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Global existence of solutions to a parabolic-parabolic chemotaxis system with subquadratic growth

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1. Chemotaxis-Growth System

In a study of the chemotactic bacterial pattern formation, Mimura and Tsujikawa [10] analyzed a parabolic-parabolic chemotaxis system with bacterial proliferation, which is of the following simplified form:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \chi \nabla \cdot (u \nabla v) + f(u) & \text{in } \Omega \times (0, \infty), \\ \tau \dfrac{\partial v}{\partial t} = \Delta v - v + g(u) & \text{in } \Omega \times (0, \infty), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & \text{in } \Omega. \end{cases} \tag{E}$$

Here, $\Omega \subset \mathbb{R}^n$ ($n = 2$ or 3) is a bounded domain with smooth boundary $\partial\Omega$, and the coefficients χ and τ are positive constants. The unknown functions $u(x, t)$ and $v(x, t)$ are the population density of biological individuals and the concentration of chemical substance in position x at time t, respectively. We here note that the limit $\tau \to 0$ of the time scale indicates the parabolic-elliptic simplification of [21] (cf. [12]). We assume that the function $f(u)$ is a real smooth function of $u \in [0, \infty)$ such that $f(0) = 0$ and

$$f(u) = u - \mu u^{\alpha} \quad \text{for sufficiently large } u;$$

and the function $g(u)$ is given by

$$g(u) = u(1+u)^{\beta-1} \quad \text{for } u \geq 0,$$

where the exponents α and β satisfy the relations $1 < \alpha \leq 2$ and $0 < \beta \leq 1$, and μ is a positive constant. The function $f(u)$ models the proliferation and the reduction in numbers due to death of bacteria as following a logistic process (we refer to the proliferation and reduction in numbers together simply as growth). When $\alpha = 2$, that is, quadratic degradation is assumed, the function $f(u)$ gives usual logistic growth. Subquadratic degradation then refers to the case of $\alpha < 2$. The function $g(u)$ models the nonlinear secretion of the chemical substance from the bacteria, whose increasing order is β [5, 11, 23].

In the context of global existence and blow-up of solutions, the degradation of the growth can be considered as an inhibitory effect from the increase of u. In fact, if we suppose that the growth is absent ($f(u) \equiv 0$) and the secretion is linear ($\beta = 1$), then

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the system reduces to the classical parabolic-parabolic Keller-Segel system [7], for which several mathematicians showed the blow-up of solutions: Herrero and Velázquez [4] ($n = 2$), Horstmann and Wang [6] ($n = 2$) and Winkler [26]($n \geq 3$). Therefore, we have known that if $f(u) \equiv 0$ or $\alpha = 1$ with a special choice of $\mu = 1$ and $\beta = 1$, then no inhibitory effect can cause a chemotactic collapse in the n-dimensional domain ($n \geq 2$). In contrast, for the parabolic-parabolic chemotaxis-growth system with $\alpha = 2$ and $\beta = 1$, the global existence of solutions is assured even if the initial total mass $\|u_0\|_{L_1}$ and the chemotactic coefficient χ are sufficiently large when $n = 2$ by one of the authors et al. [19] and $n \geq 3$ by Winkler [25]. From these results, we find that if degradation is quadratic ($\alpha = 2$) and secretion is linear ($\beta = 1$), then the blow-up of solutions is prevented independently of the space dimension. We can then conjecture that the critical degradation order α is in the interval $[1, 2]$ when $\beta = 1$; however, the corresponding result for the parabolic-parabolic system (E) has yet to be established.

We then introduced sub-linear secretion $\beta < 1$, and showed a sufficient condition for the existence of global and bounded solutions under certain relations between α and β when $n = 2$ or $n = 3$ [14]. In fact, we can obtain the following:

Theorem 1.1. *Let ε be an arbitrarily fixed exponent satisfying $0 < \varepsilon < 1/4$. For the exponents α and β, assume the relation*

$$\frac{2(n+4)}{n+6} < \alpha \leq 2, \qquad 0 < \beta < \frac{n+6}{2(n+2)}(\alpha - 1). \tag{1}$$

