数理解析研究所講究録
第 1986 巻 2016 年 100-107

Conditions for k-connected graphs to have a contractible edge

Kiyoshi Ando
National Institute of Informatics
JST ERATO Kawarabayashi Large Graph Project
kiyoshiando@nii.ac.jp

Abstract

An edge of k-connected graph is said to be k-contractible if the contraction of it results in a k-connected graph. When $k \leq 3$, each k-connected graph on more then $k+1$ vertices has a k-contractible edge. When $k \geq 4$, there are infinitely many k-connected graphs with no k-contractible edge for each k. Hence, if $k \geq 4$, we can not expect the existence of a k-contractible edge in a k-connected graph with no condition. In this note, we give a brief survey on sufficient conditions for k-connected graphs to have a k-contractible edge. We also give some recent new results on this topic.

1 Introduction

In this note, we deal with finite undirected graphs with neither loops nor multiple edges. For a graph G, let $V(G)$ and $E(G)$ denote the set of vertices of G and the set of edges of G, respectively. For an edge $e \in E(G)$, let $V(e)$ stand for the set of end vertices of e. For a vertex $x \in V(G)$, we write $N_G(x)$ for the neighborhood of x. Moreover, for a subset $X \subset V(G)$, let $N_G(X) = \cup_{x\in X} N_G(x) - X$. We denote the degree of $x \in V(G)$ by $deg_G(x)$, namely $deg_G(x) = |N_G(x)|$. We denote the set of vertices of degree i by $V_i(G)$. We denote the minimum degree of G by $\delta(G)$. For a subset X of $V(G)$, the subgraph induced by X is denoted by $G[X]$. If there is no ambiguity, we write X for $G[X]$. For a subgraph $A \subset G$, if there is no ambiguity, we write A for $V(A)$. Let K_n, P_n and C_n be the complete graph on n vertices, the path on n vertices and the cycle on n vertices, respectively. Let G and H be two graphs. Let $G \cup H$ and $G + H$ denote the union of G and H, and the join of G and H, respectively. Let K_4^- be the graph obtained from K_4 by removing one edge, that is $K_4^- = K_2 + 2K_1$. If G has no H as a subgraph, then G is said to be *H-off.* If G has no H as an induced subgraph, then G is said to be *H-free.* If H is a complete graph, then H-off and H-free are equivalent. For a connected graph G, a subset $S \subset V(G)$ is said to be a *cutset* if $G - S$ is disconnected. A cutset S is said to be an *i-cutset* if $|S| = i$.

Let k be an integer such that $k \geq 2$. Let G be a k-connected graph and let $e = xy$ be an edge of G. We consider the following operation on G. Delete both x and y, and add new vertex z and join z to each vertex in $N_G(x) \cup N_G(y) - \{x, y\}$. We call this operation *contraction* of $e = xy$ and we write the resulting graph by G/e. An edge e of G is said to be *k-contractible* if the contraction of it results in a k-connected graph. If an edge e of G is not contractible, then it is said to be *k-noncontractible*. We observe that an edge $e \in E(G)$ is k-noncontractible if and only if there is a k-cutset S such that $V(e) \subset S$. A k-connected graph G is said to be *contraction critically k-connected* if G has no k-contractible edge. A k-cutset S is said to be *trivial* if there is a vertex $x \in V_k(G)$ such that $S = N_G(x)$.

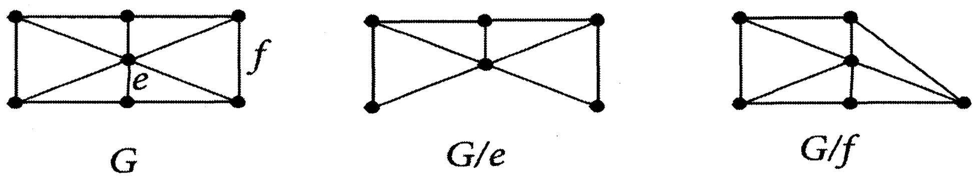

Fig.1: *Contractible edge*

In Fig. 1, G is 3-connected, e is a 3-noncontractible edge of G and f is a 3-contractible edge of G.
If $k \geq 4$, then we cannot expect the existence of a contractible edge in a k-connected graph with no condition. We consider conditions for a k-connected graph to have a k-contractible edge. We say 'k-sufficient condition' for a condition for a k-connected graph to have a k-contractible edge.

