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WEAK NORM INFLATION IN A FAMILY OF SOBOLEV SPACES FOR THE 2D EULER EQUATIONS

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1. Introduction

Consider the incompressible Euler equations[1]

$$\begin{aligned} &D_t u := u_t + u\cdot\nabla u = -\nabla\pi, \qquad t \geq 0,\ x \in \mathbb{R}^2 \\ &\operatorname{div} u = 0 \\ &u(0) = u_0 \end{aligned} \tag{1.1}$$

where $u = u(t, x)$ is the velocity of the fluid and $\pi = \pi(t, x)$ is the pressure function. The study of the Cauchy problem (1.1) has a long history going back to the work of Gyunter [5], Lichtenstein [8] and Wolibner [12] in the 1920s and 1930s. Modern expositions of those as well as more refined results that have been obtained can be found for example in the recent textbooks of Chemin [4], Majda and Bertozzi [9] or Bahouri, Chemin and Danchin [1].

Recently, in a series of papers Bourgain and Li [2, 3] introduced a method based on large Lagrangian deformations to prove strong local ill-posedness results in borderline Sobolev spaces $W^{n/p+1,p}$ for any $1 \leq p < \infty$ and in Besov spaces $B^{n/p+1}_{p,q}$ for any $1 \leq p < \infty$ and $1 < q \leq \infty$ and $n = 2$ or 3. Roughly speaking, using multi-scale vortices the authors construct a background Lagrangian flow with symmetries which they use together with careful estimates of singular integral operators to show that the flow has a large deformation effect. They then define a suitable high frequency perturbation in such a way so that its support intersects a neighbourhood of the point where the flow has the largest gradient which leads to norm inflation. In this paper we describe a different though related mechanism based on a limiting procedure that involves a scale of Banach spaces. On the one hand the norm inflation result we prove is weaker than that obtained by Bourgain and Li. On the other hand our method seems to be applicable in the borderline function spaces that were left out of the analysis in [2, 3]. Details for the case of the Besov space $B^2_{2,1}$ will appear elsewhere. Our objective here is to introduce this technique in a simpler setting of the Sobolev $W^{s,p}$ spaces. More precisely, we will prove the following result.

[1]This note may be regarded as complementary to the paper [10].

Theorem 1.1. *Let $M_j \nearrow \infty$ be a sequence of positive numbers. There is a sequence of smooth solutions $\{u_j\}_{j=1}^{\infty}$ of (1.1) with rapidly decaying initial data $\{u_{0,j}\}_{j=1}^{\infty}$ and a sequence $p_j \searrow 2$ such that*

$$\|u_{0,j}\|_{W^{2,p_j}} \lesssim 1 \quad \text{and} \quad \|u_j(t)\|_{W^{2,p_j}} \geq M_j$$

for some $0 < t \leq M_j^{-3}$.

In what follows we will need the fact that the Euler equations (1.1) are locally wellposed in the sense of Hadamard in the little Hölder space. A proof of this result is given in [11].

2. Large Lagrangian deformation

The vorticity formulation of the Cauchy problem for the Euler equations in two dimensions has the form

$$\begin{aligned} &\omega_t + u \cdot \nabla \omega = 0, \qquad\qquad t \geq 0,\ x \in \mathbb{R}^2 \\ &u = K * \omega \\ &\omega(0) = \omega_0 \end{aligned} \tag{2.2}$$

where $\omega = \partial_1 u_2 - \partial_2 u_1$ and

$$K(x) = \frac{1}{2\pi}\left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\right) \quad \text{and} \quad \nabla^{\perp} = \left(-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1}\right) \tag{2.3}$$

denote the Biot-Savart kernel and the symplectic gradient, respectively. Recall that the Lagrangian flow of u is obtained by solving

$$\begin{aligned} &\frac{d}{dt}\eta(t,x) = u(t,\eta(t,x)) \\ &\eta(0,x) = x \end{aligned} \tag{2.4}$$

and defines a curve in the group of diffeomorphisms of $\mathbb{R}^2$.

