this result does not assert uniqueness for (P). When we try to attain uniqueness for (P), we would put $(w_i, \theta_i, \boldsymbol{v}_i)$ as a solution of (P) $(i = 1, 2)$. In this case, $\|w_1 - w_2\|_{L^\infty(0,T;L^\infty(\Omega))}$ is required to be estimated, and hence so is $\|\theta_1 - \theta_2\|_{L^\infty(0,T;L^\infty(\Omega))}$. Then we need an appropriate estimate for

$$\theta_1 \cdot \nabla(\boldsymbol{v}_1 - \boldsymbol{v}_2) \quad \text{and} \quad (\theta_1 - \theta_2) \cdot \nabla \boldsymbol{v}_2. \tag{1.2}$$

This breaks down in [32] because of low regularity for solutions of the Navier-Stokes equation (see (1.1)).

The purpose of this paper is to establish existence and uniqueness for (P) with $\boldsymbol{v}$ more regular than the class (1.1). In order to decide height of regularity for $\boldsymbol{v}$ so that (1.2) can be appropriately estimated, we introduce the fractional power of the Stokes operator and its domain $D(A^\alpha)$ $(0 \le \alpha \le 1)$ (such operator A^α is dealt with by e.g., Fujiwara [7], Fujita-Morimoto [6], Ôtani [26], Mitrea-Monniaux [24], and Guermond-Salgado [14]). In fact, we will establish existence and uniqueness for (P) in a N-dimensional domain $(N = 2, 3)$, where the solution of the Navier-Stokes equation belongs the next class:

$$\boldsymbol{v} \in H^1(0,T;D(A^{\frac{1-\alpha}{2}})^*) \cap L^\infty(0,T;D(A^{\frac{\alpha}{2}})) \cap L^2(0,T;D(A^{\frac{1+\alpha}{2}}))$$

with the condition

$$\boldsymbol{v}_0 \in D(A^{\frac{\alpha}{2}}), \qquad \frac{3(N-2)}{4} < \alpha \le 1.$$

Here $D(A^{\frac{\alpha}{2}})$ is roughly a set of α-order differentiable functions satisfying divergence free.

1.2 Main results

First we introduce notation, starting with $H := L^2(\Omega)$, $V := H_0^1(\Omega)$, $\boldsymbol{H} := \boldsymbol{L}^2_\sigma(\Omega)$ and $\boldsymbol{V} := \boldsymbol{H}^1_\sigma(\Omega)$ with the standard inner products, respectively, where $\boldsymbol{L}^2_\sigma(\Omega)$ and $\boldsymbol{H}^1_\sigma(\Omega)$ are the closure of $\boldsymbol{\mathcal{D}}_\sigma(\Omega) := \{\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\Omega) = \boldsymbol{C}_0^\infty(\Omega) \mid \operatorname{div} \boldsymbol{v} = 0\}$ on $\boldsymbol{L}^2(\Omega)$ and $\boldsymbol{H}^1(\Omega)$, respectively. Here the dense and compact imbeddings $V \hookrightarrow H$ and $\boldsymbol{V} \hookrightarrow \boldsymbol{H} \hookrightarrow \boldsymbol{V}^*$ hold.

To formulate the equation for hysteresis we define the closed and convex set $K(\theta)$ and the indicator function I_θ, which are depending on $\theta \in H$, as

$$K(\theta) := \{w \in H \mid \psi_1(\theta) \le w \le \psi_2(\theta) \text{ a.e. on } \Omega\}, \qquad \theta \in H,$$
$$I_\theta(w) := \begin{cases} 0 & w \in K(\theta), \\ \infty & w \in H \setminus K(\theta), \end{cases} \qquad \theta \in H.$$

Then we introduce the subdifferential operator of ∂I_θ, which is characterized by $\xi \in \partial I_\theta(w) \Leftrightarrow (-\xi, w - z)_H \le 0$ $(z \in K(\theta))$ for $\theta \in H$ and $w \in D(\partial I_\theta) = K(\theta)$. For details on subdifferential operators we can refer to e,g., Barbu [1], [2].

On the other hand, for formulation of the Navier-Stokes equation, we define the Stokes operator $A : D(A) \subset \boldsymbol{H} \to \boldsymbol{H}$ as $A := -P\Delta$, where $D(A) := \boldsymbol{H}^2(\Omega) \cap \boldsymbol{V}$ and $P : \boldsymbol{L}^2(\Omega) \to \boldsymbol{H}$ is the Helmholtz projection. It is well-known the operator A can be extended to the following form:

$$A : \boldsymbol{V} \to \boldsymbol{V}^*, \qquad \langle A\boldsymbol{v}, \boldsymbol{z}\rangle_{\boldsymbol{V}^*,\boldsymbol{V}} := \sum_{i,j=1}^N \int_\Omega \frac{\partial v_j}{\partial x_i}\frac{\partial z_j}{\partial x_i}\,d\boldsymbol{x}, \quad \boldsymbol{v}, \boldsymbol{z} \in \boldsymbol{V}.$$

Here we introduce the fractional power of the Stokes operator A^α $(-1 \le \alpha \le 1)$, which is linear, unbounded and self-adjoint operator on $\boldsymbol{H}$. Moreover we define the Hilbert space $\boldsymbol{V}_\alpha$ as $\boldsymbol{V}_\alpha := D(A^{\frac{\alpha}{2}})$ for $0 \le \alpha \le 2$ and $\boldsymbol{V}_\alpha := \boldsymbol{V}^*_{-\alpha}$ for $-2 \le \alpha < 0$ with the inner product $(\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{V}_\alpha} := (A^{\frac{\alpha}{2}}\boldsymbol{u}, A^{\frac{\alpha}{2}}\boldsymbol{v})_{\boldsymbol{H}}$, $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}_\alpha$ for $-2 \le \alpha \le 2$, where $A^{\frac{\alpha}{2}}\boldsymbol{u} \in \boldsymbol{H}$ for $-2 \le \alpha < 0$ and $\boldsymbol{u} \in \boldsymbol{V}_\alpha$ means that $(A^{\frac{\alpha}{2}}\boldsymbol{u}, \boldsymbol{z})_{\boldsymbol{H}} = \langle \boldsymbol{u}, A^{\frac{\alpha}{2}}\boldsymbol{z}\rangle_{\boldsymbol{V}_\alpha, \boldsymbol{V}_{-\alpha}}$ for all $\boldsymbol{z} \in \boldsymbol{H}$. Then $\boldsymbol{V}_\alpha$ is a set of α-order differentiable functions as follows:

$$\boldsymbol{V}_\alpha = \begin{cases} \boldsymbol{H}^\alpha(\Omega) \cap \boldsymbol{H}, & 0 \le \alpha < \frac{1}{2}, \\ \boldsymbol{H}^\alpha_0(\Omega) \cap \boldsymbol{H}, & \frac{1}{2} \le \alpha \le 1, \\ \boldsymbol{H}^\alpha(\Omega) \cap \boldsymbol{V}, & 1 \le \alpha \le 2. \end{cases}$$

Here $\boldsymbol{H}^\alpha$ and $\boldsymbol{H}^\alpha_0$ are the fractional Sobolev spaces (see e.g., Demengel-Demengel [5]). Indeed, e.g., [14, Corollary 2.1] read the above characterization. For details on the fractional powers of the Stokes operator, we can refer to [7], [6] and [24]. Moreover note $\boldsymbol{V}_0 = \boldsymbol{H}$, $\boldsymbol{V}_1 = \boldsymbol{V}$ and the compact and dense imbeddings $\boldsymbol{V}_\alpha \hookrightarrow \boldsymbol{H} \hookrightarrow \boldsymbol{V}_{-\alpha}$ for $0 \le \alpha \le 1$. In this paper, we regard A as the following form for all $0 \le \alpha \le 1$:

$$A : \boldsymbol{V}_{1+\alpha} \to \boldsymbol{V}_{-1+\alpha},$$
$$\langle A\boldsymbol{v}, \boldsymbol{z}\rangle_{\boldsymbol{V}_{-1+\alpha}, \boldsymbol{V}_{1-\alpha}} = \left(A^{\frac{1+\alpha}{2}}\boldsymbol{v}, A^{\frac{1-\alpha}{2}}\boldsymbol{z}\right)_{\boldsymbol{H}}, \quad \boldsymbol{v} \in \boldsymbol{V}_{1+\alpha}, \boldsymbol{z} \in \boldsymbol{V}_{1-\alpha}.$$

Moreover we define the operator B as for all $0 \le \alpha \le 1$,

$$B : \boldsymbol{V}_\alpha \times \boldsymbol{V}_{1+\alpha} \to \boldsymbol{V}_{-1+\alpha},$$

$$\langle B(\boldsymbol{u},\boldsymbol{v}), \boldsymbol{z}\rangle_{\boldsymbol{V}_{-1+\alpha},\boldsymbol{V}_{1-\alpha}} := \int_\Omega ((\boldsymbol{u}\cdot\nabla)\boldsymbol{v})\boldsymbol{z}\, d\boldsymbol{x} = \sum_{i,j=1}^{N} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} z_j \, d\boldsymbol{x},$$

$$(\boldsymbol{u},\boldsymbol{v}) \in \boldsymbol{V}_\alpha \times \boldsymbol{V}_{1+\alpha},\ \boldsymbol{z} \in \boldsymbol{V}_{1-\alpha}.$$

Here (3.3) in Lemma 3.1 in Section 3 guarantees B operates $\boldsymbol{V}_\alpha \times \boldsymbol{V}_{1+\alpha}$ on $\boldsymbol{V}_{-1+\alpha}$.

Under the above setting we provide a definition of solutions.

Definition 1.1. A triplet $(w,\theta,\boldsymbol{v})$ is called a *solution* to (P) if the followings hold:

(D1) $w \in \mathcal{C}_1(T;\theta) := \{w \in H^1(0,T;H) \mid w(t) \in K(\theta(t)) \text{ for all } t \in [0,T]\}$,
$\theta \in \mathcal{C}_2(T) := H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;L^\infty(\Omega))$,
$\boldsymbol{v} \in \mathcal{C}_3(T) := H^1(0,T;\boldsymbol{V}_{-1+\alpha}) \cap L^\infty(0,T;\boldsymbol{V}_\alpha) \cap L^2(0,T;\boldsymbol{V}_{1+\alpha})$;

(D2)
$$\begin{aligned} &dw/dt + \partial I_\theta(w) \ni 0 && \text{in } H && \text{a.e. on } (0,T),\\ &d\theta/dt - \Delta\theta + \boldsymbol{v}\cdot\nabla\theta + w = f && \text{in } H && \text{a.e. on } (0,T),\\ &d\boldsymbol{v}/dt + A\boldsymbol{v} + B(\boldsymbol{v},\boldsymbol{v}) = P\boldsymbol{g}(\theta) && \text{in } \boldsymbol{V}_{-1+\alpha} && \text{a.e. on } (0,T);\end{aligned}$$

(D3) $(w(0),\theta(0),\boldsymbol{v}(0)) = (w_0,\theta_0,\boldsymbol{v}_0)$ in $H \times H \times \boldsymbol{H}$.

Now we are in a position to state the main results. Assume the following conditions:

(A1) $\psi_1, \psi_2 \in C^1(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R})$, $\psi_1 \le \psi_2$ on $\mathbb{R}$;

(A2) $f \in L^2(0,T;H) \cap L^1(0,T;L^\infty(\Omega))$, $\boldsymbol{g} \in \mathrm{Lip}(\mathbb{R};\mathbb{R}^N)$;

(A3) $w_0 \in K(\theta_0)$, $\theta_0 \in V \cap L^\infty(\Omega)$, $\boldsymbol{v}_0 \in \boldsymbol{V}_\alpha$.

Under the above assumption with the condition

$$\frac{3(N-2)}{4} < \alpha \le 1 \tag{1.3}$$

we establish solvability of global in time solutions in 2D and local in time solutions in 3D.

Theorem 1.1. *Let* $N = 2$, $0 < T < \infty$ *and* $0 < \alpha \le 1$, *Suppose* (A1)–(A3). *Then there exists a unique solution* $(w,\theta,\boldsymbol{v})$ *to* (P). *Furthermore, if* $(w_i,\theta_i,\boldsymbol{v}_i)$ *is a solution with the initial data* $(w_{0,i},\theta_{0,i},\boldsymbol{v}_{0,i})$ $(i = 1,2)$, *then continuous dependence of solutions on initial data holds*:

$$\begin{aligned}(1.4)\quad &\|w_1 - w_2\|_{L^\infty(0,T;L^\infty(\Omega))} + \|\theta_1 - \theta_2\|_{L^\infty(0,T;L^\infty(\Omega))} + \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{L^\infty(0,T;\boldsymbol{V}_\alpha)}\\ &\le C_0\left(\|w_{0,1} - w_{0,2}\|_{L^\infty(\Omega)} + \|\theta_{0,1} - \theta_{0,2}\|_V + \|\theta_{0,1} - \theta_{0,2}\|_{L^\infty(\Omega)} + \|\boldsymbol{v}_{0,1} - \boldsymbol{v}_{0,2}\|_{\boldsymbol{V}_\alpha}\right),\end{aligned}$$

where $C_0 > 0$ *is a constant, which increases depending on increase of* $\max_{i=1,2}\|\theta_{0,i}\|_H$, $\|\theta_{0,2}\|_V$ *and* $\max_{i=1,2}\|\boldsymbol{v}_{0,i}\|_{\boldsymbol{V}_\alpha}$.