2 Degree k-sufficient conditions

A *fragment* A is a non-empty union of components of $G - S$, where S is a k-cutset of G for which $V(G) - (A \cup S) \neq \phi$. Mader [11], [12] showed that every contraction critically k-connected graph has a fragment A whose neighbourhood contains an edge for which $|A| \leq \frac{1}{2}(k-1)$.

By generalizing methods of Mader, Egawa [4] proved that every contraction critically k-connected graph has a fragment A such that $|A| \leq \frac{1}{4}k$, and gave the following minimum-degree k-sufficient condition. He also showed that the bound is sharp.

Theorem 1 *Let $k \geq 2$ be an integer, and let G be a k-connected graph with $\delta(G) \geq \lfloor \frac{5}{4}k \rfloor$. Then G has a k-contractible edge, unless $k = 2$ or 3 and G is isomorphic to K_{k+1}.*

Kriesell [8] extended Theorem 1 and proved the following degree-sum k-sufficient condition.

Theorem 2 *Let G be a k-connected graph for which* $\deg(v)+\deg(w) \geq 2\lfloor\frac{5}{4}k\rfloor - 1$, *for any pair v, w of distinct vertices of G. Then G contains a k-contractible edge.*

Solving a conjecture in [8], Su and Yuan [13] proved the following stronger result.

Theorem 3 *Let G be a contraction-critical k-connected graph with $k \geq 8$. Then G has two adjacent vertices v, w for which* $\deg(v) + \deg(w) \leq 2\lfloor\frac{5}{4}k\rfloor - 2$.

The following interesting result is also due to Kriesell [10].

Theorem 4 *For every $k \geq 1$, there exists a number $f(k)$ for which every k-connected graph with average degree at least $f(k)$ has a k-contractible edge.*

In [10], he showed that $f(k) \leq ck^2 \log k$ for some constant c, and posed the following conjecture.

Conjecture A *There exists a constant c for which every finite k-connected graph with average degree at least ck^2 has a k-contractible edge.*

Another problem is to determine the best value of $f(k)$, for a given value of k. Up to now, we have no 5-contraction-critical graph whose average degree exceeds $\frac{25}{2}$. Hence we pose the following.

Conjecture B *The average degree of every 5-contraction-critical graph is less than $\frac{25}{2}$.*

If Conjecture B is true, the bound $\frac{25}{2}$ is sharp. We construct a series of 5-connected graphs whose average degree tend to $\frac{25}{2}$. To construct the 5-connected graphs, we introduce K_4^--configuration. Let $S = \{a_1, a_2, v, b_1, b_2\}$ be a 5-cutset of a 5-connected graph G, and let A be a component of $G - S$ such that $V(A) \subseteq V_5(G)$, $|V(A)| = 4$, and $G[A] \cong K_4^-$ say, $A = \{x_1, x_2, y_1, y_2\}$, with edges within A and between A and S exactly as in Fig. 2; there may be edges between the vertices of S. We call this configuration, $G[V(A) \cup S]$, a *K_4^--configuration with centre v.* Note that $\{x_1, x_2, y_1, y_2\} \subseteq V_5(G)$ and that the edges in Fig. 2 other than vx_1 and vy_1 are all trivially 5-noncontractible. Moreover, we can find two nontrivial 5-cutsets, $\{x_1, x_2, v, b_1, b_2\}$ and $\{y_1, y_2, v, a_1, a_2\}$ that contain $V(vx_1)$ and $V(vy_1)$, respectively. Hence all edges in Fig. 2 are 5-noncontractible. Note finally that if there is an edge between vertices of S, then it is 5-noncontractible, since S is a 5-cutset of G.