Then, for each initial function $0 \leq u_0 \in H^{(n/2)-1}(\Omega) \subset L_n(\Omega)$ and $0 \leq v_0 \in W \subset C(\overline{\Omega})$, the problem (E) *admits a unique global solution (u, v) in the function space*

$$\begin{cases} 0 \leq u \in C([0,\infty); H^{(n/2)-1}(\Omega)) \cap C((0,\infty); H^3_N(\Omega)) \cap C^1((0,\infty); H^1(\Omega)), \\ 0 \leq v \in C([0,\infty); W) \cap C((0,\infty); H^{4+\varepsilon}_{N^2}(\Omega)) \cap C^1((0,\infty); H^{2+\varepsilon}_N(\Omega)). \end{cases}$$

Here, the function spaces $H^s_N(\Omega)$, $H^s_{N^2}(\Omega)$ and W are defined by

$$H^s_N(\Omega) = \left\{ w \in H^s(\Omega);\ \frac{\partial w}{\partial n} = 0 \text{ on } \partial\Omega \right\} \quad \text{for } s > \frac{3}{2},$$

$$H^s_{N^2}(\Omega) = \left\{ w \in H^2_N(\Omega);\ \Delta w \in H^{s-2}_N(\Omega) \right\} \quad \text{for } s > \frac{7}{2},$$

$$W = \begin{cases} H^{1+\varepsilon}(\Omega) & \text{when } n = 2, \\ H^{(3/2)+\varepsilon}_N(\Omega) & \text{when } n = 3, \end{cases}$$

for some fixed exponent $0 < \varepsilon < 1/4$.

In this report, we show a sketch of a proof of the theorem and also the asymptotic behavior of the solutions. In the paper [14], the global attractors and the exponential attractors also were constructed. Indeed, we derived the higher order $H^2 \times H^3$-uniform estimates of the solutions (u, v), and then proved not only the boundedness of the solutions but also the existence of a ball in the $H^2 \times H^3$ topology in which any solutions that start from a bounded set of the universal space shall eventually be contained within a finite time. We call the set an absorbing set. The omega-limit set of the absorbing set is the global attractor (e.g. [22]). Hence, the dynamics including many complex pattern formations reduces to the restricted region in the function space and all the orbits traced out by the solutions converge to the global attractor as $t \to \infty$.

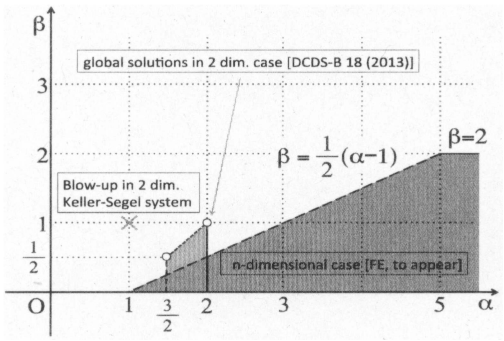

Figure 1: Region of global existence in α-β plane [13, 14, 15]. The X mark denotes an occurrence of a blow-up of solutions in the Keller-Segel system ($\alpha = \mu = \beta = 1$). The critical degradation order α between global existence and blow-up has not been found for (E).

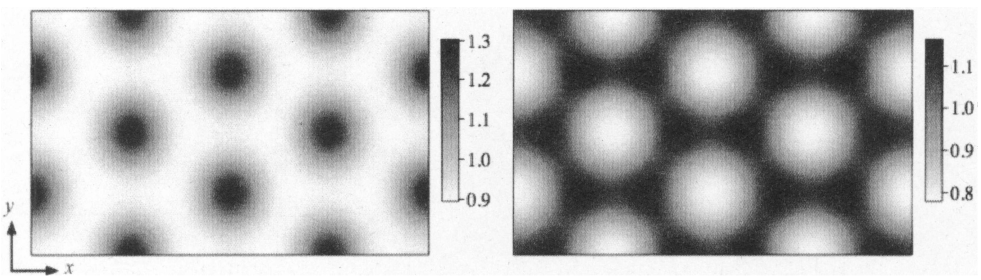

Figure 2: Dot and hexagonal pattern formation of solutions to the system (E). A dot pattern (left) and a hexagonal pattern (right) [9, 16]. In actual phenomena, some dot patterns have been observed; on the other hand, hexagonal patterns have not been observed, as far as the authors know.