Theorem 1.2. *Let $N = 3$, $0 < T < \infty$ and $\frac{3}{4} < \alpha \le 1$, Suppose* (A1)–(A3). *Put*

$$T_* = T_*(\psi_1, \psi_2, f, \boldsymbol{g}, \theta_0, \boldsymbol{v}_0) := \delta\gamma^{-\frac{4}{2\alpha-1}} \wedge T,$$

where $\delta > 0$ is a constant small enough, and $\gamma = \gamma(\psi_1, \psi_2, f, \boldsymbol{g}, \theta_0, \boldsymbol{v}_0) > 0$ is defined as

$$\gamma := \|\boldsymbol{v}_0\|_{\boldsymbol{V}_\alpha} + \|P\boldsymbol{g}(0)\|_{\boldsymbol{H}} + \|\boldsymbol{g}'\|_{\boldsymbol{L}^\infty(\mathbb{R})}\left(\|\theta_0\|_{L^\infty(\Omega)} + \|f\|_{L^1(0,T;L^\infty(\Omega))} + \max_{i=1,2}|\psi_i(0)|\right).$$

Then there exists a unique solution $(w, \theta, \boldsymbol{v})$ to (P) *with $T = T_*$. Furthermore, the continuous dependence of solutions on initial data* (1.4) *holds where $T = T_*$ and C_0 increases depending on increase of* $\max_{i=1,2}\|\theta_{0,i}\|_H$, $\|\theta_{0,2}\|_V$, $\|\theta_{0,2}\|_{L^\infty(\Omega)}$ *and* $\max_{i=1,2}\|\boldsymbol{v}_{0,i}\|_{\boldsymbol{V}_\alpha}$.

Remark 1.1. Let $N = 2, 3$ and $\alpha = 1$. Let $(w, \theta, \boldsymbol{v})$ be a solution to (P) for some $0 < T < \infty$. In light of the well-known fact $\boldsymbol{H}^\perp = \{\nabla\pi \in \boldsymbol{L}^2(\Omega) \mid \pi \in H^1(\Omega)\}$ (see e.g., Temam [29, Theorem 1.4 in Chapter I]), there exists a function π satisfying $\nabla\pi \in L^2(0, T; \boldsymbol{L}^2(\Omega))$ such that $\partial\boldsymbol{v}/\partial t - \Delta\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = \boldsymbol{g}(\theta) - \nabla\pi$ in $\boldsymbol{L}^2(\Omega)$.

2 Orientation

The proof of the main results proceeds in the following three steps.

1. In Section 4 we show existence and uniqueness of solutions to

$$\begin{cases} dw/dt + \partial I_\theta(w) \ni 0 & \text{in } H \quad \text{a.e. on } (0,T),\\ d\theta/dt - \Delta\theta + \boldsymbol{v}\cdot\nabla\theta + w = f & \text{in } H \quad \text{a.e. on } (0,T),\\ (w(0), \theta(0)) = (w_0, \theta_0) & \text{in } H \times H \end{cases}$$

with some estimates for θ with fixed $\boldsymbol{v}$. Hence we have the mapping $S_1 : \boldsymbol{v} \mapsto \theta$.

2. In Section 5 we also establish solvability for

$$\begin{cases} d\boldsymbol{v}/dt + A\boldsymbol{v} + B(\boldsymbol{v}, \boldsymbol{v}) = P\boldsymbol{g}(\theta) & \text{in } \boldsymbol{V}_{-1+\alpha} \quad \text{a.e. on } (0,T),\\ \boldsymbol{v}(0) = \boldsymbol{v}_0 \in \boldsymbol{V}_\alpha & \text{in } \boldsymbol{H} \end{cases}$$

with estimates for $\boldsymbol{v}$ with fixed θ. Thus the mapping $S_2 : \theta \mapsto \boldsymbol{v}$ appears.

3. In Section 6 we combine the above two problems by virtue of the contraction mapping principle for the mapping $S := S_2 \circ S_1$. The cornerstone of estimates toward contractivity of S is appropriate estimates for

$$\boldsymbol{v}_1 \cdot \nabla(\theta_1 - \theta_2) \quad \text{or} \quad (\boldsymbol{v}_1 - \boldsymbol{v}_2)\cdot\nabla\theta_2$$

by adopting the semigroup of the Dirichlet Laplacian and its properties.

In Sections 4 and 6 we will use the following norm. For a Banach space X and $k > 0$, we introduce an equivalent norm $\|\cdot\|_{L_k^\infty(0,T;X)}$ on the Lebesgue space $L^\infty(0, T; X)$ as follows:

$$\|u\|_{L_k^\infty(0,T;X)} := \sup_{t\in(0,T)} \|u(t)\|_X e^{-kt}, \quad u \in L^\infty(0, T; X), \tag{2.1}$$

with

$$\|\cdot\|_{L^\infty(0,T;X)} e^{-kT} \le \|\cdot\|_{L_k^\infty(0,T;X)} \le \|\cdot\|_{L^\infty(0,T;X)}.$$

Especially, in Section 6 we adopt $\|\cdot\|_{L_k^\infty(0,T;L^\infty(\Omega))}$ as the metric function of the contraction mapping principle.

3 Estimates for the convective terms

The space $\boldsymbol{V}_\alpha$ causes not a few complication when we estimate the convective terms $\boldsymbol{v}\cdot\nabla\theta$ or $B(\boldsymbol{u},\boldsymbol{v})$. The following lemma gives estimates for the convective terms.

Lemma 3.1. *The following holds*:

$$
\begin{aligned}
&(3.1) && \|\boldsymbol{v}\cdot\nabla\theta\|_H \le c_0\|\boldsymbol{v}\|_{\boldsymbol{V}_\alpha}\|\theta\|_V^{\rho}\|\Delta\theta\|_H^{1-\rho}, && \boldsymbol{v}\in\boldsymbol{V}_\alpha,\ \theta\in H^2(\Omega)\cap V,\\
&(3.2) && \|\boldsymbol{v}\cdot\nabla\theta\|_{L^\sigma(\Omega)} \le c_0\|\boldsymbol{v}\|_{\boldsymbol{V}_\alpha}\|\theta\|_{L^\infty(\Omega)}^{1/2}\|\Delta\theta\|_H^{1/2}, && \boldsymbol{v}\in\boldsymbol{V}_\alpha,\ \theta\in H^2(\Omega)\cap V,\\
&(3.3) && \|B(\boldsymbol{u},\boldsymbol{v})\|_{\boldsymbol{V}_{-1+\alpha}} \le c_0\|\boldsymbol{u}\|_{\boldsymbol{V}_\alpha}\|\boldsymbol{v}\|_{\boldsymbol{V}_\alpha}^{\rho}\|\boldsymbol{v}\|_{\boldsymbol{V}_{1+\alpha}}^{1-\rho}, && \boldsymbol{u}\in\boldsymbol{V}_\alpha,\ \boldsymbol{v}\in\boldsymbol{V}_{1+\alpha},\\
&(3.4) && \|B(\boldsymbol{u},\boldsymbol{u})\|_{\boldsymbol{V}_{-\frac{N}{2}+\alpha}} \le c_0\|\boldsymbol{u}\|_{\boldsymbol{H}}^{1/2}\|\boldsymbol{u}\|_{\boldsymbol{V}}^{1/2}\|\boldsymbol{u}\|_{\boldsymbol{V}_\alpha}^{1/2}\|\boldsymbol{u}\|_{\boldsymbol{V}_{1+\alpha}}^{1/2}, && \boldsymbol{u}\in\boldsymbol{V}_{1+\alpha},\\
&(3.5) && \|B(\boldsymbol{u},\boldsymbol{u})\|_{\boldsymbol{V}_{-\tau}} \le c_0\|\boldsymbol{u}\|_{\boldsymbol{H}}\|\boldsymbol{v}\|_{\boldsymbol{V}_{1+\alpha}}, && \boldsymbol{u}\in\boldsymbol{H},\ \boldsymbol{v}\in\boldsymbol{V}_{1+\alpha}.
\end{aligned}
$$

where $\rho\in(0,1]$ and $\sigma,\tau\in[1,\infty]$ are defined as

$$
\rho := \begin{cases} 1-\frac{N}{2}+\alpha, & N=2,\ 0<\alpha<1 \quad \text{or} \quad N=3,\ \frac{1}{2}<\alpha\le 1,\\ \frac{1}{2}, & N=2,\ \alpha=1.\end{cases}
$$

$$
\sigma := \begin{cases} \left(\frac{3}{4}-\frac{\alpha}{N}\right)^{-1}, & N=2,\ 0<\alpha<1 \quad \text{or} \quad N=3,\ \frac{1}{2}<\alpha\le 1,\\ 2, & N=2,\ \alpha=1,\end{cases}
$$

$$
\tau := \begin{cases} \frac{N}{2}-\alpha, & N=2,\ 0<\alpha<1 \quad \text{or} \quad N=3,\ \frac{1}{2}<\alpha\le 1,\\ \frac{1}{2}, & N=2,\ \alpha=1,\end{cases}
$$

and $c_0>0$ is a constant.

Proof. For simplicity use the notation $\|\cdot\|_p := \|\cdot\|_{L^p(\Omega)}$ or $\|\cdot\|_p := \|\cdot\|_{\boldsymbol{L}^p(\Omega)}$ and let $c>0$ denote certain constant. We use the Hölder inequality, the Sobolev inequality and the Gagliardo-Nirenberg inequality through the proof. At that time we choose N and α so as not to satisfy (3.1) with $\rho=0$ nor (3.3) with $\rho=0$. (See e.g., [5] for the Sobolev imbedding theorem with fractional orders).

First we see that

$$
\|\boldsymbol{v}\cdot\nabla\theta\|_H \le \|\boldsymbol{v}\|_{\left(\frac{1}{2}-\frac{\alpha}{N}\right)^{-1}}\|\nabla\theta\|_{\left(\frac{\alpha}{N}\right)^{-1}} \le c\|\boldsymbol{v}\|_{\boldsymbol{V}_\alpha}\|\theta\|_V^{1-\frac{N}{2}+\alpha}\|\Delta\theta\|_H^{\frac{N}{2}-\alpha}.
$$

Here note that $\frac{1}{2}-\frac{\alpha}{N}\neq 0$ and $\frac{1}{2}-\frac{1}{N}<\frac{\alpha}{N}\le\frac{1}{2}$. On the other hand, if $N=2$ and $\alpha=1$, then (by using the Poincaré inequality if needed) it follows that

$$
\begin{aligned}
(3.6)\qquad \|\boldsymbol{v}\cdot\nabla\theta\|_H &\le \|\boldsymbol{v}\|_4\|\nabla\theta\|_4 \le c\|\boldsymbol{v}\|_{\boldsymbol{H}}^{1/2}\|\boldsymbol{v}\|_{\boldsymbol{V}}^{1/2}\|\theta\|_V^{1/2}\|\Delta\theta\|_H^{1/2}\\
&\le c\|\boldsymbol{v}\|_{\boldsymbol{V}}\|\theta\|_V^{1/2}\|\Delta\theta\|_H^{1/2}.
\end{aligned}
$$

Hence the desired inequality (3.1) holds. We also see that

$$
\|\boldsymbol{v}\cdot\nabla\theta\|_{\left(\frac{3}{4}-\frac{\alpha}{N}\right)^{-1}} \le \|\boldsymbol{v}\|_{\left(\frac{1}{2}-\frac{\alpha}{N}\right)^{-1}}\|\nabla\theta\|_{\left(\frac{1}{4}\right)^{-1}} \le c\|\boldsymbol{v}\|_{\boldsymbol{V}_\alpha}\|\theta\|_{L^\infty(\Omega)}^{1/2}\|\Delta\theta\|_H^{1/2}.
$$

Here note that $\frac{1}{2}-\frac{\alpha}{N}\neq 0$. This inequality and (3.6) yield the desired inequality (3.2). Next it follows that for all $\boldsymbol{z}\in\boldsymbol{V}_{1-\alpha}$,

$$\begin{aligned}\left|\int_\Omega((\boldsymbol{u}\cdot\nabla)\boldsymbol{v})\boldsymbol{z}\right| &\leq \|\boldsymbol{u}\|_{(\frac{1}{2}-\frac{\alpha}{N})^{-1}}\|\nabla\boldsymbol{v}\|_{(\frac{1}{N})^{-1}}\|\boldsymbol{z}\|_{(\frac{1}{2}-\frac{1-\alpha}{N})^{-1}}\\ &\leq c\|\boldsymbol{u}\|_{\boldsymbol{V}_\alpha}\|\boldsymbol{v}\|_{\boldsymbol{V}_\alpha}^{1-\frac{N}{2}+\alpha}\|\boldsymbol{v}\|_{\boldsymbol{V}_{1+\alpha}}^{\frac{N}{2}-\alpha}\|\boldsymbol{z}\|_{\boldsymbol{V}_{1-\alpha}}.\end{aligned}$$

Here note that $\frac{1}{2}-\frac{\alpha}{N}\neq 0$, $\frac{1}{2}-\frac{1}{N}<\frac{1}{N}-\frac{1-\alpha}{N}\leq\frac{1}{2}+\frac{1-\alpha}{N}$ and $\frac{1}{2}-\frac{1-\alpha}{N}\neq 0$. On the other hand, if $N=2$ and $\alpha=1$, then we also have

$$\begin{aligned}\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\|_{\boldsymbol{H}} &\leq \|\boldsymbol{u}\|_4\|\nabla\boldsymbol{v}\|_4\\ &\leq c\|\boldsymbol{u}\|_{\boldsymbol{H}}^{1/2}\|\boldsymbol{u}\|_{\boldsymbol{V}}^{1/2}\|\boldsymbol{v}\|_{\boldsymbol{V}}^{1/2}\|\boldsymbol{v}\|_{\boldsymbol{V}_2}^{1/2}\leq c\|\boldsymbol{u}\|_{\boldsymbol{V}}\|\boldsymbol{v}\|_{\boldsymbol{V}}^{1/2}\|\boldsymbol{v}\|_{\boldsymbol{V}_2}^{1/2}.\end{aligned}$$