In order to construct a flow with a large deformation gradient we need to choose a suitable initial vorticity. Given a smooth radial bump function $0 \leq \psi \leq 1$ with support in the ball $B(0, 1/4)$ let

$$\psi_0(x_1, x_2) = \sum_{\varepsilon_1, \varepsilon_2 = \pm 1} \varepsilon_1 \varepsilon_2 \psi(x_1 - \varepsilon_1, x_2 - \varepsilon_2). \tag{2.5}$$

Next, fix a positive integer N_0 and for any numbers $p > 0$ and $M \gg 1$ and any integer $N \geq 1$ define

$$\omega_0(x) = M^{-2} N^{-\frac{1}{p}} \sum_{N_0 \leq k \leq N_0 + N} \psi_k(x), \tag{2.6}$$

where

$$\psi_k(x) = 2^{(-1+\frac{2}{p})k} \psi_0(2^k x)$$

whose supports are disjoint for any $k \geq 1$ and contained in the sets

$$\operatorname{supp} \psi_k \subset \bigcup_{\varepsilon_1, \varepsilon_2 = \pm 1} B\big((\varepsilon_1 2^{-k}, \varepsilon_2 2^{-k}), 2^{-(k+2)}\big). \tag{2.7}$$

It follows that ω_0 is smooth, compactly supported and odd in both variables x_1, x_2 and furthermore

Lemma 2.1. *If $2 < p < \infty$ then for any integer $N \geq 1$ we have*

$$\|\omega_0\|_{W^{1,p}} \lesssim M^{-2}. \tag{2.8}$$

Proof. A straightforward calculation is omitted. ∎

Let $u = K * \omega$ be the associated velocity field and consider its Lagrangian flow $\eta(t)$ given by (2.4). It is not difficult to verify that $\eta(t)$ is smooth and preserves the coordinate axes x_1, x_2 as well as the symmetries of the initial vorticity ω_0. In fact, it is hyperbolic near the origin (which is a stagnation point) and we have the following estimate

Proposition 2.2. *Let $M \gg 1$. For any sufficiently large integer $N \geq 1$ and any $2 < p < \infty$ sufficiently close to 2 we have*

$$\sup_{0 \leq t \leq M^{-3}} \|D\eta(t)\|_\infty \geq M. \tag{2.9}$$

Proof. Cf. [10]; Prop. 6. ∎

Recall that the little Hölder space $c^{1,\alpha}$ is a closed subspace of the Hölder space $C^{1,\alpha}$ and consists of those functions that in addition satisfy the vanishing condition $\lim_{h\to 0} \sup_{|x-y|<h} |x-y|^{-\alpha}|Df(x) - Df(y)| = 0$. It is a Banach space and is equipped with the same norm $\|\cdot\|_{1,\alpha}$ as the standard Hölder space. We will also need the following wellposedness result

Proposition 2.3. *The data-to-solution map of the Euler equations (1.1) is continuous in $c^{1,\alpha} \cap L^2$ for any $0 < \alpha < 1$.*

Proof. Cf. [11]. Note that the restriction to L^2 is needed to ensure appropriate decay at infinity. ∎

3. Proof of Theorem 1.1.

Let $M_j \nearrow \infty$ and choose sequences $N_j \geq 1$ and $2 < p_j < \infty$ for which the estimate (2.9) of Proposition 2.2 holds.[2] Proceeding as in [2] we will perturb the initial vorticity (2.6) and work with the associated Lagrangian flows. Theorem 1.1 will be a direct consequence of the following result.

Theorem 3.1. *There is a sequence $\omega_{0,j}^n$ of smooth, compactly supported functions with the following properties*

1. *there is a constant $C > 0$ such that $\|\omega_{0,j}^n\|_{W^{1,p_j}} \leq C$, and*
2. *there is $0 < t^* \leq M_j^{-3}$ such that $\|\omega_j^n(t^*)\|_{W^{1,p_j}} \geq M_j^{1/3}$*

for all sufficiently large positive integers n.

Proof of Theorem 3.1. First, given $M_j \gg 1$ observe that if $\|\omega_j(t)\|_{W^{1,p_j}} \geq M_j^{1/3}$ for some $0 < t \leq M_j^{-3}$ then there is nothing to prove and therefore we may assume that

$$\|\omega_j(t)\|_{W^{1,p_j}} \leq M_j^{1/3} \qquad \text{for all } \ 0 \leq t \leq M_j^{-3}. \tag{3.10}$$

Since $p_j > 2$ the associated velocity field $u_j = \nabla^\perp \Delta^{-1} \omega_j$ is C^1 and therefore so is its flow $\eta_j(t) = (\eta_j^1(t), \eta_j^2(t))$. Using (2.9) we can then pick $0 \leq t^* \leq M_j^{-3}$ and $x^* = (x_1^*, x_2^*) \in \mathbb{R}^2$ for which the absolute value of one of the entries in the Jacobi matrix $D\eta_j(t^*, x^*)$ is at