An exponential attractor contains the global attractor and attracts the orbits at an exponential rate. Moreover, its fractal dimension is finite [3]. We thus find that the dynamics of solutions exponentially converges to the restricted compact region, of which the degree of freedom is finite. It is known that such characteristics have some advantages for numerical computations (e.g. [22]). Of the results including numerical computations of chemotaxis-growth systems, we cite here only a selection of the articles and books: the famous book on mathematical biology by Murray [11]; one-dimensional pattern formations by Kurata et al. [8], Okuda and Osaki [18], Painter and Hillen [20] and Uemichi and Osaki [24]; two-dimensional pattern formations by Aida et al. [1], Okuda and Osaki [17] and Kuto et al. [9] (see Fig. 2); and three-dimensional pattern formations by Narumi and Osaki [16] (see Figs. 2 and 3).

2. Global Existence of Solutions

After showing the local unique existence of solutions, we show the global existence of solutions by obtaining several a priori estimates.

Step1 (L_1-uniform estimates of u). By integrating the first equation of (E) in Ω,

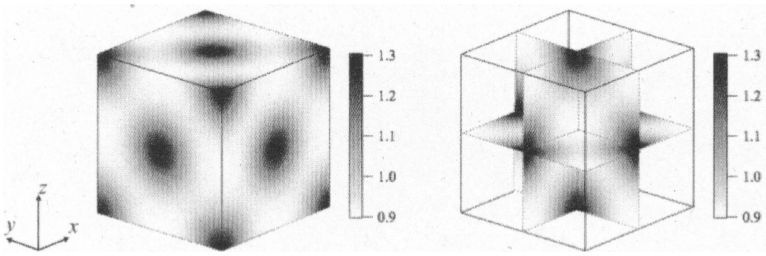

Figure 3: Three-dimensional face centered cubic (FCC) pattern formation in the chemotaxis-growth system (E) [16].

we obtain

$$\frac{d}{dt}\int_{\Omega} u\,dx = \int_{\Omega} f(u)\,dx \le \int_{\Omega}(a - \frac{\mu}{2}u^{\alpha})dx \le \int_{\Omega}(a_1 - u)dx.$$

From Gronwall's inequality, we obtain $\|u\|_{L_1} \le e^{-t}\|u_0\|_{L_1} + a_1|\Omega|(1-e^{-t})$, where a_1 is a constant. At the same time, we have

$$\int_0^t e^{-\omega(t-s)}\int_{\Omega} u^{\alpha}dxds \le \frac{2\omega}{\mu}\left\{\left(\frac{a}{\omega}+a_1\right)|\Omega| + \|u_0\|_{L_1}\right\} \le C(1+\|u_0\|_{L_1}).$$

Step 2 (H^1-uniform estimate of v: degradation vs. secretion). By multiplying the second equation of (E) by $-\Delta v$ and integrating the result over Ω, under the assumption $0 < 2\beta \le \alpha$, we obtain

$$\begin{aligned}\frac{\tau}{2}\frac{d}{dt}\int_{\Omega}|\nabla v|^2dx &\le -\frac{1}{2}\int_{\Omega}(\Delta v)^2dx - \int_{\Omega}|\nabla v|^2dx + \frac{1}{2}\int_{\Omega}(1+u)^{2\beta}dx \\ &\le -\frac{1}{2}\int_{\Omega}(\Delta v)^2dx - \int_{\Omega}|\nabla v|^2dx + C\int_{\Omega}(1+u^{\alpha})dx.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\int_{\Omega}|\nabla v|^2dx &\le e^{-2t/\tau}\int_{\Omega}|\nabla v_0|^2dx + C\int_0^t e^{-2(t-s)/\tau}\int_{\Omega}(1+u^{\alpha})dxds \\ &\le e^{-2t/\tau}\int_{\Omega}|\nabla v_0|^2dx + C(1+\|u_0\|_{L_1}).\end{aligned}$$

Here, we note that the assumptions $0 < \beta \le \alpha/2$ and $\alpha < 2$ imply sub-linear secretion $\beta < 1$.