Hence the desired inequality (3.3) is obtained. Moreover it follows that for all $\boldsymbol{z}\in\boldsymbol{V}_{\frac{N}{2}-\alpha}$,

$$\begin{aligned}\left|\int_\Omega((\boldsymbol{u}\cdot\nabla)\boldsymbol{u})\boldsymbol{z}\right| &\leq \|\boldsymbol{u}\|_{(\frac{1}{2}-\frac{\alpha}{2N})^{-1}}\|\nabla\boldsymbol{u}\|_{(\frac{1}{2}-\frac{\alpha}{2N})^{-1}}\|\boldsymbol{z}\|_{(\frac{\alpha}{N})^{-1}}\\ &\leq c\|\boldsymbol{u}\|_{\boldsymbol{H}}^{1/2}\|\boldsymbol{u}\|_{\boldsymbol{V}_\alpha}^{1/2}\|\boldsymbol{u}\|_{\boldsymbol{V}}^{1/2}\|\boldsymbol{u}\|_{\boldsymbol{V}_{1+\alpha}}^{1/2}\|\boldsymbol{z}\|_{\boldsymbol{V}_{\frac{N}{2}-\alpha}}.\end{aligned}$$

Thus we obtain the desired inequality (3.4). Finally we see that for all $\boldsymbol{z}\in\boldsymbol{V}_{\frac{N}{2}-\alpha}$,

$$\begin{aligned}\left|\int_\Omega((\boldsymbol{u}\cdot\nabla)\boldsymbol{v})\boldsymbol{z}\right| &\leq \|\boldsymbol{u}\|_{(\frac{1}{2})^{-1}}\|\nabla\boldsymbol{v}\|_{(\frac{1}{2}-\frac{\alpha}{N})^{-1}}\|\boldsymbol{z}\|_{(\frac{\alpha}{N})^{-1}}\\ &\leq c\|\boldsymbol{u}\|_{\boldsymbol{H}}\|\boldsymbol{v}\|_{\boldsymbol{V}_{1+\alpha}}\|\boldsymbol{z}\|_{\boldsymbol{V}_{\frac{N}{2}-\alpha}}.\end{aligned}$$

Here note that $\frac{1}{2}-\frac{\alpha}{N}\neq 0$. On the other hand, if $N=2$ and $\alpha=1$, then it follows that for all $\boldsymbol{z}\in\boldsymbol{V}_{\frac{1}{2}}$,

$$\begin{aligned}\left|\int_\Omega((\boldsymbol{u}\cdot\nabla)\boldsymbol{v})\boldsymbol{z}\right| &\leq \|\boldsymbol{u}\|_2\|\boldsymbol{v}\|_4\|\boldsymbol{z}\|_4\\ &\leq c\|\boldsymbol{u}\|_{\boldsymbol{H}}\|\boldsymbol{v}\|_{\boldsymbol{V}}^{1/2}\|\boldsymbol{v}\|_{\boldsymbol{V}_2}^{1/2}\|\boldsymbol{z}\|_{\boldsymbol{V}_{\frac{1}{2}}}\leq c\|\boldsymbol{u}\|_{\boldsymbol{H}}\|\boldsymbol{v}\|_{\boldsymbol{V}}\|\boldsymbol{z}\|_{\boldsymbol{V}_{\frac{1}{2}}}.\end{aligned}$$

Therefore we derive the desired inequality (3.5). □

4 Heat equation with hysteresis

The following proposition provides solvability for the heat equation with hysteresis with some estimates in the case $N=2,3$.

Proposition 4.1. *Let* $N=2,3$ *and* $0<T<\infty$. *Let* C_1, C_2 *and* C_3 *be as in* Definition 1.1. *Assume* (A1), (A2) *and* (A3). *Then for all* $\boldsymbol{v}\in C_3(T)$, *there exists a unique solution* (w,θ) *satisfying* $w\in C_1(T;\theta)$ *and* $\theta\in C_2(T)$ *such that*

$$\text{(H)}\qquad\begin{cases}dw/dt+\partial I_\theta(w)\ni 0 & \text{in } H \quad \text{a.e. on } (0,T),\\ d\theta/dt-\Delta\theta+\boldsymbol{v}\cdot\nabla\theta+w=f & \text{in } H \quad \text{a.e. on } (0,T),\\ (w(0),\theta(0))=(w_0,\theta_0) & \text{in } H\times H,\end{cases}$$

and moreover the following holds:

$$\|\theta\|_{L^\infty(0,T;H)} \le M_1 = M_1(\|\theta_0\|_H), \tag{4.1}$$

$$\|\theta\|_{L^\infty(0,T;L^\infty(\Omega))} \le M_2 = M_2(\|\theta_0\|_{L^\infty(\Omega)}), \tag{4.2}$$

$$\|\theta\|^2_{L^\infty(0,T;V)} + \|\Delta\theta\|^2_{L^2(0,T;H)} \le M_3 = M_3(\|\theta_0\|_V, \|\boldsymbol{v}\|_{L^\infty(0,T;\boldsymbol{V}_\alpha)}), \tag{4.3}$$

Furthermore, if (w_i, θ_i) *is a solution with* $\boldsymbol{v} = \boldsymbol{v}_i$, $w_0 = w_{0,i}$ *and* $\theta_0 = \theta_{0,i}$ $(i = 1, 2)$, *then the following holds for all* $t \in [0, T]$:

$$\begin{aligned} &\|(\theta_1 - \theta_2)(t)\|_V^2 + \|\Delta(\theta_1 - \theta_2)\|^2_{L^2(0,t;H)} \\ &\qquad \le C_2 \left(\|\theta_{0,1} - \theta_{0,2}\|_V^2 + \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|^2_{L^\infty(0,t;\boldsymbol{V}_\alpha)} + \|w_1 - w_2\|^2_{L^2(0,t;H)} \right), \end{aligned} \tag{4.4}$$

$$\|(w_1 - w_2)(t)\|_{L^\infty(\Omega)} \le \|w_{0,1} - w_{0,2}\|_{L^\infty(\Omega)} + C_3 \|\theta_1 - \theta_2\|_{L^\infty(0,t;L^\infty(\Omega))}. \tag{4.5}$$

Here $M_1, M_2, M_3, C_1, C_2, C_3 > 0$ *are constants. In particular,*

- M_1 *increases depending on increase of* $\|\theta_0\|_H$. *Specifically*

$$M_1 := C_1 \left(\|\theta_0\|_H + \|f\|_{L^1(0,T;H)} + \max_{i=1,2} |\psi_i(0)| \right); \tag{4.6}$$

- M_2 *increases depending on increase of* $\|\theta_0\|_{L^\infty(\Omega)}$;
- M_3 *increases depending on increase of* $\|\theta_0\|_V$ *and* $\|\boldsymbol{v}\|_{L^\infty(0,T;\boldsymbol{V}_\alpha)}$;
- C_2 *increases depending on increase of* $\min_{i=1,2} \|\theta_{0,i}\|_V$ *and* $\max_{i=1,2} \|\boldsymbol{v}_i\|_{L^\infty(0,T;\boldsymbol{V}_\alpha)}$.

Proof. The proof would be completed by referring to the statement and the proof of [32, Lemma 3.1 and Propositions 3.2 and 5.1].

First existence and uniqueness for (H) would be obtained by almost the same argument of [32, Proof of Proposition 5.1] via [32, Lemma 3.1 and Proposition 3,2]. It suffices to only note (3.1) in Lemma 3.1 and replace the definition of k in [32, Lemma 3.1 and Proposition 3,2] with $k(t) := k_0 \int_0^t \|\boldsymbol{v}(r)\|^{2/\rho}_{\boldsymbol{V}_\alpha} dr$, where ρ is defined in Lemma 3.1.

Next letting (w, θ) be a solution to (H), we show the estimates (4.1), (4.2) and (4.3). Multiplying the second equation in (H) by $\theta(t)$, we see that for a.a. $t \in (0, T)$,

$$\|\theta(t)\|_H \frac{d}{dt} \|\theta(t)\|_H + \|\theta(t)\|_V^2 \le (\|f(t)\|_H + \|w(t)\|_H) \|\theta(t)\|_H.$$

In view of the condition $w \in K(\theta)$ and Lipschitz continuity of ψ_1, ψ_2 integrating the above inequality implies that for all $t \in [0, T]$,

$$\begin{aligned} \|\theta(t)\|_H &\le \|\theta_0\|_H + \|f\|_{L^1(0,t;H)} + \|w\|_{L^1(0,t;H)} \\ &\le \|\theta_0\|_H + \|f\|_{L^1(0,t;H)} + t|\Omega|^{1/2} \max_{i=1,2} |\psi_i(0)| + \max_{i=1,2} \|\psi_i'\|_{L^\infty(\mathbb{R})} \|\theta\|_{L^1(0,t;H)}. \end{aligned}$$

Multiply it by e^{-kt}, take the supremum as $t \in (0, T)$ and note that

$$\|\theta\|_{L^1(0,t;H)} e^{-kt} = \int_0^t \|\theta(s)\|_H e^{-ks} e^{k(s-t)} ds \le \frac{1}{k} \|\theta\|_{L_k^\infty(0,t;H)}$$

(see (2.1) for the definition of $\|\cdot\|_{L_k^\infty(0,T;H)}$). Then we deduce that

$$\|\theta\|_{L_k^\infty(0,T;H)} \le \|\theta_0\|_H + \|f\|_{L^1(0,T;H)} + T|\Omega|^{1/2} \max_{i=1,2} |\psi_i(0)| + \frac{1}{k} \max_{i=1,2} \|\psi_i'\|_{L^\infty(\mathbb{R})} \|\theta\|_{L_k^\infty(0,T;H)}.$$

Thus the desired inequality (4.1) holds for $k > 0$ large enough. On the other hand, applying [32, Eq. (3.5) in Lemma 3.1] ($h = f - w$, $u_0 = \theta_0$ and $u = \theta$) implies that for $t \in [0, T]$,

$$\|\theta(t)\|_{L^\infty(\Omega)} \le \|\theta_0\|_{L^\infty(\Omega)} + \|f\|_{L^1(0,t;L^\infty(\Omega))} + \|w\|_{L^1(0,t;L^\infty(\Omega))}.$$

By a similar argument toward (4.1) as above (replace H with $L^\infty(\Omega)$) we also deduce the desired inequality (4.2). Moreover apply [32, Eq. (3.4) in Lemma 3.1] ($h = f - w$, $u_0 = \theta_0$ and $u = \theta$). Then we have

$$\|\theta\|^2_{L^\infty(0,T;V)} + \|\Delta\theta\|^2_{L^2(0,T;H)} \le ce^{c\|\boldsymbol{v}\|^{2/\rho}_{L^\infty(0,T;\boldsymbol{V}_\alpha)}} \left(\|\theta_0\|^2_V + \|f\|^2_{L^2(0,T;H)} + \|w\|^2_{L^2(0,T;H)}\right),$$

where $c > 0$ is a constant and ρ is defined in Lemma 3.1. Then using the condition $w \in K(\theta)$, i.e.,

$$\|w\|_{L^2(0,T;H)} \le |Q|^{1/2} \max_{i=1,2} |\psi_i(0)| + \max_{i=1,2} \|\psi_i'\|_{L^\infty(\Omega)} \|\theta\|_{L^2(0,T;H)}$$

and plugging (4.1), we obtain the desired inequality (4.3).

Finally letting (w_i, θ_i) be a solution with $\boldsymbol{v} = \boldsymbol{v}_i$, $w_0 = w_{0,i}$ and $\theta_0 = \theta_{0,i}$ ($i = 1, 2$), we show the estimates (4.4) and (4.5). By applying [32, Eq. (3.4) of Lemma 3.1] ($h = -(\boldsymbol{v}_1 - \boldsymbol{v}_2) \cdot \nabla\theta_2 - (w_1 - w_2)$, $\boldsymbol{v} = \boldsymbol{v}_1$, $u_0 = \theta_{0,1} - \theta_{0,2}$ and $u = \theta_1 - \theta_2$) we deduce that for all $t \in [0, T]$,

$$\begin{aligned}&\|(\theta_1 - \theta_2)(t)\|^2_V + \|\Delta(\theta_1 - \theta_2)\|^2_{L^2(0,t;H)} \\ &\quad \le ce^{c\|\boldsymbol{v}_1\|^{2/\rho}_{L^\infty(0,T;\boldsymbol{V}_\alpha)}} \left(\|\theta_{0,1} - \theta_{0,2}\|^2_V + \|(\boldsymbol{v}_1 - \boldsymbol{v}_2) \cdot \nabla\theta_2\|^2_{L^2(0,t;H)} + \|w_1 - w_2\|^2_{L^2(0,t;H)}\right),\end{aligned}$$

where $c > 0$ is a constant and ρ is defined in Lemma 3.1. Here (3.1) in Lemma 3.1 and (4.3) imply

$$\begin{aligned}\|(\boldsymbol{v}_1 - \boldsymbol{v}_2) \cdot \nabla\theta_2\|^2_{L^2(0,t;H)} &\le c_0^2 \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|^2_{L^\infty(0,t;\boldsymbol{V}_\alpha)} \|\theta_2\|^{2\rho}_{L^\infty(0,t;V)} \|\Delta\theta_2\|^{2-2\rho}_{L^{2-2\rho}(0,t;H)} \\ &\le c_0^2 T^\rho M_3 \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|^2_{L^\infty(0,t;\boldsymbol{V}_\alpha)},\end{aligned}$$

where $M_3 = M_3(\|\theta_{0,2}\|_V, \|\boldsymbol{v}_2\|_{L^\infty(0,T;\boldsymbol{V}_\alpha)})$ is defined as (4.3). Then the desired inequality (4.4) is obtained. The estimate (4.5) is proved by a similar way as in the proof of [17, Lemma 3.1] or [32, Lemma 2.1]. Indeed, we would show $\frac{d}{dt}\|w_\pm(t)\|^2_H \le 0$, where

$$w_\pm(t) := \left[w_1(t) - w_2(t) \mp \|w_{0,1} - w_{0,2}\|_{L^\infty(\Omega)} \mp \max_{i=1,2} \|\psi_i(\theta_1) - \psi_i(\theta_2)\|_{L^\infty(0,T;L^\infty(\Omega))}\right]^\pm,$$

and hence the desired inequality (4.5) holds. □

5 Navier-Stokes equations

In this section we provide the solvability with estimates for

$$(\mathrm{NS})_\alpha \qquad \begin{cases} d\boldsymbol{v}/dt + A\boldsymbol{v} + B(\boldsymbol{v}, \boldsymbol{v}) = P\boldsymbol{g}(\theta) & \text{in } \boldsymbol{V}_{-1+\alpha} \quad \text{a.e. on } (0, T), \\ \boldsymbol{v}(0) = \boldsymbol{v}_0 \in \boldsymbol{V}_\alpha & \text{in } \boldsymbol{H}. \end{cases}$$

The following two propositions show solvability for $(\mathrm{NS})_\alpha$ with $\frac{N-2}{2} < \alpha \le 1$ for $N = 2, 3$.