[2]For example, it is sufficient to take $N_j = 10M_j^{10}$ and $p_j = 2\frac{N_0+N_j}{N_0+N_j-1}$.

least as large as M_j and by continuity deduce that in a sufficiently small δ-neighbourhood of x^* we have

$$\left|\frac{\partial \eta_j^2}{\partial x_2}(t^*, x)\right| \geq M_j \qquad \text{for all} \quad |x - x^*| < \delta. \tag{3.11}$$

To construct a sequence of high-frequency perturbations of the initial vorticity in W^{1,p_j} consider a smooth bump function $\hat{\chi} \in C_c^\infty(\mathbb{R}^2)$ in the Fourier space with support in the unit ball and such that $0 \leq \hat{\chi} \leq 1$ and $\int_{\mathbb{R}^2} \hat{\chi}(\xi)\, d\xi = 1$. Let $\xi_0 = (2, 0)$ and define

$$\hat{\rho}(\xi) = \hat{\chi}(\xi - \xi_0) + \hat{\chi}(\xi + \xi_0), \qquad \xi \in \mathbb{R}^2.$$

Observe that $\hat{\rho}$ is supported in the union of the unit balls with centers at ξ_0 and $-\xi_0$ and satisfies

$$\rho(0) = \int_{\mathbb{R}^2} \hat{\rho}(\xi)\, d\xi = 2 \tag{3.12}$$

and for any $k \in \mathbb{Z}_+$ and $\lambda > 0$ define

$$\beta_j^{k,\lambda}(x) = \frac{\lambda^{-1+\frac{2}{p_j}}}{\sqrt{k}} \sum_{\varepsilon_1, \varepsilon_2 = \pm 1} \varepsilon_1 \varepsilon_2 \rho(\lambda(x - x_\varepsilon^*)) \sin k x_1 \tag{3.13}$$

where $x_\varepsilon^* = (\varepsilon_1 x_1^*, \varepsilon_2 x_2^*)$.

To proceed we need the following estimates.

Lemma 3.2. *Let $2 < p_j < \infty$, $2 \leq p \leq \infty$ and $\sigma > 0$. For any sufficiently large positive integer k and $\lambda > 0$ we have*

1. $\|\beta_j^{k,\lambda}\|_{W^{1,p_j}} \lesssim \lambda^{-1}\sqrt{k}$
2. $\|\Delta^{\frac{1+\sigma}{2}} \partial_l \Delta^{-1} \beta_j^{k,\lambda}\|_{L^p} \lesssim k^{-\frac{1}{2}} \lambda^{-1+\frac{2}{p_j}-\frac{2}{p}} (\lambda^\sigma + k^\sigma)$
3. $\|\partial_l \Delta^{-1} \beta_j^{k,\lambda}\|_{L^p} \lesssim k^{-\frac{1}{2}} \lambda^{-2+\frac{2}{p_j}-\frac{2}{p}}$

where $l = 1, 2$.

Proof. The first estimate is routine and therefore we focus on the remaining two. It will be convenient to use the Fourier transform

$$\hat{\beta}_j^{k,\lambda}(\xi) = \frac{1}{2i} k^{-\frac{1}{2}} \lambda^{-3+\frac{2}{p_j}} \sum_{\varepsilon_1, \varepsilon_2 = \pm 1} \sum_{m=1}^{2} (-1)^{m+1} \varepsilon_1 \varepsilon_2 \hat{\rho}\big(\lambda^{-1} \xi_m^k\big) e^{-2\pi i \langle x_\varepsilon^*, \xi_m^k \rangle} \tag{3.14}$$

where $\xi_m^k = \big(\xi_1 + \frac{(-1)^m}{2\pi} k, \xi_2\big)$. Let p' be the conjugate Lebesgue exponent of p. Using the Hausdorff-Young inequality we have

$$\begin{aligned}
\left\|\Delta^{\frac{1+\sigma}{2}} \partial_l \Delta^{-1} \beta_j^{k,\lambda}\right\|_{L^p} &\lesssim \left\||\cdot|^\sigma \hat{\beta}_j^{k,\lambda}\right\|_{L^{p'}} \\
&\lesssim k^{-\frac{1}{2}} \lambda^{-3+\frac{2}{p_j}} \sum_{m=1}^{2} \left(\int_{\mathbb{R}^2} |\xi|^{\sigma p'} |\hat{\rho}(\lambda^{-1}\xi_m^k)|^{p'}\, d\xi\right)^{1/p'}
\end{aligned}$$