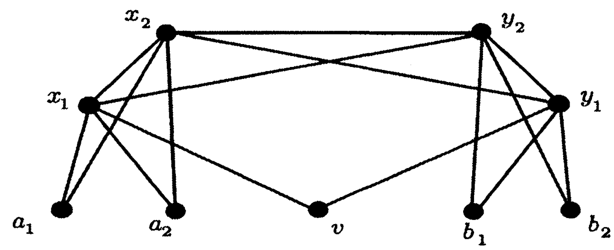

Fig. 2. A K_4^--configuration with centre x

Let ℓ be a suitable integer which is divisible by 10. Then,. by the results due to Hanani(1961), we can find $\ell(\ell-1)/20$ many copies of K_5's whose edge sets partitions $E(K_\ell)$. We apply K_4^--attaching on each K_5 and let G_ℓ be the resulting graph. Then G_ℓ is contraction-critically 5-connected. Let ξ denote the number of K_4^--attachings in G_ℓ. Then we observe that $\xi = \frac{1}{10}\frac{\ell(\ell-1)}{2} \approx \frac{\ell^2}{20}$ and $|V(G_\ell)| = 4 \times \xi + \ell \approx \frac{\ell^2}{5}$. Moreover we observe that $|E(G_\ell)| = |E(K_\ell)| + 15 \times \xi \approx \frac{5\ell^2}{4}$. Hence we have $\lim_{\ell\to\infty}$ (Avrage degree of G_ℓ) $= \lim_{\ell\to\infty} \frac{2|E(G_\ell)|}{|V(G_\ell)|} = \frac{25}{2}$. This series of graphs shows that $f(5)$ does not exceed the value $\frac{25}{2}$.

3 Forbidden-subgraph k-sufficient conditions

From Mader's result that every contraction critically k-connected graph has a fragment A whose neighbourhood contains an edge for which $|A| \leq \frac{1}{2}(k-1)$, we can see that 'triangle-free' is a k-sufficient condition; Thomassen [14] pointed out this condition.

Theorem 5 *Every k-connected triangle-free graph has a k-contractible edge.*

The following result due to Egawa, Enomoto and Saito [5] gives a bound on the number of k-contractible edges in a k-connected triangle-free graph.

Theorem 6 *Every k-connected triangle-free graph with n vertices and m edges contains* $\min\{n + \frac{3}{2}k^2 - 3k, m\}$ *k-contractible edges.*

From Theorem 5 we know that every contraction critically k-connected graph has triangles. The following result due to Kriesell [9] is a substantial improvement on a bound $\frac{1}{3}n$ found by Mader [12].

Theorem 7 *Let G be a k-connected graph of order n with no contractible edges. Then G contains at least $\frac{2}{3}n$ triangles.*

In view of Theorem 6, a k-connected 'triangle-free' graph has many k-contractible edges, indicating the possible existence of a weaker k-sufficient condition involving forbidden subgraphs.

In this direction, Kawarabayashi [7] showed the following.

Theorem 8 *For an odd integer $k \geq 3$, every $K_1 + (K_2 \cup P_3)$-off k-connected graph has a k-contractible edge.*

Since $K_1 + (K_2 \cup P_3)$ contains K_3, Theorem 8 is an extension of Theorem 5 when k is odd.

Recall $K_4^- = K_2 + 2K_1$. We call the graph $K_1 + 2K_2$ a *bowtie*.

Ando, Kaneko, Kawarabayashi and Yoshimoto [2] proved that every k-connected bowtie-off graph has a k-contractible edge.

Theorem 9 *Every k-connected bowtie-off graph has a k-contractible edge.*

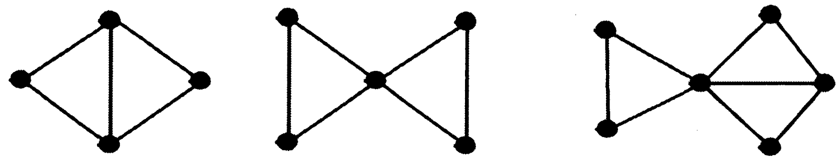

Fig. 3. K_4^-, bowtie and $K_1 + (K_2 \cup P_3)$

Theorem 9 is also an extension of Theorem 5.