Proposition 5.1. *Let* $N = 2$, $0 < T < \infty$ *and* $0 < \alpha \leq 1$. *Let* $\mathcal{C}_2$ *and* $\mathcal{C}_3$ *be as in* Definition 1.1. *Assume* (A2) *and* (A3). *Then for all* $\theta \in \mathcal{C}_2(T)$, *there exists a unique solution* $\boldsymbol{v} \in \mathcal{C}_3(T)$ *to* $(\mathrm{NS})_\alpha$. Moreover the following holds:

$$(5.1)\qquad \|\boldsymbol{v}\|^2_{L^\infty(0,T;\boldsymbol{V}_\alpha)} + \|\boldsymbol{v}\|^2_{L^2(0,T;\boldsymbol{V}_{1+\alpha})} \leq M_4 = M_4(\|\boldsymbol{v}_0\|_{\boldsymbol{V}_\alpha}, \|\theta\|_{L^2(0,T;H)}).$$

Furthermore, if $\boldsymbol{v}_i$ is a solution with $\theta = \theta_i$ and $\boldsymbol{v}_0 = \boldsymbol{v}_{0,i}$ $(i = 1, 2)$, then the following holds for all $t \in [0, T]$:

$$(5.2)\qquad \|\boldsymbol{v}_1(t) - \boldsymbol{v}_2(t)\|^2_{\boldsymbol{V}_\alpha} + \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|^2_{L^2(0,t;\boldsymbol{V}_{1+\alpha})} \leq C_4 \left(\|\boldsymbol{v}_{0,1} - \boldsymbol{v}_{0,2}\|^2_{\boldsymbol{V}_\alpha} + \|\theta_1 - \theta_2\|^2_{L^2(0,t;H)}\right).$$

Here $M_4, C_4 > 0$ are constants. In particular,

- M_4 increases depending on increase of $\|\boldsymbol{v}_0\|_{\boldsymbol{V}_\alpha}$ and $\|\theta\|_{L^2(0,T;H)}$;
- C_4 increases depending on increase of $\max_{i=1,2} \|\boldsymbol{v}_{0,i}\|_{\boldsymbol{V}_\alpha}$ and $\max_{i=1,2} \|\theta_i\|_{L^2(0,T;H)}$.

Proposition 5.2. *Let* $N = 3$, $0 < T < \infty$ *and* $\frac{1}{2} < \alpha \leq 1$. *Let* $\mathcal{C}_2$ *and* $\mathcal{C}_3$ *be as in* Definition 1.1. *Assume* (A2) *and* (A3). *Put*

$$T_0 = T_0(\theta, \boldsymbol{v}_0) := \delta \left(\|\boldsymbol{v}_0\|_{\boldsymbol{V}_\alpha} + \|P\boldsymbol{g}(0)\|_{\boldsymbol{H}} + \|\boldsymbol{g}'\|_{\boldsymbol{L}^\infty(\mathbb{R})}\|\theta\|_{L^\infty(0,T;H)}\right)^{-\frac{4}{2\alpha-1}} \wedge T.$$

Then for all $\theta \in \mathcal{C}_2(T)$, *there exists a unique solution* $\boldsymbol{v} \in \mathcal{C}_3(T_0)$ *to* $(\mathrm{NS})_\alpha$. Moreover the following holds:

$$(5.1)'\qquad \|\boldsymbol{v}\|^2_{L^\infty(0,T_0;\boldsymbol{V}_\alpha)} + \|\boldsymbol{v}\|^2_{L^2(0,T_0;\boldsymbol{V}_{1+\alpha})} \leq M_4' = M_4'(\|\boldsymbol{v}_0\|_{\boldsymbol{V}_\alpha}, \|\theta\|_{L^\infty(0,T;H)}).$$

Furthermore, if $\boldsymbol{v}_i$ is a solution with $\theta = \theta_i$ and $\boldsymbol{v}_0 = \boldsymbol{v}_{0,i}$ $(i = 1, 2)$, then the following holds for all $t \in [0, T_0]$:

$$(5.2)'\qquad \|\boldsymbol{v}_1(t) - \boldsymbol{v}_2(t)\|^2_{\boldsymbol{V}_\alpha} + \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|^2_{L^2(0,t;\boldsymbol{V}_{1+\alpha})} \leq C_4' \left(\|\boldsymbol{v}_{0,1} - \boldsymbol{v}_{0,2}\|^2_{\boldsymbol{V}_\alpha} + \|\theta_1 - \theta_2\|^2_{L^2(0,t;H)}\right).$$

Here δ, $M_4', C_4' > 0$ are constants. In particular,

- M_4' increases depending on increase of $\|\boldsymbol{v}_0\|_{\boldsymbol{V}_\alpha}$ and $\|\theta\|_{L^\infty(0,T;H)}$;
- C_4' increases depending on increase of $\max_{i=1,2} \|\boldsymbol{v}_{0,i}\|_{\boldsymbol{V}_\alpha}$ and $\max_{i=1,2} \|\theta_i\|_{L^\infty(0,T;H)}$.

Remark 5.1. $T_0(\theta, \boldsymbol{v}_0)$ is bounded below by T_* (defined in Theorem 1.2) uniformly on $\theta \in L^\infty(0, T; H)$ which is the second part of solutions to (H) (see (4.1) with (4.6)).

Proof of Propositions 5.1 and 5.2. Let $N = 2, 3$, $0 < T < \infty$ and $\frac{N-2}{2} < \alpha \leq 1$. From Lipschitz continuity of $\boldsymbol{g}$ we see that for all $t \in [0, T]$,

$$(5.3)\qquad \|P\boldsymbol{g}(\theta(t))\|_{\boldsymbol{H}} \leq \|P\boldsymbol{g}(0)\|_{\boldsymbol{H}} + \|\boldsymbol{g}'\|_{\boldsymbol{L}^\infty(\mathbb{R})}\|\theta(t)\|_{\boldsymbol{H}}.$$

Use it when we estimate $\|Pg(\theta(t))\|_{H}$. First using (5.4) as below, we prove the estimate (5.1) (for $N=2$) or (5.1)′ (for $N=3$). Suppose v is a solution to $(\mathrm{NS})_\alpha$ and multiply the equation in $(\mathrm{NS})_\alpha$ by $A^\alpha v$. Then we see that for a.a. $t\in(0,T)$,

$$
\begin{aligned}
(5.4)\qquad \frac{1}{2}\frac{d}{dt}\|v(t)\|_{V_\alpha}^2+\|v(t)\|_{V_{1+\alpha}}^2 &\\
&\le \left(\|B(v(t),v(t))\|_{V_{-1+\alpha}}+\|Pg(\theta(t))\|_{V_{-1+\alpha}}\right)\|A^\alpha v(t)\|_{V_{1-\alpha}}\\
&\le \left(\|B(v(t),v(t))\|_{V_{-1+\alpha}}+c_1\|Pg(\theta(t))\|_{H}\right)\|v(t)\|_{V_{1+\alpha}},
\end{aligned}
$$

where $c_1>0$ is a constant. By the way note that the following estimate holds:

$$
(5.5)\qquad \|v\|_{L^\infty(0,T;H)}^2+\|v\|_{L^2(0,T;V)}^2\le M_5=M_5(\|v_0\|_H,\|\theta\|_{L^2(0,T;H)}),
$$

where $M_5>0$ is a constant, which increases depending on increase of $\|v_0\|_H$, $\|\theta\|_{L^2(0,T;H)}$. Indeed, multiplying the equation in $(\mathrm{NS})_0$ by v with the standard argument yields the inequality (5.5). For details refer to, e.g., [30, Chapter 3.1]. Note (5.3) if needed.

Now we put $N=2$ and show (5.1). We see from (5.4) with (3.4) in Lemma 3.1 that for a.a. $t\in(0,T)$,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|v(t)\|_{V_\alpha}^2+\|v(t)\|_{V_{1+\alpha}}^2 &\\
&\le c_0\|v(t)\|_H^{1/2}\|v(t)\|_V^{1/2}\|v(t)\|_{V_\alpha}^{1/2}\|v(t)\|_{V_{1+\alpha}}^{3/2}+c_1\|Pg(\theta(t))\|_H\|v(t)\|_{V_{1+\alpha}}\\
&\le c_1'\left(\|v(t)\|_H^2\|v(t)\|_V^2\|v(t)\|_{V_\alpha}^2+\|Pg(\theta(t))\|_H^2\right)+\frac{1}{2}\|v(t)\|_{V_{1+\alpha}}^2,
\end{aligned}
$$

where $c_1'>0$ is a constant depending only on c_0 and c_1. Using the Gronwall lemma and (5.5), we deduce that for all $t\in[0,T]$,

$$
\begin{aligned}
\|v(t)\|_{V_\alpha}^2+\|v\|_{L^2(0,t;V_{1+\alpha})}^2 &\le e^{2c_1'\int_0^t\|v(r)\|_H^2\|v(r)\|_V^2\,dr}\left(\|v_0\|_{V_\alpha}^2+2c_1'\|Pg(\theta)\|_{L^2(0,t;H)}^2\right)\\
&\le e^{2c_1'M_5^2}\left(\|v_0\|_{V_\alpha}^2+2c_1'\|Pg(\theta)\|_{L^2(0,t;H)}^2\right).
\end{aligned}
$$

Hence the desired inequality (5.1) holds.

On the other hand, we put $N=3$ and show (5.1)′ similarly to [29, Proof of Theorem 3.11 in Chapter III]. It follows from (5.4) with (3.3) in Lemma 3.1 that for a.a. $t\in(0,T)$,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|v(t)\|_{V_\alpha}^2+\|v(t)\|_{V_{1+\alpha}}^2 &\le c_0\|v(t)\|_{V_\alpha}^{1+\rho}\|v(t)\|_{V_{1+\alpha}}^{2-\rho}+c_1\|Pg(\theta(t))\|_H\|v(t)\|_{V_{1+\alpha}}\\
&\le c_1''\left(\|v(t)\|_{V_\alpha}^{2(\frac{1}{\rho}+1)}+\|Pg(\theta(t))\|_H^2\right)+\frac{1}{2}\|v(t)\|_{V_{1+\alpha}}^2,
\end{aligned}
$$

where $c_1''>0$ is a constant depending only on c_0 and c_1, and $\rho:=\alpha-\frac{1}{2}$ is defined in Lemma 3.1. Thus we see that for a.a. $t\in(0,T)$,

$$
(5.6)\qquad \frac{d}{dt}\|v(t)\|_{V_\alpha}^2+\|v(t)\|_{V_{1+\alpha}}^2\le 2c_1''\left(\|v(t)\|_{V_\alpha}^{2(\frac{1}{\rho}+1)}+\|Pg(\theta(t))\|_H^2\right).
$$

Now we let $z(t) := \max\{\|v(t)\|_{V_\alpha}^2, \|v_0\|_{V_\alpha}^2, 2c_1''c\|Pg(\theta)\|_{L^\infty(0,T;H)}^2\}$ for $t \in [0,T]$, where $c > 0$ is a constant satisfying $\|\cdot\|_{V_\alpha} \le c\|\cdot\|_{V_{1+\alpha}}$. Then for a.a. $t \in (0,T)$,

$$\frac{d}{dt}z(t) = \begin{cases} \frac{d}{dt}\|v(t)\|_{V_\alpha}^2 & \text{if } \|v(t)\|_{V_\alpha}^2 \ge \max\left\{\|v_0\|_{V_\alpha}^2, 2c_1''c\|Pg(\theta)\|_{L^\infty(0,T;H)}^2\right\}, \\ 0 & \text{if } \|v(t)\|_{V_\alpha}^2 < \max\left\{\|v_0\|_{V_\alpha}^2, 2c_1''c\|Pg(\theta)\|_{L^\infty(0,T;H)}^2\right\}. \end{cases}$$

Hence (5.6) implies that $\frac{d}{dt}z(t) \le 2c_1''z(t)^{\frac{1}{\rho}+1}$ for a.a. $t \in (0,T)$. Moreover it follows that for all $\varepsilon > 0$,

$$\frac{d}{dt}(z(t)+\varepsilon)^{-\frac{1}{\rho}} = -\frac{1}{\rho}(z(t)+\varepsilon)^{-(\frac{1}{\rho}+1)}\frac{d}{dt}z(t) \ge -\frac{2c_1''}{\rho}.$$

Integrating it yields that for all $t \in [0, T_\varepsilon]$,

$$(z(t)+\varepsilon)^{-\frac{1}{\rho}} \ge (z(0)+\varepsilon)^{-\frac{1}{\rho}} - \frac{2c_1''}{\rho}\cdot T_\varepsilon \ge 2^{-\frac{1}{\rho}}(z(0)+\varepsilon)^{-\frac{1}{\rho}}, \tag{5.7}$$

where $T_\varepsilon := \frac{\rho}{2c_1''}(1-2^{-\frac{1}{\rho}})(z(0)+\varepsilon)^{-\frac{1}{\rho}} \wedge T$. Thus taking a limit of (5.7) to the power of $-\rho$ as $\varepsilon \downarrow 0$, we see that for all $t \in [0, T_0]$,

$$\|v(t)\|_{V_\alpha}^2 \le z(t) \le 2z(0) = 2\max\left\{\|v_0\|_{V_\alpha}^2, 2c_1''\|Pg(\theta)\|_{L^\infty(0,T;H)}^2\right\}. \tag{5.8}$$

Here note that

$$\lim_{\varepsilon\downarrow 0} T_\varepsilon = \delta\max\left\{\|v_0\|_{V_\alpha}^2, 2c_1''\|Pg(\theta)\|_{L^\infty(0,T;H)}^2\right\}^{-\frac{2}{2\alpha-1}} \wedge T \ge T_0,$$

where $\delta := \frac{2\alpha-1}{4c_1''}(1-2^{-\frac{2}{2\alpha-1}})$. Then by integrating (5.6) and using (5.8) we obtain the desired inequality (5.1)$'$.