Changing the variables and estimating further we obtain

$$\lesssim k^{-\frac{1}{2}}\lambda^{-3+\frac{2}{p_j}+\frac{2}{p'}}\sum_{m=1}^{2}\left(\int_{\mathbb{R}^2}\left((\xi_1-\frac{(-1)^m}{2\pi}k)^2+\xi_2^2\right)^{\sigma p'/2}|\hat{\rho}(\lambda^{-1}\xi)|^{p'}d\xi\right)^{1/p'}$$

$$\lesssim k^{-\frac{1}{2}}\lambda^{-1+\frac{2}{p_j}-\frac{2}{p}}\sum_{m=1}^{2}\left(\int_{\mathbb{R}^2}\left((\lambda\xi_1-\frac{(-1)^m}{2\pi}k)^2+(\lambda\xi_2)^2\right)^{\sigma p'/2}|\hat{\rho}(\zeta)|^{p'}d\xi\right)^{1/p'}$$

$$\lesssim k^{-\frac{1}{2}}\lambda^{-1+\frac{2}{p_j}-\frac{2}{p}}\big(\lambda^{\sigma}+k^{\sigma}\big).$$

By the same calculation as above we also have

$$\|\partial_l\Delta^{-1}\beta_j^{k,\lambda}\|_{L^p}\lesssim\||\cdot|^{-1}\hat{\beta}_j^{k,\lambda}\|_{L^{p'}}$$

$$\lesssim k^{-\frac{1}{2}}\lambda^{-1+\frac{2}{p_j}-\frac{2}{p}}\sum_{m=1}^{2}\int_{\operatorname{supp}\{\hat{\rho}\}}\left((\lambda\xi_1-\tfrac{(-1)^m}{2\pi}k)^2+(\lambda\xi_2)^2\right)^{-p'/2}|\hat{\rho}(\xi)|^{p'}\,d\xi$$

$$\lesssim k^{-\frac{1}{2}}\lambda^{-2+\frac{2}{p_j}-\frac{2}{p}}$$

for any sufficiently large k and λ. ∎

Next, setting $\beta_j^n=\beta_j^{k,\lambda}$ where $k=\lambda^2$ and $\lambda=3n$ with $n\gg 1$ and using (3.11) and (3.12) we have

Lemma 3.3. *Let $2<p_j<\infty$, $0<t^*\le M_j^{-3}$ and $n\gg 1$ be as above. Then*

1. $\|\partial_2\beta_j^n\partial_1\eta_j^2(t^*)\|_{L^{p_j}}\lesssim Cn^{-1}$
2. $\|\partial_1\beta_j^n\partial_2\eta_j^2(t^*)\|_{L^{p_j}}\gtrsim M\big(1+\mathcal{O}(n^{-\frac{1}{2}})\big)-Cn^{-1}$

where C depends on $\|\hat{\rho}\|_{L^{p'_j}}$ and $\sup_{0\le t\le 1}\|u_j(t)\|_{C^1}$ and where $\frac{1}{p'_j}+\frac{1}{p_j}=1$.

Proof. Cf. [10]; Lem. 11. ∎

Consider the sequence of initial vorticities $\omega_{0,j}^n(x)=\omega_{0,j}(x)+\beta_j^n(x)$ with $n\ge 1$. By Lemma 2.1 and Lemma 3.2 (part 1) it is clearly in W^{1,p_j} for any $2<p_j<\infty$ and so let $\omega_j^n(t)\in C([0,1],W^{1,p_j}(\mathbb{R}^2))$ be the corresponding solution of the vorticity equations (2.2). Observe that choosing the parameter $0<\sigma<\frac{1}{2}$ ensures that both right hand sides of the expressions in parts 2 and 3 of Lemma 3.2 converge to zero as $n\to\infty$. Furthermore, choosing $p>2/\sigma$ and $0<\alpha<\sigma-2/p$ and using continuity of the solution map in little Hölder spaces in Lemma 2.3 we obtain

$$\sup_{0\le t\le 1}\|\nabla^{\perp}\Delta^{-1}(\omega_j^n(t)-\omega_j(t))\|_{C^1}\lesssim\sup_{0\le t\le 1}\|\nabla^{\perp}\Delta^{-1}(\omega_j^n(t)-\omega_j(t))\|_{1,\alpha}\longrightarrow 0 \tag{3.15}$$

as $n\to\infty$ and therefore standard application of Gronwall's inequality to the flow equation (2.4) gives

$$\theta_n=\sup_{0\le t\le 1}\|\eta_j^n(t)-\eta_j(t)\|_{C^1}\longrightarrow 0\quad\text{as } n\to\infty \tag{3.16}$$

where $\eta_j^n(t)$ is the flow of the velocity field $u_j^n=\nabla^{\perp}\Delta^{-1}\omega_j^n$ corresponding to the initial vorticity $\omega_{0,j}^n$.