Step 3 ($L_{\theta} \times H^2$-uniform estimate: chemotaxis vs. degradation). By multiplying the first equation of (E) by $(1+u)^{\theta-1}$ and integrating the result over Ω, we have

$$\begin{aligned}&\frac{1}{\theta}\frac{d}{dt}\int_{\Omega}(1+u)^{\theta}dx \\ &= -(\theta-1)\int_{\Omega}(1+u)^{\theta-2}|\nabla u|^2dx + \chi(\theta-1)\int_{\Omega}u(1+u)^{\theta-2}\nabla u\cdot\nabla v dx \\ &\qquad + \int_{\Omega}(1+u)^{\theta-1}f(u)dx \\ &\le -(\theta-1)\int_{\Omega}(1+u)^{\theta-2}|\nabla u|^2dx + \chi\int_{\Omega}(1+u)^{\theta}|\Delta v|dx - \int_{\Omega}u^{\alpha+\theta-1}dx + C.\end{aligned}$$

The chemotaxis term can be estimated as

$$\int_\Omega (1+u)^\theta |\Delta v| dx \le \|(1+u)^\theta\|_{L_{\frac{2(n+4)}{n+6}}} \|\Delta v\|_{L_{\frac{2(n+4)}{n+2}}}$$
$$\le C_n \|(1+u)^\theta\|_{L_{\frac{2(n+4)}{n+6}}} \|\Delta v\|_{H^{\frac{n}{n+4}}} \le C_n \|(1+u)^\theta\|_{L_{\frac{2(n+4)}{n+6}}} \|v\|_{H^1}^{\frac{2}{n+4}} \|v\|_{H^3}^{\frac{n+2}{n+4}}$$
$$\le \eta \|v\|_{H^3}^2 + C_\eta \|(1+u)^\theta\|_{L_{\frac{2(n+4)}{n+6}}}^{\frac{2(n+4)}{n+6}} \|v\|_{H^1}^{\frac{4}{n+6}}.$$

We here adopt the assumption "chemotaxis < degradation", that is,

$$\frac{2(n+4)}{n+6}\theta < \alpha + \theta - 1 \quad \Leftrightarrow \quad \theta < \frac{n+6}{n+2}(\alpha - 1).$$

Then, we obtain

$$\frac{1}{\theta}\frac{d}{dt}\int_\Omega (1+u)^\theta dx \le -(\theta - 1)\int_\Omega (1+u)^{\theta-2}|\nabla u|^2 dx + \eta \int_\Omega |\nabla \Delta v|^2 dx$$
$$- \frac{\mu}{4}\int_\Omega u^{\alpha+\theta-1} dx + \psi\left(\|v\|_{H^1} + \eta^{-1}\right).$$

Meanwhile, by applying operator ∇ to the second equation of (E), multiplying by $\nabla\Delta v$, and integrating the result over Ω, we have

$$\frac{\tau}{2}\frac{d}{dt}\int_\Omega |\Delta v|^2 dx = -\int_\Omega |\nabla\Delta v|^2 dx - \int_\Omega |\Delta v|^2 dx - \int_\Omega \nabla\Delta v \cdot g'(u)\nabla u dx$$
$$\le -\frac{1}{2}\int_\Omega |\nabla\Delta v|^2 dx - \int_\Omega |\Delta v|^2 dx + \frac{\beta^2}{2}\int_\Omega (u+1)^{2\beta-2}|\nabla u|^2 dx.$$

With an additional assumption $2\beta \le \theta$, we obtain an $L_\theta \times H^2$-uniform estimate.

Step 4 ($L_{2\theta}$-uniform estimate of u). From the first equation, we have

$$\frac{1}{2\theta}\frac{d}{dt}\int_\Omega (1+u)^{2\theta} dx \le -(2\theta - 1)\int_\Omega (1+u)^{2\theta-2}|\nabla u|^2 dx$$
$$+ \chi \int_\Omega (1+u)^{2\theta}|\Delta v| dx + C\int_\Omega (1 - u^{\alpha+2\theta-1}) dx.$$

The chemotaxis term can be estimated as

$$\int_\Omega (1+u)^{2\theta}|\Delta v| dx \le \|(1+u)^\theta\|_{L_4}^2 \|\Delta v\|_{L_2}$$
$$\le C\left(\|(1+u)^\theta\|_{H^1}^{\frac{3n}{2n+4}} \|(1+u)^\theta\|_{L_1}^{\frac{4-n}{2n+4}}\right)^2 \|\Delta v\|_{L_2}$$
$$\le \eta \|(1+u)^\theta\|_{H^1}^2 + C_\eta \|1+u\|_{L_\theta}^{2\theta} \|v\|_{H^2}^{\frac{2n+4}{4-n}}.$$

Then, we have a recurrence relation $\theta_{j+1} = 2\theta_j$ of θ, which allows L_θ-uniform estimates with arbitrary θ (cf. [2]).