Next letting $N = 2,3$, we prove the estimate (5.2) (for $N=2$) or (5.2)$'$ (for $N=3$). For simplicity we let T_0 (defined in the case $N=3$) be denoted by T. Suppose v_i is a solution with $\theta = \theta_i$ and $v_0 = v_{0,i}$ to (NS)$_\alpha$ $(i=1,2)$ and take the difference between the equation for $i=1$ and $i=2$. For simplicity put $\theta := \theta_1 - \theta_2$, $v_0 := v_{0,1} - v_{0,2}$ and $v := v_1 - v_2$. Then it follows that

$$\begin{cases} dv/dt + Av + B(v_1, v) + B(v, v_2) = Pg(\theta_1) - Pg(\theta_2) & \text{in } V_{-1+\alpha}, \\ v(0) = v_0 \in V_\alpha & \text{in } H. \end{cases}$$

Multiply it by $A^\alpha v$ and use (3.3) in Lemma 3.1. Then we see that for a.a. $t \in (0,T)$,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|v(t)\|_{V_\alpha}^2 + \|v(t)\|_{V_{1+\alpha}}^2 \\ &\le \left(\|B(v_1(t), v(t))\|_{V_{-1+\alpha}} + \|B(v(t), v_2(t))\|_{V_{-1+\alpha}} \right.\\ &\qquad \left. + \|Pg(\theta_1(t)) - Pg(\theta_2(t))\|_{V_{-1+\alpha}}\right)\|A^\alpha v(t)\|_{V_{1-\alpha}} \\ &\le c_0\|v_1(t)\|_{V_\alpha}\|v(t)\|_{V_\alpha}^\rho\|v(t)\|_{V_{1+\alpha}}^{2-\rho} + c_0\|v(t)\|_{V_\alpha}\|v_2(t)\|_{V_\alpha}^\rho\|v_2(t)\|_{V_{1+\alpha}}^{1-\rho}\|v(t)\|_{V_{1+\alpha}} \\ &\qquad + c_2\|g'\|_{L^\infty(\mathbb{R})}\|\theta(t)\|_H\|v(t)\|_{V_{1+\alpha}} \\ &\le c_2'\left(\|v_1(t)\|_{V_\alpha}^{\frac{2}{\rho}}\|v(t)\|_{V_\alpha}^2 + \|v(t)\|_{V_\alpha}^2\|v_2(t)\|_{V_\alpha}^{2\rho}\|v_2(t)\|_{V_{1+\alpha}}^{2-2\rho} + \|\theta(t)\|_H^2\right) \\ &\qquad + \frac{1}{2}\|v(t)\|_{V_{1+\alpha}}^2, \end{aligned}$$

where $c_2, c_2' > 0$ are constants. In particular, c_2' depends only on c_0, c_2 and $\|g'\|_{L^\infty(\mathbb{R})}$. From the Gronwall lemma and (5.1) (for $N = 2$) or (5.1)$'$ (for $N = 3$) we deduce that for all $t \in [0, T]$,

$$\begin{aligned}
&\|v(t)\|_{V_\alpha}^2 + \|v\|_{L^2(0,t;V_{1+\alpha})}^2 \\
&\quad \le \exp\left[2c_2'\left(T\|v_1\|_{L^\infty(0,T;V_\alpha)}^{\frac{2}{\rho}} + T^\rho \|v_2\|_{L^\infty(0,T;V_\alpha)}^{2\rho}\|v_2\|_{L^2(0,T;V_{1+\alpha})}^{2-2\rho}\right)\right] \\
&\qquad\qquad\qquad\qquad \times \left(\|v_0\|_{V_\alpha}^2 + \|\theta\|_{L^2(0,t;H)}^2\right) \\
&\quad \le \exp\left[2c_2'\left(TM_4(\|v_{0,1}\|_{V_\alpha}, \|\theta_1\|_{L^2(0,T;H)})^{\frac{1}{\rho}} + T^\rho M_4(\|v_{0,2}\|_{V_\alpha}, \|\theta_2\|_{L^2(0,T;H)})\right)\right] \\
&\qquad\qquad\qquad\qquad \times \left(\|v_0\|_{V_\alpha}^2 + \|\theta\|_{L^2(0,t;H)}^2\right).
\end{aligned}$$

Here, in the case $N = 3$, replace $M_4(\|v_{0,i}\|_{V_\alpha}, \|\theta_i\|_{L^2(0,T;H)})$ by $M_4'(\|v_{0,i}\|_{V_\alpha}, \|\theta_i\|_{L^\infty(0,T;H)})$ ($i = 1, 2$). Hence we obtain the desired inequality (5.2) (for $N = 2$) or (5.2)$'$ (for $N = 3$), which also implies uniqueness for $(\mathrm{NS})_\alpha$.

Finally let $\theta \in \mathcal{C}_2(T)$ and $v_0 \in V_\alpha$ as in (A3). Then we prove existence for $(\mathrm{NS})_\alpha$ for $N = 2, 3$. We apply the Galerkin approximation similarly as in [29]. It is well-known that for a Hilbert basis $\{e_n\} \subset V$ of the topology on H and $v_{0,n} \in E_n := \mathrm{span}\{e_1, \ldots, e_n\}$ (which is the space spanned by $e_1, \ldots, e_n$) there exists a solution

$$v_n(t) = \sum_{k=1}^{n} v_{n,k}(t) e_k \in E_n, \quad t \in [0, T], \tag{5.9}$$

where $v_{n,k}(t) \in \mathbb{R}$, such that for each $k = 1, \ldots, n$,

$$\begin{cases} \langle dv_n/dt(t) + Av_n(t) + B(v_n(t), v_n(t)), e_k \rangle_{V^*, V} \\ \qquad\qquad\qquad = \langle Pg(\theta(t)), e_k \rangle_{V^*, V} \quad \text{a.a. } t \in (0, T), \\ v_n(0) = v_{0,n} \in E_n. \end{cases} \tag{5.10}$$

Here we decide $\{e_n\}$ and $v_{0,n}$ as follows. By virtue of the Riesz representation theorem for V_α, we have the continuous operator $\Lambda : V_{-\alpha} \to V_\alpha$ such that

$$(\Lambda u, z)_{V_\alpha} = \langle u, z \rangle_{V_{-\alpha}, V_\alpha} \quad \text{for all } z \in V_\alpha,$$

and hence the compact imbeddings $V_\alpha \hookrightarrow H \hookrightarrow V_{-\alpha}$ yield that Λ is a compact operator on H. Moreover self-adjointness of $\Lambda : H \to H$ is easily seen. Thus H has a Hilbert basis $\{e_n\}$ composed of eigenfuctions of Λ with the eigenvalues $\{\lambda_n^{-1}\}$ satisfying $\lambda_n > 0$. That is,

$$\left(A^{\frac{\alpha}{2}} e_n, A^{\frac{\alpha}{2}} z\right)_H = \lambda_n (e_n, z)_H \qquad \text{for all } z \in V_\alpha. \tag{5.11}$$

Now we regularize $e_n \in V_\alpha$. It follows from (5.11) that for all $z \in V$,

$$\left(A^{\frac{1}{2}} A^{\frac{-1+\alpha}{2}} e_n, A^{\frac{1}{2}} z\right)_H = \left(A^{\frac{\alpha}{2}} e_n, A^{\frac{\alpha}{2}} A^{\frac{1-\alpha}{2}} z\right)_H = \left(\lambda_n e_n, A^{\frac{1-\alpha}{2}} z\right)_H = \left(\lambda_n A^{\frac{1-\alpha}{2}} e_n, z\right)_H.$$

Thus $A^{\frac{-1+\alpha}{2}}\boldsymbol{e}_n$ satisfies the following:

$$\begin{cases} -\Delta\left(A^{\frac{-1+\alpha}{2}}\boldsymbol{e}_n\right) + \nabla\pi = \lambda_n A^{\frac{1-\alpha}{2}}\boldsymbol{e}_n & \text{in } \Omega, \\ \operatorname{div}\left(A^{\frac{-1+\alpha}{2}}\boldsymbol{e}_n\right) = 0 & \text{in } \Omega, \\ A^{\frac{-1+\alpha}{2}}\boldsymbol{e}_n = 0 & \text{on } \Gamma. \end{cases}$$

Apply the regularization for the above elliptic problem with $\lambda_n A^{\frac{1-\alpha}{2}}\boldsymbol{e}_n \in \boldsymbol{V}_{2\alpha-1} \subset \boldsymbol{H}$. Then we have $A^{\frac{-1+\alpha}{2}}\boldsymbol{e}_n \in \boldsymbol{V}_2$, i.e., $\boldsymbol{e}_n \in \boldsymbol{V}_{1+\alpha} \subset \boldsymbol{V}_{2\alpha}$. Therefore (5.11) yields that

$$(5.11)' \qquad A^\alpha \boldsymbol{e}_n = \lambda_n \boldsymbol{e}_n \quad \text{in } \boldsymbol{H}.$$

Moreover (5.10) has a solution (5.9), and hence for each $k = 1, \dots, n$,

$$(5.10)' \quad \begin{cases} \langle d\boldsymbol{v}_n/dt(t) + A\boldsymbol{v}_n(t) + B(\boldsymbol{v}_n(t), \boldsymbol{v}_n(t)), \boldsymbol{e}_k\rangle_{\boldsymbol{V}_{-1+\alpha}, \boldsymbol{V}_{1-\alpha}} \\ \qquad\qquad = \langle P\boldsymbol{g}(\theta(t)), \boldsymbol{e}_k\rangle_{\boldsymbol{V}_{-1+\alpha}, \boldsymbol{V}_{1-\alpha}} \quad \text{a.a. } t \in (0,T), \\ \boldsymbol{v}_n(0) = \boldsymbol{v}_{0,n} \in E_n. \end{cases}$$

Now we define $\boldsymbol{v}_{0,n} \in E_n$ as $\boldsymbol{v}_{0,n} := P_n \boldsymbol{v}_0$ where $P_n : \boldsymbol{V}_{-\alpha} \to E_n$ is defined as $P_n \boldsymbol{u} := \sum_{k=1}^n \langle \boldsymbol{u}, \boldsymbol{e}_k\rangle_{\boldsymbol{V}_{-\alpha}, \boldsymbol{V}_\alpha} \boldsymbol{e}_k$ for $\boldsymbol{u} \in \boldsymbol{V}_{-\alpha}$. In light of (5.11)′, P_n is the orthogonal projection on E_n of each topology on $\boldsymbol{V}_{-\alpha}$, $\boldsymbol{H}$ and $\boldsymbol{V}_\alpha$. Then P_n would satisfy the following conditions:

$$(5.12) \qquad \|P_n \boldsymbol{u}\|_{\boldsymbol{V}_\beta} \le \|\boldsymbol{u}\|_{\boldsymbol{V}_\beta}, \qquad \boldsymbol{u} \in \boldsymbol{V}_\beta \quad (\beta \in \{-\alpha, 0, \alpha\}),$$

$$(5.13) \qquad P_n \boldsymbol{u} \to \boldsymbol{u} \quad \text{in } \boldsymbol{V}_\alpha, \qquad \boldsymbol{u} \in \boldsymbol{V}_\alpha.$$

The standard property of orthogonal projections implies (5.12). On the other hand, if $\boldsymbol{u} \in \boldsymbol{V}_{2\alpha}$, then (5.13) holds since (5.11)′ yields that

$$A^\alpha P_n \boldsymbol{u} = \sum_{k=1}^n (\boldsymbol{u}, \boldsymbol{e}_k)_{\boldsymbol{H}} A^\alpha \boldsymbol{e}_k = \sum_{k=1}^n (\boldsymbol{u}, A^\alpha \boldsymbol{e}_k)_{\boldsymbol{H}} \boldsymbol{e}_k = \sum_{k=1}^n (A^\alpha \boldsymbol{u}, \boldsymbol{e}_k)_{\boldsymbol{H}} \boldsymbol{e}_k = P_n A^\alpha \boldsymbol{u}$$
$$\to A^\alpha \boldsymbol{u} \quad \text{in } \boldsymbol{H}.$$

In the case $\boldsymbol{u} \in \boldsymbol{V}_\alpha$, we also have (5.13). Indeed, take arbitrary $\varepsilon > 0$. Then there is $\boldsymbol{u}_\varepsilon \in \boldsymbol{V}_{2\alpha}$ such that $\|\boldsymbol{u} - \boldsymbol{u}_\varepsilon\|_{\boldsymbol{V}_\alpha} < \varepsilon$, and hence

$$\begin{aligned} \|P_n \boldsymbol{u} - \boldsymbol{u}\|_{\boldsymbol{V}_\alpha} &\le \|P_n(\boldsymbol{u} - \boldsymbol{u}_\varepsilon)\|_{\boldsymbol{V}_\alpha} + \|P_n \boldsymbol{u}_\varepsilon - \boldsymbol{u}_\varepsilon\|_{\boldsymbol{V}_\alpha} + \|\boldsymbol{u}_\varepsilon - \boldsymbol{u}\|_{\boldsymbol{V}_\alpha} \\ &< \|P_n \boldsymbol{u}_\varepsilon - \boldsymbol{u}_\varepsilon\|_{\boldsymbol{V}_\alpha} + 2\varepsilon. \end{aligned}$$

Therefore we obtain $\limsup_{n\to\infty} \|P_n \boldsymbol{u} - \boldsymbol{u}\|_{\boldsymbol{V}_\alpha} \le 2\varepsilon$, which implies (5.13).