Since $K_1 + P_4$ contains a bowtie, the following Theorem 10 is an extension of Theorem 9 (see [6]).

Theorem 10 *Let $k \geq 5$, and let G be a k-connected $(K_1 + P_4)$-off graph. If $G[V_k(G)]$ is bowtie-off, then G has a k-contractible edge.*

The following Theorem 11 is another extension of Theorem 5 (see [3]). Note that if $s = t = 1$, then Theorem 11 is equivalent to Theorem 5. Also note that '$K_4^- \cong K_2 + 2K_1$' and 'bowtie $\cong K_1 + 2K_2$'; hence $K_2 + sK_1$ and $K_1 + tK_2$ may be regarded as a generalized K_4^- and a generalized bowtie, respectively.

Theorem 11 *For $k \geq 5$, take two positive integers s and t with $s(t-1) < k$. If a k-connected graph G contains neither $K_2 + sK_1$ nor $K_1 + tK_2$, then G contains a k-contractible edge.*

We cannot replace the condition $s(t-1) < k$ by $s(t-1) \leq k$ in Theorem 11.

4 Some k-sufficient conditions involving degree and forbidden-subgraph

Theorems 5, 10 and 11 deal with forbidden-subgraph k-sufficient conditions. On the other hand, Theorem 1 gives a minimum-degree k-sufficient condition. However, if we restrict ourselves to a class of graphs that satisfy some forbidden-subgraph conditions, then we may relax the minimum-degree bound in Theorem 1. The following forbidden-subgraph condition relaxes the minimum-degree bound (see [3]). Let K_5^- be the graph obtained from K_5 by removing one edge.

Theorem 12 *For $k \geq 5$, let G be a k-connected graph which contains neither K_5^- nor $5K_1 + P_3$. If $\delta(G) \geq k+1$, then G has a k-contractible edge.*

Note that if $k \geq 5$, then $\lfloor \frac{5}{4}k \rfloor \geq k+1$. Since there is a k-regular contraction critically k-connected graph which contains neither K_5^- nor $5K_1+P_3$, we cannot replace $\delta(G) \geq k+1$ by $\delta(G) \geq k$ in Theorem 12. In this sense, the minimum-degree bound in Theorem 12 is sharp.

Yang and Sun [15] present the following as a counter part of a Kawarabyashi's result.

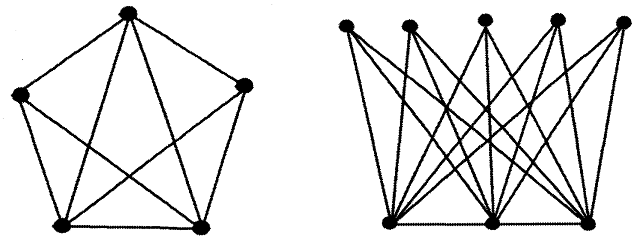

Fig. 4. K_5^- and $5K_1 + P_3$

Theorem 13 *Let G be a K_4^--off k-connected graph with $k \geq 5$. If $V_k(G)$ is independent, then G has a k-contractible edge.*

The degree condition of Theorem 13, '$V_k(G)$ is independent' is equivalent to the condition that '$d_G(W) \geq 2k+1$ for any connected subgraph W of G with $|W| = 2$'. Here we consider the more general degree condition that '$d_G(W) \geq f(k)$ for any connected subgraph W of G with $|W| = m$'.

Using a condition of this type, we have the following (see [1]).