By similar arguments to the above, we can also obtain the higher order uniform estimate [14]. □

3. Dynamical System and Attractors

By the above results, we can show the existence of attractors in the dynamical system of the solutions. In fact, let us define the universal space H as

$$H = L_2(\Omega) \times H^1(\Omega).$$

The initial functions are taken in the following set:

$$K = \{(u, v) \in H^{\frac{n}{2}-1}(\Omega) \times W;\ u \geq 0,\ v \geq 0\}, \quad 0 < \varepsilon < \frac{1}{4}.$$

Then, the global unique solutions belong to $\mathcal{D} = H^2_N(\Omega) \times H^3_N(\Omega)$, which shows that a continuous semigroup $S(t) : K \to K$ such that $(u_0, v_0) \mapsto (u(t), v(t)) \in K \cap \mathcal{D}$ can be defined. From the higher order uniform estimate in $\mathcal{D}$ [14], we have an absorbing set $\mathcal{B}$; that is, for every bounded set $B \subset K$, there exists a time t_B that may depend on B such that $\bigcup_{t \geq t_B} S(t)B \subset \mathcal{B}$. More precisely, we can show the following:

Proposition 3.1. *A bounded ball $\mathcal{B}$ of K*

$$\mathcal{B} = \{(u, v) \in H^2_N(\Omega) \times H^3_N(\Omega); \|u\|_{H^2} + \|v\|_{H^3} \leq C,\ u \geq 0,\ v \geq 0\} \subset K$$

is an absorbing set of the dynamical system $(S(t), K, H)$. Here, the constant C is a universal constant, which is suitably determined from the a priori estimates.

From the existence of the absorbing set $\mathcal{B}$, we can construct a positively invariant set $\mathcal{H}$ as

$$\mathcal{H} = \overline{\bigcup_{t \geq t_{\mathcal{B}}} S(t)\mathcal{B}},$$

whose topology of the closure is of K. Therefore, the asymptotic behavior of the solutions is reduced to the eventual dynamical system $(S(t), \mathcal{H}, H)$. In the dynamical system, the global attractor $\mathcal{A}$, which is a compact and invariant set in H and attracts every bounded subset of $\mathcal{H}$, is obtained as the ω-limit set of $\mathcal{B}$: $\mathcal{A} = \bigcap_{t \geq 0} \overline{\bigcup_{s \geq t} S(t)\mathcal{B}}$. A subset $\mathcal{M} \subset \mathcal{H}$ is called the exponential attractor for $(S(t), \mathcal{H}, H)$ if $\mathcal{A} \subset \mathcal{M} \subset \mathcal{H}$; $\mathcal{M}$ is a compact subset of H and is invariant for $S(t)$; $\mathcal{M}$ has finite fractal dimension $d_F(\mathcal{M})$; and $h(S(t)\mathcal{H}, \mathcal{M}) \leq c_0 \exp(-c_1 t)$ for $t \geq 0$ with some constants $c_0, c_1 > 0$. Here, $h(B_0, B_1) = \sup_{U \in B_0} \inf_{V \in B_1} \|U - V\|_H$ denotes the Hausdorff pseudodistance of two sets B_0 and B_1. Eden et al. [3, Proposition 3.1 and Theorem 3.1] showed that under the above setting, if some Lipschitz conditions hold for the equations, then an exponential attractor $\mathcal{M}$ exists for the dynamical system $(S(t), \mathcal{H}, H)$. We can verify the Lipschitz conditions by arguments quite similar to those in [19, Section 5]. Then we obtain the existence theorem of the global attractor and an exponential attractor as follows:

Theorem 3.2. *There exist the global attractor $\mathcal{A}$ and an exponential attractor $\mathcal{M}$ for the dynamical system $(S(t), \mathcal{H}, H)$ of the chemotaxis-growth system* (E).

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