Combining (3.16) with the fact that the Lagrangian flows are volume-preserving and using conservation of vorticity we now have

$$\begin{aligned} \|\omega_j^n(t^*)\|_{W^{1,p_j}} &\gtrsim \|d\omega_{0,j}^n(\nabla^\perp \eta_j^{n,2}(t^*))\|_{L^{p_j}} \gtrsim \|d\omega_{0,j}^n(\nabla^\perp \eta_j^2(t^*))\|_{L^{p_j}} - \theta_n \|d\omega_{0,j}^n\|_{L^{p_j}} \\ &\gtrsim \|d\beta_j^n(\nabla^\perp \eta_j^2(t^*))\|_{L^{p_j}} - \|d\omega_{0,j}(\nabla^\perp \eta_j^2(t^*))\|_{L^{p_j}} - \theta_n \|d\omega_{0,j}^n\|_{L^{p_j}}. \end{aligned} \tag{3.17}$$

Finally, it suffices to observe that by the assumption (3.10) the middle term on the right of (3.17) is bounded by

$$\|d\omega_{0,j}(\nabla^\perp \eta_j^2(t^*))\|_{L^{p_j}} \lesssim \|\omega_j(t^*)\|_{W^{1,p_j}} \lesssim M_j^{1/3}$$

while by Lemma 3.3 we have

$$\|d\beta_j^n(\nabla^\perp \eta_j^2(t^*))\|_{L^{p_j}} \gtrsim \|\partial_1 \beta_j^n \partial_2 \eta_j^2(t^*)\|_{L^{p_j}} - \|\partial_2 \beta_j^n \partial_1 \eta_j^2(t^*)\|_{L^{p_j}} \gtrsim M_j$$

for any sufficiently large $n \gg 1$. The proof of Theorem 3.1 is completed. ∎

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References

1. H. Bahouri, J. Chemin and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Springer, New York 2011.
2. J. Bourgain and D. Li, *Strong ill-posedness of the incompressible Euler equations in borderline Sobolev spaces*, to appear in Invent. math.; preprint arXiv:1307.7090 [math.AP].
3. J. Bourgain and D. Li, *Strong illposedness of the incompressible Euler equation in integer C^m spaces*, Geom. funct. anal. **25** (2015), 1-86; preprint arXiv:1405.2847 [math.AP].
4. J. Chemin, *Perfect incompressible fluids*, Oxford University Press, New York 1998.
5. N. Gyunter, *On the motion of a fluid contained in a given moving vessel*, (Russian), Izvestia AN USSR, Sect. Phys. Math. (1926-8).
6. T. Kato and G. Ponce, *Well-posedness of the Euler and Navier-Stokes equations in the Lebesgue spaces $L^p_s(\mathbf{R}^2)$*, Rev. Mat. Iberoamericana **2** (1986), 73-88.
7. T. Kato and G. Ponce, *On nonstationary flows of viscous and ideal fluids in $L^p_s(\mathbb{R}^2)$*, Duke Math. J. **55** (1987), 487-499.
8. L. Lichtenstein, *Uber einige Existenzprobleme der Hydrodynamik unzusamendruckbarer, reibunglosiger Flussigkeiten und die Helmholtzischen Wirbelsatze*, Math. Zeit. **23** (1925), **26** (1927), **28** (1928), **32** (1930).
9. A. Majda and A. Bertozzi, *Vorticity and Incompressible Flow*, Cambridge University Press, Cambridge 2002.
10. G. Misiołek and T. Yoneda, *Local ill-posedness of the incompressible Euler equations in C^1 and $B^1_{\infty,1}$*, to appear in Math. Ann.; preprint arXiv:1405.1943 [math.AP] and arXiv:1405.4933 [math.AP] (2014).
11. G. Misiołek and T. Yoneda, *The solution map of the incompressible Euler equations in Hölder spaces*, preprint (2015).
12. W. Wolibner, *Un theoréme sur l'existence du mouvement plan d'un fluide parfait, homogéne, incompressible, pendant un temps infiniment long*, Math. Z. **37** (1933), 698-726.

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