Theorem 14 *Let G be a k-connected graph with $k \geq 5$ having neither $K_1 + C_4$ nor $K_2 + (K_1 \cup K_2)$. If $\deg_G(W) \geq 3k+1$ for any connected subgraph W of G with $|W| = 3$, then G has a k-contractible edge.*

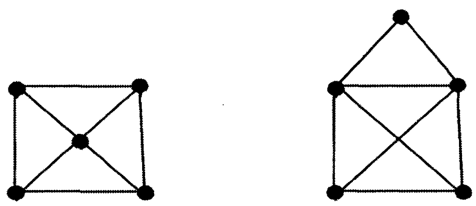

Fig. 5. $K_1 + C_4$, bowtie and $K_2 + (K_1 \cup K_2)$

Note that both $K_1 + C_4$ and $K_2 + (K_1 \cup K_2)$ contains K_4^-. And the degree condition '$\deg_G(W) \geq 3k+1$ for any connected subgraph W of G with $|W| = 3$' is much weaker than the degree condition that '$V_k(G)$ is independent'. Hence Theorem 14 extends both degree and forbidden-subgraph conditions of Theorem 13.

Yang and Sun [16] also present the following result.

Theorem 15 *Let k be an integer such that $k \geq 5$, and let G be a (K_1+C_4)-off k-connected graph. If $d_G(W) \geq 2k+2$ for any connected subgraph W of G with $|W| = 2$, then G has a k-contractible edge.*

We think that we can relax the degree bound of Theorem 15 by one.

Conjecture C *Let k be an integer such that $k \geq 5$, and let G be a (K_1+C_4)-off k-connected graph. If $d_G(W) \geq 2k+1$ for any connected subgraph W of G with $|W| = 2$, then G has a k-contractible edge.*

The conjecture C is still open, however we obtain the following weaker result, which is an extension of Theorem 15.

Theorem 16 *Let k be an integer such that $k \geq 5$, and let G be a (K_1+C_4)-off k-connected graph. If $d_G(W) \geq 3k+2$ for any connected subgraph W of G with $|W| = 3$, then G has a k-contractible edge.*

References

[1] K. Ando, Some degree and forbidden subgraph conditions for a graph to have a k-contractible edg, *Discrete Math.* in print.

[2] K. Ando, A. Kaneko, K. Kawarabayashi and K. Yoshimoto, Contractible edges and bowties in a k-connected graph, *Ars Combin.* **64** (2002), 239–247.

[3] K. Ando and K. Kawarabayashi, Some forbidden subgraph conditions for a graph to have a k-contractible edge, *Discrete Math.* **267** (2003), 3–11.

[4] Y. Egawa, Contractible edges in n-connected graphs with minimum degree greater than or equal to $\frac{5n}{4}$, *Graphs Combin.* **7** (1991), 15–21.

[5] Y. Egawa, H. Enomoto and A. Saito, Contractible edges in triangle-free graphs, *Combinatorica* **6** (1986), 269–274.

[6] K. Kawarabayashi, Note on k-contractible edges in k-connected graphs, *Australas. J. Combin.* **24** (2001), 165–168.

[7] K. Kawarabayashi, Contractible edges and triangles in k-connected graphs, *J. Combin. Theory (B)* **85** (2002), 207–221.

[8] M. Kriesell, A degree sum condition for the existence of a contractible edge in a κ-connected graph, *J. Combin. Theory (B)* **82** (2001), 81–101.

[9] M. Kriesell, Triangle density and contractibility, *Combin. Probab. Comput.* **14** (2005), 133–146.

[10] M. Kriesell, Average degree and contractibility, *J. Graph Theory* **51** (2006), 205–224.

[11] W. Mader, Disjunkte Fragmente in kritisch n-fach zusammenhängende Graphen, *Europ. J. Combin.* **6** (1985), 353–359.

[12] W. Mader, Generalizations of critical connectivity of graphs, *Discrete Math.* **72** (1988), 267–283.

[13] J. Su and X. Yuan, A new degree sum condition for the existence of a contractible edge in a κ-connected graph, *J. Combin. Theory (B)* **96** (2006), 276–295.

[14] C. Thomassen, Non-separating cycles in k-connected graphs, *J. Graph Theory* **5** (1981), 351–354.

[15] Y. Yang and L. Sun, Contractible Edges in k-Connected Graphs with Some Forbidden Subgraphs, *Graphs Combin.* **30** (2014), 1607–1614.

[16] Y. Yang and L. Sun, preprint.