Now multiplying the equation in (5.10)′ by $v_{n,k}(t)$ and taking addition as $k = 1, \dots, n$ (namely $\sum_{k=1}^n v_{n,k}(t) \times (5.10)'$) with (5.9) implies

$$\langle d\boldsymbol{v}_n/dt + A\boldsymbol{v}_n + B(\boldsymbol{v}_n, \boldsymbol{v}_n), \boldsymbol{v}_n\rangle_{\boldsymbol{V}_{-1+\alpha}, \boldsymbol{V}_{1-\alpha}} = \langle P\boldsymbol{g}(\theta), \boldsymbol{v}_n\rangle_{\boldsymbol{V}_{-1+\alpha}, \boldsymbol{V}_{1-\alpha}}.$$

Similarly $\sum_{k=1}^n \lambda_k v_{n,k}(t) \times (5.10)'$ with (5.11)′ implies

$$\langle d\boldsymbol{v}_n/dt + A\boldsymbol{v}_n + B(\boldsymbol{v}_n, \boldsymbol{v}_n), A^\alpha \boldsymbol{v}_n\rangle_{\boldsymbol{V}_{-1+\alpha}, \boldsymbol{V}_{1-\alpha}} = \langle P\boldsymbol{g}(\theta), A^\alpha \boldsymbol{v}_n\rangle_{\boldsymbol{V}_{-1+\alpha}, \boldsymbol{V}_{1-\alpha}}.$$

Therefore by noting the above two equations and almost the same calculation toward (5.1) (for $N=2$) or (5.1)$'$ (for $N=3$) it follows from (5.12) with $\beta=\alpha$ that

$$\begin{aligned}\|\boldsymbol{v}_n\|^2_{L^\infty(0,T;\boldsymbol{V}_\alpha)}+\|\boldsymbol{v}_n\|^2_{L^2(0,T;\boldsymbol{V}_{1+\alpha})}&\le M_4(\|\boldsymbol{v}_{0,n}\|_{\boldsymbol{V}_\alpha},\|\theta\|_{L^2(0,T;H)})\\&\le M_4(\|\boldsymbol{v}_0\|_{\boldsymbol{V}_\alpha},\|\theta\|_{L^2(0,T;H)}).\end{aligned}$$

Here, in the case $N=3$, replace $M_4(\|\boldsymbol{v}_0\|_{\boldsymbol{V}_\alpha},\|\theta\|_{L^2(0,T;H)})$ by $M_4'(\|\boldsymbol{v}_0\|_{\boldsymbol{V}_\alpha},\|\theta\|_{L^\infty(0,T;H)})$. Hence there exists subsequence of $\{\boldsymbol{v}_n\}$ (still denoted by $\{\boldsymbol{v}_n\}$) with the limit function $\boldsymbol{v}\in L^\infty(0,T;\boldsymbol{V}_\alpha)\cap L^2(0,T;\boldsymbol{V}_{1+\alpha})$ and

$$\begin{aligned}\boldsymbol{v}_n\to\boldsymbol{v}\qquad&\text{weakly* in } L^\infty(0,T;\boldsymbol{V}_\alpha),\\\boldsymbol{v}_n\to\boldsymbol{v}\qquad&\text{weakly in } L^2(0,T;\boldsymbol{V}_{1+\alpha}).\end{aligned}$$

Moreover it follows from the characterization of $A:\boldsymbol{V}_{1+\alpha}\to\boldsymbol{V}_{-1+\alpha}$ and (3.3) in Lemma 3.1 that there exists $\boldsymbol{\xi}\in L^{\frac{2}{1-\rho}}(0,T;\boldsymbol{V}_{-1+\alpha})$ and

$$\begin{aligned}A\boldsymbol{v}_n\to A\boldsymbol{v}\qquad&\text{weakly in } L^2(0,T;\boldsymbol{V}_{-1+\alpha}),\\B(\boldsymbol{v}_n,\boldsymbol{v}_n)\to\boldsymbol{\xi}\qquad&\text{weakly in } L^{\frac{2}{1-\rho}}(0,T;\boldsymbol{V}_{-1+\alpha}).\end{aligned}$$

We show $\boldsymbol{\xi}=B(\boldsymbol{v},\boldsymbol{v})$ later. Therefore we have

$$\begin{aligned}\boldsymbol{h}_n&:=-A\boldsymbol{v}_n-B(\boldsymbol{v}_n,\boldsymbol{v}_n)+P\boldsymbol{g}(\theta)\\&\to-A\boldsymbol{v}-\boldsymbol{\xi}+P\boldsymbol{g}(\theta)=:\boldsymbol{h}\qquad\text{weakly in } L^2(0,T;\boldsymbol{V}_{-1+\alpha}).\end{aligned}\tag{5.14}$$

Here the equation in (5.10)$'$ yields $v'_{n,k}(t)=\langle\boldsymbol{h}_n(t),\boldsymbol{e}_k\rangle_{\boldsymbol{V}_{-\alpha},\boldsymbol{V}_\alpha}$, and hence

$$\frac{d}{dt}\boldsymbol{v}_n(t)=\sum_{k=1}^n v'_{n,k}(t)\boldsymbol{e}_k=\sum_{k=1}^n\langle\boldsymbol{h}_n(t),\boldsymbol{e}_k\rangle_{\boldsymbol{V}_{-\alpha},\boldsymbol{V}_\alpha}\boldsymbol{e}_k=P_n\boldsymbol{h}_n(t).$$

Thus (5.12) with $\beta=-\alpha$ implies that

$$\|d\boldsymbol{v}_n/dt\|_{L^2(0,T;\boldsymbol{V}_{-\alpha})}=\|P_n\boldsymbol{h}_n\|_{L^2(0,T;\boldsymbol{V}_{-\alpha})}\le\|\boldsymbol{h}_n\|_{L^2(0,T;\boldsymbol{V}_{-\alpha})}.$$

Since $\{\boldsymbol{h}_n\}$ is bounded in $L^2(0,T;\boldsymbol{V}_{-\alpha})$, so is $\{d\boldsymbol{v}_n/dt\}$, and hence

$$d\boldsymbol{v}_n/dt\to d\boldsymbol{v}/dt\qquad\text{weakly in } L^2(0,T;\boldsymbol{V}_{-\alpha}).\tag{5.15}$$

Then the Lions-Aubin compact theorem (see e.g., Simon [27, Corollary 4]) yields

$$\boldsymbol{v}_n\to\boldsymbol{v}\qquad\qquad\text{in } C([0,T];\boldsymbol{H}).$$

Moreover we have

$$B(\boldsymbol{v}_n,\boldsymbol{v}_n)\to B(\boldsymbol{v},\boldsymbol{v})\quad\text{weakly in } L^2(0,T;\boldsymbol{V}_{-\tau}),$$

where τ is defined in Lemma 3.1. Indeed, in view of (3.5) in Lemma 3.1 we see that for all $\boldsymbol{\zeta} \in L^2(0,T;\boldsymbol{V}_\tau)$,

$$\begin{aligned}
&\left|\langle B(\boldsymbol{v}_n,\boldsymbol{v}_n) - B(\boldsymbol{v},\boldsymbol{v}),\boldsymbol{\zeta}\rangle_{L^2(0,T;\boldsymbol{V}_{-\tau}),L^2(0,T;\boldsymbol{V}_\tau)}\right| \\
&\qquad = \left|\langle B(\boldsymbol{v}_n - \boldsymbol{v},\boldsymbol{v}_n) + B(\boldsymbol{v},\boldsymbol{v}_n - \boldsymbol{v}),\boldsymbol{\zeta}\rangle_{L^2(0,T;\boldsymbol{V}_{-\tau}),L^2(0,T;\boldsymbol{V}_\tau)}\right| \\
&\qquad \le c_0\|\boldsymbol{v}_n - \boldsymbol{v}\|_{C([0,T];\boldsymbol{H})}\|\boldsymbol{v}_n\|_{L^2(0,T;\boldsymbol{V}_{1+\alpha})}\|\boldsymbol{\zeta}\|_{L^2(0,T;\boldsymbol{V}_\tau)} \\
&\qquad\qquad + \langle B(\boldsymbol{v},\boldsymbol{v}_n - \boldsymbol{v}),\boldsymbol{\zeta}\rangle_{L^2(0,T;\boldsymbol{V}_{-\tau}),L^2(0,T;\boldsymbol{V}_\tau)} \\
&\qquad \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Therefore $\boldsymbol{\xi} = B(\boldsymbol{v},\boldsymbol{v})$. Now take arbitrary $\boldsymbol{\zeta} \in L^2(0,T;\boldsymbol{V}_\alpha)$, multiply the equation in (5.10)′ by $\sum_{k=1}^n(\boldsymbol{\zeta}(t),\boldsymbol{e}_k)_{\boldsymbol{H}}$ and integrate over $[0,T]$ (namely $\int_0^T \sum_{k=1}^n(\boldsymbol{\zeta}(t),\boldsymbol{e}_k)_{\boldsymbol{H}}\times$(5.10)′). Then we have

$$\langle d\boldsymbol{v}_n/dt, P_n\boldsymbol{\zeta}\rangle_{L^2(0,T;\boldsymbol{V}_{-\alpha}),L^2(0,T;\boldsymbol{V}_\alpha)} = \langle \boldsymbol{h}_n, P_n\boldsymbol{\zeta}\rangle_{L^2(0,T;\boldsymbol{V}_{-\alpha}),L^2(0,T;\boldsymbol{V}_\alpha)}\,.$$

Passage to the limit of the above relation with (5.13), (5.14) and (5.15) yields that

$$\langle d\boldsymbol{v}/dt, \boldsymbol{\zeta}\rangle_{L^2(0,T;\boldsymbol{V}_{-\alpha}),L^2(0,T;\boldsymbol{V}_\alpha)} = \langle \boldsymbol{h}, \boldsymbol{\zeta}\rangle_{L^2(0,T;\boldsymbol{V}_{-\alpha}),L^2(0,T;\boldsymbol{V}_\alpha)}\,,$$

and hence $d\boldsymbol{v}/dt = \boldsymbol{h} \in L^2(0,T;\boldsymbol{V}_{-1+\alpha})$ holds from the arbitrariness of $\boldsymbol{\zeta} \in L^2(0,T;\boldsymbol{V}_\alpha)$. This concludes existence since $\boldsymbol{v}$ is a solution to $(\mathrm{NS})_\alpha$. □

Remark 5.2. Let $N = 2,3$, $0 < T < \infty$ and $\alpha = 1$. It is well-known that $(\mathrm{NS})_1$ has a (strong) solution $\boldsymbol{v} \in H^1(0,T;\boldsymbol{H}) \cap L^\infty(0,T;\boldsymbol{V}) \cap L^2(0,T;\boldsymbol{V}_2)$ with an initial data $\boldsymbol{v}_0 \in \boldsymbol{V}$ (see e.g., [29, Theorem 3.10 or 3.11 in Chapter III], [30, Theorem 3.2]). Concerning the (global in time) existence in Proposition 5.1 ($N = 2$), we would prove via another approximation instead of the Galerkin approximation. Indeed, for $\boldsymbol{v}_0 \in \boldsymbol{V}_\alpha$ take $\{\boldsymbol{v}_{0,n}\} \in \boldsymbol{V}$ such that $\boldsymbol{v}_{0,n} \to \boldsymbol{v}_0$ in $\boldsymbol{V}_\alpha$ and consider the approximate solution $\boldsymbol{v}_n \in H^1(0,T;\boldsymbol{H}) \cap L^\infty(0,T;\boldsymbol{V}) \cap L^2(0,T;\boldsymbol{V}_2)$ with the initial data $\boldsymbol{v}_{0,n} \in \boldsymbol{V}$. Then a similar calculation guarantees the existence. However concerning Proposition 5.2 ($N = 3$), the same way toward the (local in time) existence would break down since $T_0(\theta,\boldsymbol{v}_{0,n})$ decreases depending on increase of $\|\boldsymbol{v}_{0,n}\|_{\boldsymbol{V}}$ and there is a possibility $T_0(\theta,\boldsymbol{v}_{0,n})$ tends to 0.

6 Proof of the main theorems

In this section $e^{t\Delta}$ denotes the semigroup of the Dirichlet Laplacian Δ for $t \in [0,T]$. See e.g., Cazenave-Haraux [3] for such semigroup and its properties.

Lemma 6.1. *For all* $\xi \in L^p(0,T;L^q(\Omega))$ *with*

$$\frac{1}{p} + \frac{N}{2}\cdot\frac{1}{q} < 1 \tag{6.1}$$

the following estimate holds for $t \in [0,T]$:

$$\int_0^t \|e^{(t-s)\Delta}\xi(s)\|_{L^\infty(\Omega)}\,ds \le c_0 t^{1-\frac{1}{p}-\frac{N}{2q}}\|\xi\|_{L^p(0,t;L^q(\Omega))},$$

where $c_0 > 0$ *is a constant.*

Proof. The standard estimate for the heat kernel and the Hölder inequality yield that

$$\int_0^t \|e^{(t-s)\Delta}\xi(s)\|_{L^\infty(\Omega)}\,ds \le c\int_0^t (t-s)^{-\frac{N}{2}\cdot\frac{1}{q}}\|\xi(s)\|_{L^q(\Omega)}\,ds$$
$$\le c\left(\int_0^t (t-s)^{-\frac{N}{2}\cdot\frac{1}{q}\cdot p'}\right)^{1/p'} ds\|\xi\|_{L^p(0,t;L^q(\Omega))},$$

where $c > 0$ is a constant. Here the necessary and sufficient condition for integrability of $(t-s)^{-\frac{N}{2}\cdot\frac{1}{q}\cdot p'}$ on $(0,t)$ is that $-\frac{N}{2}\cdot\frac{1}{q}\cdot p' > -1$, namely (6.1), and hence the desired inequality is obtained. □

Proof of Theorems 1.1 and 1.2. Let $N = 2, 3$, $0 < T < \infty$ and $\frac{3(N-2)}{4} < \alpha \le 1$. Suppose (A1)–(A3). Even if $N = 3$, we let T_* be denoted with T for simplicity. Fixing $\theta \in L^\infty(0,T;L^\infty(\Omega))$, we see from Proposition 5.1 (for $N = 2$) or Proposition 5.2 (for $N = 3$) that there exists a unique solution $\boldsymbol{v}(=: S_1(\theta))$ to the Navier-Stokes equation. On the other hand, Proposition 4.1 gives a unique solution $(w,\theta)(=: (S_2'(\boldsymbol{v}), S_2(\boldsymbol{v})))$ to the heat equation with the hysteresis with fixed $\boldsymbol{v}$. That is, Proposition 5.1 or 5.2 and Proposition 4.1 provide the following mappings:

$$\begin{aligned}
&S_1 : \theta \in X(T) \mapsto \boldsymbol{v} \in \mathcal{C}_3(T) && (\boldsymbol{v} \text{ is the solution to } (\mathrm{NS})_\alpha \text{ for } \theta),\\
&S_2 : \boldsymbol{v} \in \mathcal{C}_3(T) \mapsto \theta \in X(T) && (\theta \text{ is the second part of the solution to (H) for } \boldsymbol{v}),\\
&S_2' : \boldsymbol{v} \in \mathcal{C}_3(T) \mapsto w \in \mathcal{C}_1(T;\theta) && (w \text{ is the first part of the solution to (H) for } \boldsymbol{v}),
\end{aligned}$$

where $X(T) \subset L^\infty(0,T;L^\infty(\Omega))$ is defined below. Moreover we consider the well-defined mapping

$$S := S_2 \circ S_1 : \overline{\theta} \in X(T) \mapsto \boldsymbol{v} \in \mathcal{C}_3(T) \mapsto \widetilde{\theta} \in X(T).$$

In other words, for fixed $\overline{\theta}$ there exists a unique solution $(w,\widetilde{\theta},\boldsymbol{v})$ such that

$$\begin{cases}
dw/dt + \partial I_{\widetilde{\theta}}(w) \ni 0 & \text{in } H \quad \text{a.e. on } (0,T),\\
d\widetilde{\theta}/dt - \Delta\widetilde{\theta} + \boldsymbol{v}\cdot\nabla\widetilde{\theta} + w = f & \text{in } H \quad \text{a.e. on } (0,T),\\
d\boldsymbol{v}/dt + A\boldsymbol{v} + B(\boldsymbol{v},\boldsymbol{v}) = P\boldsymbol{g}(\overline{\theta}) & \text{in } \boldsymbol{V}_{-1+\alpha} \quad \text{a.e. on } (0,T),\\
(w(0),\theta(0),\boldsymbol{v}(0)) = (w_0,\theta_0,\boldsymbol{v}_0) & \text{in } H\times H\times \boldsymbol{H}.
\end{cases}$$

In order to establish existence we apply the contraction mapping principle with the complete metric space $(X(T), d)$ as

$$X(T) := \left\{\theta \in L^\infty(0,T;L^\infty(\Omega)) \mid \|\theta\|_{L^\infty(0,T;H)} \le M_1(\|\theta_0\|_H)\right\},$$
$$d(\theta_1,\theta_2) := \|\theta_1 - \theta_2\|_{L^\infty_k(0,T;L^\infty(\Omega))},$$

where $M_1(\|\theta_0\|_H) > 0$ is defined as (4.6) in Proposition 4.1 and $\|\cdot\|_{L^\infty_k(0,T;L^\infty(\Omega))}$ is defined as (2.1) with $k > 0$ large enough. Note the relation $S_1(X(T)) \subset \mathcal{C}_3(T)$ as above. Actually, in the case $N = 3$, the relation $S_1(X(T_*)) \subset \mathcal{C}_3(T_*)$ eventually holds since the relation $\theta \in X(T_*)$ implies $T_* \le T_0(\theta,\boldsymbol{v}_0)$, and hence $S_1(\theta) \in \mathcal{C}_3(T_0(\theta,\boldsymbol{v}_0)) \subset \mathcal{C}_3(T_*)$. (From now on we let T_* be denoted by T for simplicity.) Now let $\overline{\theta} \in X(T)$ and put $\boldsymbol{v} := S_1(\overline{\theta})$,

$\widetilde{\theta} := S_2(\boldsymbol{v})$ and $w := S_2'(\boldsymbol{v})$. Then Proposition 5.1 (for $N = 2$) or 5.2 (for $N = 3$) implies

$$\begin{aligned}(6.2)\qquad \|\boldsymbol{v}\|^2_{L^\infty(0,T;\boldsymbol{V}_\alpha)} + \|\boldsymbol{v}\|^2_{L^2(0,T;\boldsymbol{V}_{1+\alpha})} &\le M_4(\|\boldsymbol{v}_0\|_{\boldsymbol{V}_\alpha}, \|\overline{\theta}\|_{L^2(0,T;H)})\\ &\le M_4(\|\boldsymbol{v}_0\|_{\boldsymbol{V}_\alpha}, T^{1/2}M_1(\|\theta_0\|_H))\\ &=: M_4'' = M_4''(\|\theta_0\|_H, \|\boldsymbol{v}_0\|_{\boldsymbol{V}_\alpha}).\end{aligned}$$

Here, in the case $N = 3$, replace $M_4(\|\boldsymbol{v}_0\|_{\boldsymbol{V}_\alpha}, T^{1/2}M_1(\|\theta_0\|_H))$ by $M_4'(\|\boldsymbol{v}_0\|_{\boldsymbol{V}_\alpha}, M_1(\|\theta_0\|_H))$. Hence M_4'' increases depending on increase of $\|\theta_0\|_H$ and $\|\boldsymbol{v}_0\|_{\boldsymbol{V}_\alpha}$. Moreover Proposition 4.1 yields

$$(6.3)\qquad \|\widetilde{\theta}\|_{L^\infty(0,T;H)} \le M_1(\|\theta_0\|_H),$$

$$(6.4)\qquad \|\widetilde{\theta}\|_{L^\infty(0,T;L^\infty(\Omega))} \le M_2(\|\theta_0\|_{L^\infty(\Omega)}),$$

$$\begin{aligned}(6.5)\qquad \|\widetilde{\theta}\|^2_{L^\infty(0,T;V)} + \|\Delta\widetilde{\theta}\|^2_{L^2(0,T;H)} &\le M_3(\|\theta_0\|_V, \|\boldsymbol{v}\|_{L^\infty(0,T;\boldsymbol{V}_\alpha)})\\ &\le M_3(\|\theta_0\|_V, M_4''(\|\theta_0\|_H, \|\boldsymbol{v}_0\|_{\boldsymbol{V}_\alpha})^{1/2})\\ &=: M_3' = M_3'(\|\theta_0\|_V, \|\boldsymbol{v}_0\|_{\boldsymbol{V}_\alpha}),\end{aligned}$$

where M_3' increases depending on increase of $\|\theta_0\|_V$ and $\|\boldsymbol{v}_0\|_{\boldsymbol{V}_\alpha}$. Thus the estimate (6.3) yields the relation $\widetilde{\theta} \in X(T)$, and hence guarantees $S(X(T)) \subset X(T)$.

Now we show contractivity of S. Let $\overline{\theta}_1, \overline{\theta}_2 \in X$ and put $\boldsymbol{v}_i := S_1(\overline{\theta}_i)$, $\widetilde{\theta}_i := S_2(\boldsymbol{v}_i)$ and $w_i := S_2'(\boldsymbol{v}_i)$ $(i = 1, 2)$. For simplicity put $w := w_1 - w_2$, $\widetilde{\theta} := \widetilde{\theta}_1 - \widetilde{\theta}_2$, $\overline{\theta} := \overline{\theta}_1 - \overline{\theta}_2$, $\boldsymbol{v} := \boldsymbol{v}_1 - \boldsymbol{v}_2$, $w_0 := w_{0,1} - w_{0,2}$, $\theta_0 := \theta_{0,1} - \theta_{0,2}$ and $\boldsymbol{v}_0 := \boldsymbol{v}_{0,1} - \boldsymbol{v}_{0,2}$. Here in view of the estimate (4.5) and the estimate (5.2) (for $N = 2$) or (5.2)$'$ (for $N = 3$) we see that for $t \in [0, T]$:

$$(6.6)\qquad \|w(t)\|_{L^\infty(\Omega)} \le \|w_0\|_{L^\infty(\Omega)} + C_3\|\widetilde{\theta}\|_{L^\infty(0,t;L^\infty(\Omega))},$$

$$(6.7)\qquad \|\boldsymbol{v}(t)\|^2_{\boldsymbol{V}_\alpha} + \|\boldsymbol{v}\|^2_{L^2(0,t;\boldsymbol{V}_{1+\alpha})} \le C_4''\left(\|\boldsymbol{v}_0\|^2_{\boldsymbol{V}_\alpha} + \|\overline{\theta}\|^2_{L^2(0,t;H)}\right),$$

where

$$C_4'' := \begin{cases} C_4(\max_{i=1,2}\|\boldsymbol{v}_{0,i}\|_{\boldsymbol{V}_\alpha}, T^{1/2}\max_{i=1,2}M_1(\|\theta_{0,i}\|_H)), & N = 2,\\ C_4'(\max_{i=1,2}\|\boldsymbol{v}_{0,i}\|_{\boldsymbol{V}_\alpha}, \max_{i=1,2}M_1(\|\theta_{0,i}\|_H)), & N = 3,\end{cases}$$

which increases depending on increase of $\max_{i=1,2}\|\theta_{0,i}\|_H$ and $\max_{i=1,2}\|\boldsymbol{v}_{0,i}\|_{\boldsymbol{V}_\alpha}$. Moreover plugging (6.7) into the estimate (4.4) implies that for all $t \in [0, T]$,

$$\begin{aligned}(6.8)\qquad \|\widetilde{\theta}(t)\|^2_V + \|\Delta\widetilde{\theta}\|^2_{L^2(0,t;H)} &\le C_2\left(\|\theta_0\|^2_V + \|\boldsymbol{v}\|^2_{L^\infty(0,t;\boldsymbol{V}_\alpha)} + \|w\|^2_{L^2(0,t;H)}\right)\\ &\le C_2'\left(\|\theta_0\|^2_V + \|\boldsymbol{v}_0\|^2_{\boldsymbol{V}_\alpha} + \|\overline{\theta}\|^2_{L^2(0,t;H)} + \|w\|^2_{L^2(0,t;H)}\right),\end{aligned}$$

where

$$\begin{aligned}C_2' := C_2(\min_{i=1,2}\|\theta_{0,i}\|_V, \max_{i=1,2}M_4''(\|\theta_{0,i}\|_H, \|\boldsymbol{v}_{0,i}\|_{\boldsymbol{V}_\alpha})^{1/2})\\ \times\left(C_4''(\max_{i=1,2}\|\theta_{0,i}\|_H, \max_{i=1,2}\|\boldsymbol{v}_{0,i}\|_{\boldsymbol{V}_\alpha}) \vee 1\right).\end{aligned}$$

C_2' increases depending on increase of $\max_{i=1,2}\|\theta_{0,i}\|_H$, $\min_{i=1,2}\|\theta_{0,i}\|_V$, $\max_{i=1,2}\|\boldsymbol{v}_{0,i}\|_{\boldsymbol{V}_\alpha}$, Moreover we estimate $\|\theta\|_{L^\infty(0,T;L^\infty(\Omega))}$ as follows. By taking the difference of the heat equations, we see that

$$\begin{cases} d\widetilde{\theta}/dt - \Delta\widetilde{\theta} + \boldsymbol{v}_1\cdot\nabla\widetilde{\theta} + \boldsymbol{v}\cdot\nabla\widetilde{\theta}_2 + w = 0 & \text{in } H \quad \text{a.e. on } (0,T), \\ \widetilde{\theta}(0) = \theta_0 & \text{in } H, \end{cases}$$

and hence we obtain the following integral equation for $t\in[0,T]$:

$$\widetilde{\theta}(t) = e^{t\Delta}\theta_0 - \int_0^t e^{(t-s)\Delta}(\boldsymbol{v}_1\cdot\nabla\widetilde{\theta})(s)\,ds - \int_0^t e^{(t-s)\Delta}(\boldsymbol{v}\cdot\nabla\widetilde{\theta}_2)(s)\,ds - \int_0^t e^{(t-s)\Delta}w(s)\,ds.$$

Here we apply Lemma 6.1 to $\boldsymbol{v}_1\cdot\nabla\widetilde{\theta}, \boldsymbol{v}\cdot\nabla\widetilde{\theta}_2 \in L^4(0,T;L^\sigma(\Omega))$ and $w\in L^2(0,T;L^\infty(\Omega))$, where σ is defined in Lemma 3.1. See (3.2) in Lemma 3.1 and note that

$$\frac{1}{4}+\frac{N}{2}\cdot\frac{1}{\sigma}<1 \iff \frac{1}{4}+\frac{N}{2}\cdot\left(\frac{3}{4}-\frac{\alpha}{N}\right)<1 \iff \alpha > \frac{3(N-2)}{4}.$$

This is exactly the condition assumed as (1.3). Then it follows that for all $t\in[0,T]$,

$$\begin{aligned} \|\widetilde{\theta}(t)\|_{L^\infty(\Omega)} &\le \|\theta_0\|_{L^\infty(\Omega)} + \int_0^t \left\|e^{(t-s)\Delta}(\boldsymbol{v}_1\cdot\nabla\widetilde{\theta})(s)\right\|_{L^\infty(\Omega)} ds \\ &\quad + \int_0^t \left\|e^{(t-s)\Delta}(\boldsymbol{v}\cdot\nabla\widetilde{\theta}_2)(s)\right\|_{L^\infty(\Omega)} ds + \int_0^t \left\|e^{(t-s)\Delta}w(s)\right\|_{L^\infty(\Omega)} ds \\ &\le \|\theta_0\|_{L^\infty(\Omega)} \\ &\quad + c_0\left[t^{\frac{3}{4}-\frac{1}{\sigma}}\left(\|\boldsymbol{v}_1\cdot\nabla\widetilde{\theta}\|_{L^4(0,t;L^\sigma)} + \|\boldsymbol{v}\cdot\nabla\widetilde{\theta}_2\|_{L^4(0,t;L^\sigma)}\right) + t^{\frac{1}{2}}\|w\|_{L^2(0,t;L^\infty(\Omega))}\right]. \end{aligned}$$

By using (3.2) in Lemma 3.1 with (6.2) and (6.8) we see that for all $t\in[0,T]$,

$$\begin{aligned} &\|\boldsymbol{v}_1\cdot\nabla\widetilde{\theta}\|_{L^4(0,t;L^\sigma(\Omega))} \\ &\quad \le c_0\|\boldsymbol{v}_1\|_{L^\infty(0,t;\boldsymbol{V}_\alpha)}\|\widetilde{\theta}\|^{1/2}_{L^\infty(0,t;L^\infty(\Omega))}\|\Delta\widetilde{\theta}\|^{1/2}_{L^\infty(0,t;H)} \\ &\quad \le c_0(M_4'')^{1/2}(C_2')^{1/4}\|\widetilde{\theta}\|^{1/2}_{L^\infty(0,t;L^\infty(\Omega))}\left(\|\theta_0\|_V^2 + \|\boldsymbol{v}_0\|^2_{\boldsymbol{V}_\alpha} + \|\overline{\theta}\|^2_{L^2(0,t;H)} + \|w\|^2_{L^2(0,t;H)}\right)^{1/4} \\ &\quad \le C_2''\|\widetilde{\theta}\|^{1/2}_{L^\infty(0,t;L^\infty(\Omega))}\left(\|\theta_0\|_V + \|\boldsymbol{v}_0\|_{\boldsymbol{V}_\alpha} + \|\overline{\theta}\|_{L^2(0,t;H)} + \|w\|_{L^2(0,t;H)}\right)^{1/2}, \end{aligned}$$

where

$$C_2'' := c_0M_4''(\|\theta_{0,1}\|_H, \|\boldsymbol{v}_{0,1}\|_{\boldsymbol{V}_\alpha})^{1/2}\times C_2'(\max_{i=1,2}\|\theta_{0,i}\|_H, \min_{i=1,2}\|\theta_{0,i}\|_V, \max_{i=1,2}\|\boldsymbol{v}_{0,i}\|_{\boldsymbol{V}_\alpha})^{1/4}.$$

C_2'' increases depending on increase of $\max_{i=1,2}\|\theta_{0,i}\|_H$, $\min_{i=1,2}\|\theta_{0,i}\|_V$, $\max_{i=1,2}\|\boldsymbol{v}_{0,i}\|_{\boldsymbol{V}_\alpha}$. Similarly, it follows from (3.2) in Lemma 3.1 with (6.4), (6.5) and (6.7) that for all $t\in[0,T]$,

$$\begin{aligned} \|\boldsymbol{v}\cdot\nabla\widetilde{\theta}_2\|_{L^4(0,t;L^\sigma(\Omega))} &\le c_0\|\boldsymbol{v}\|_{L^\infty(0,t;\boldsymbol{V}_\alpha)}\|\widetilde{\theta}_2\|^{1/2}_{L^\infty(0,t;L^\infty(\Omega))}\|\Delta\widetilde{\theta}_2\|^{1/2}_{L^2(0,t;H)} \\ &\le c_0(C_4'')^{1/2}M_2^{1/2}(M_3')^{1/4}\left(\|\boldsymbol{v}_0\|^2_{\boldsymbol{V}_\alpha} + \|\overline{\theta}\|^2_{L^2(0,t;H)}\right)^{1/2} \\ &\le C_4'''\left(\|\boldsymbol{v}_0\|_{\boldsymbol{V}_\alpha} + \|\overline{\theta}\|_{L^2(0,t;H)}\right), \end{aligned} \tag{6.9}$$

where

$$C_4''' := c_0 C_4''(\max_{i=1,2} \|\theta_{0,i}\|_H, \max_{i=1,2} \|\boldsymbol{v}_{0,i}\|_{\boldsymbol{V}_\alpha})^{1/2} \times M_2(\|\theta_{0,2}\|_{L^\infty(\Omega)})^{1/2} \times M_3'(\|\theta_{0,2}\|_V, \|\boldsymbol{v}_{0,2}\|_{\boldsymbol{V}_\alpha})^{1/4},$$

i.e., C_4''' increases depending on increase of $\max_{i=1,2} \|\theta_{0,i}\|_H$, $\|\theta_{0,2}\|_V$, $\|\theta_{0,2}\|_{L^\infty(\Omega)}$ and $\max_{i=1,2} \|\boldsymbol{v}_{0,i}\|_{\boldsymbol{V}_\alpha}$. Therefore by combining the above three inequalities it follows that for all $t \in [0,T]$,

$$\|\widetilde{\theta}(t)\|_{L^\infty(\Omega)} \le C_5 \left(\|\theta_0\|_V + \|\theta_0\|_{L^\infty(\Omega)} + \|\boldsymbol{v}_0\|_{\boldsymbol{V}_\alpha} + \|\overline{\theta}\|_{L^2(0,t;H)} + \|w\|_{L^2(0,t;L^\infty(\Omega))} \right),$$

where $C_5 > 0$ is a constant, which increases depending on increase of

$$C_2''(\max_{i=1,2} \|\theta_{0,i}\|_H, \min_{i=1,2} \|\theta_{0,i}\|_V, \max_{i=1,2} \|\boldsymbol{v}_{0,i}\|_{\boldsymbol{V}_\alpha}),$$
$$C_4'''(\max_{i=1,2} \|\theta_{0,i}\|_H, \|\theta_{0,2}\|_V, \|\theta_{0,2}\|_{L^\infty(\Omega)}, \max_{i=1,2} \|\boldsymbol{v}_{0,i}\|_{\boldsymbol{V}_\alpha}),$$

and hence on increase of $\max_{i=1,2} \|\theta_{0,i}\|_H$, $\|\theta_{0,2}\|_V$, $\|\theta_{0,2}\|_{L^\infty(\Omega)}$ and $\max_{i=1,2} \|\boldsymbol{v}_{0,i}\|_{\boldsymbol{V}_\alpha}$. Moreover in view of (6.6) we see that for all $t \in [0,T]$,

$$\begin{aligned} \|w(t)\|_{L^\infty(\Omega)} &\le \|w_0\|_{L^\infty(\Omega)} \\ &\quad + C_3 C_5 \left(\|\theta_0\|_V + \|\theta_0\|_{L^\infty(\Omega)} + \|\boldsymbol{v}_0\|_{\boldsymbol{V}_\alpha} + \|\overline{\theta}\|_{L^2(0,t;H)} + \|w\|_{L^2(0,t;L^\infty(\Omega))} \right). \end{aligned}$$

Multiplying the above two inequalities by e^{-kt} and taking the supremum as $t \in (0,T)$ (see (2.1) for the definition $\|\cdot\|_{L^\infty_k(0,T;L^\infty(\Omega))}$), we deduce that

$$\begin{aligned} \|\widetilde{\theta}\|_{L^\infty_k(0,T;L^\infty(\Omega))} &\le C_5 \Bigg(\|\theta_0\|_V + \|\theta_0\|_{L^\infty(\Omega)} + \|\boldsymbol{v}_0\|_{\boldsymbol{V}_\alpha} + \frac{1}{(2k)^{1/2}} \|\overline{\theta}\|_{L^\infty_k(0,T;L^\infty(\Omega))} \\ &\qquad + \frac{1}{(2k)^{1/2}} \|w\|_{L^\infty_k(0,T;L^\infty(\Omega))} \Bigg), \\ \|w\|_{L^\infty_k(0,T;L^\infty(\Omega))} &\le \|w_0\|_{L^\infty(\Omega)} \\ &\quad + C_3 C_5 \Bigg(\|\theta_0\|_V + \|\theta_0\|_{L^\infty(\Omega)} + \|\boldsymbol{v}_0\|_{\boldsymbol{V}_\alpha} + \frac{1}{(2k)^{1/2}} \|\overline{\theta}\|_{L^\infty_k(0,T;L^\infty(\Omega))} \\ &\qquad + \frac{1}{(2k)^{1/2}} \|w\|_{L^\infty_k(0,T;L^\infty(\Omega))} \Bigg). \end{aligned}$$

Then taking $k > 0$ large enough for example, $k := \frac{9}{2}(C_5 \vee C_5 C_3)^2$, we see from the above two inequalities that

$$\begin{aligned} \|\widetilde{\theta}\|_{L^\infty_k(0,T;L^\infty(\Omega))} &\le C_5 \left(\|\theta_0\|_V + \|\theta_0\|_{L^\infty(\Omega)} + \|\boldsymbol{v}_0\|_{\boldsymbol{V}_\alpha} \right) \\ &\quad + \frac{1}{3} \left(\|\overline{\theta}\|_{L^\infty_k(0,T;L^\infty(\Omega))} + \|w\|_{L^\infty_k(0,T;L^\infty(\Omega))} \right) \\ \|w\|_{L^\infty_k(0,T;L^\infty(\Omega))} &\le \frac{3}{2} \|w_0\|_{L^\infty(\Omega)} \\ &\quad + \frac{3}{2} C_5 C_3 \left(\|\theta_0\|_V + \|\theta_0\|_{L^\infty(\Omega)} + \|\boldsymbol{v}_0\|_{\boldsymbol{V}_\alpha} \right) + \frac{1}{2} \|\overline{\theta}\|_{L^\infty_k(0,T;L^\infty(\Omega))}. \end{aligned}$$

By combining the above two inequalities it follows that

$$\|\widetilde{\theta}\|_{L^\infty_k(0,T;L^\infty(\Omega))} \le C_5' \left(\|w_0\|_{L^\infty(\Omega)} + \|\theta_0\|_V + \|\theta_0\|_{L^\infty(\Omega)} + \|\boldsymbol{v}_0\|_{\boldsymbol{V}_\alpha}\right) + \frac{1}{2}\|\overline{\theta}\|_{L^\infty_k(0,T;L^\infty(\Omega))}, \tag{6.10}$$

where $C_5' := C_5 + \frac{1}{2} + \frac{1}{2}C_5C_3$, which increases depending on increase of $\max_{i=1,2}\|\theta_{0,i}\|_H$, $\|\theta_{0,2}\|_V$, $\|\theta_{0,2}\|_{L^\infty(\Omega)}$ and $\max_{i=1,2}\|\boldsymbol{v}_{0,i}\|_{\boldsymbol{V}_\alpha}$.

Finally we conclude the proof. (6.10) with $w_{0,1} = w_{0,2}$, $\theta_{0,1} = \theta_{0,2}$, $\boldsymbol{v}_{0,1} = \boldsymbol{v}_{0,2}$ yields

$$d(\widetilde{\theta}_1, \widetilde{\theta}_2) = \|\widetilde{\theta}_1 - \widetilde{\theta}_2\|_{L^\infty_k(0,T;L^\infty(\Omega))} \le \frac{1}{2}\|\overline{\theta}_1 - \overline{\theta}_2\|_{L^\infty_k(0,T;L^\infty(\Omega))} = \frac{1}{2}d(\overline{\theta}_1, \overline{\theta}_2),$$

i.e., $S : X(T) \to X(T)$ is a contraction mapping. By virtue of the contraction mapping principle, there exists $\theta \in X(T)$ such that $S(\theta) = \theta$. Moreover put $\boldsymbol{v} := S_1(\theta)$, $w := S_2'(\theta)$. Then $\theta = S_2(\boldsymbol{v})$, and hence $(w, \theta, \boldsymbol{v})$ is a solution to (P). This guarantees existence for (P). On the other hand, continuous dependence of solutions on initial data (1.4) would be proved by almost the same calculation (see [33]), and hence this conmpletes the proof. □

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