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NOTES ON ELEMENTARY SUBMODEL SPACES

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1. Introduction

Throughout this note, θ will denote a sufficiently large regular cardinal. $\mathcal{H}_\theta$ will denote the set of all sets x with hereditariry cardinality $< \theta$. $X = \langle X, \tau \rangle$ will denote a topological space, and M an elementary submodel of $\mathcal{H}_\theta$ with $X \in M$.

The elementary submodel space was introduced and studied in Junqueira-Tall [1].

Definition 1.1. For X and M, the *elementary submodel space* X_M is the space $X \cap M$ with topology generated by the family $\{O \cap M : O \in \tau\}$.

The elementary submodel spaces reflect many properties of the original spaces and vise verse, for instance:

Fact 1.2 ([1]). *(1) $(n = 0, 1, 2, 3, 3\frac{1}{2})$ X is T_n $\iff$ X_M is T_n.*
(2) If X_M is compact then X is compact.

In [2], Tall showed that if X_M satisfies some $\mathbb{R}$-like properties, then X_M must be the same to the original space X:

Theorem 1.3 (Tall [2]). *If X_M is locally compact, metrizable, separable, and uncountable, then $X_M = X$. In particular if X_M is homeomorphic to the real line $\mathbb{R}$, then $X_M = X$.*

Among this result, Tall asked the following question:

Question 1.4 (An irrational problem). *Suppose X_M is completely metrizable, separable, and uncountable. Does $X_M = X$? Or, if X_M is homeomorphic to the irrationals $\mathbb{R} \setminus \mathbb{Q}$, does $X_M = X$?*

In [2] and [3], Tall gave some partial answers of an irrational problem, and it turned out that if $0^\#$ does not exist, then the affirmative answer of an irrational problem holds, so it is consistent with ZFC. However it is unknown whether the negative answer of an irrational question is consistent with ZFC. In this notes, we will give another partial answer of an irrational problem under certain cardinal arithmetic assumption.

Theorem 1.5. *Suppose $2^\omega > \omega_1$ and $2^\kappa = \kappa^+$ for every cardinal $\kappa \geq 2^\omega$. For every space X, if X_M is completely metrizable, separable, and uncountable, then $X = X_M$.*

2. Proofs

Let X be a topological space. Recall that a sequence $\langle x_i : i < \delta\rangle$ in X is *left-separated* (*right-separated*, respectively) if there is a sequence $\langle O_i : i < \delta\rangle$ of open sets such that $x_i \in O_i$ but $x_j \notin O_i$ for every $j < i$ ($j > i$, respectively). The following is well-known:

Fact 2.1. *Let κ be a regular cardinal.*

(1) X has a left-separated sequence of length κ if and only if there is a subset Y of X such that Y has no subset of size $< \kappa$ which is dense in Y.

(2) X has a right-separated sequence of length κ if and only if there are a subset Y of X and a family $\mathcal{U}$ of open sets such that $\mathcal{U}$ covers Y but every subfamily of $\mathcal{U}$ of size $< \kappa$ does not cover Y.

The following lemmas may be folklores, but we prove it for the reader's convenience.

Lemma 2.2. *Let κ be a regular cardinal. Let X be a topological space and suppose that :*

(1) For every subset Y and open cover $\mathcal{U}$ of Y, there is a subcover of $\mathcal{U}$ of size $< \kappa$ which covers Y, and

(2) For every point $x \in X$, there is a family $\mathcal{U}$ of open sets such that $|\mathcal{U}| < \kappa$ and $\{x\} = \bigcap \mathcal{U}$.

Then $|X| \leq 2^{<\kappa}$.

Proof. Take $M \prec \mathcal{H}_\theta$ such that $\kappa, X \in M$, $|M| = 2^{<\kappa}$, $\kappa \subseteq M$, and ${}^{<\kappa}M \subseteq M$. It is enough to see that $X \subseteq M$.

Suppose, to the contrary, that $X \not\subseteq M$. Fix $x^* \in X \setminus M$. For $y \in X \cap M$, take a family of open sets $\mathcal{U}_y \in M$ with $\{y\} = \bigcap \mathcal{U}_y$ and $|\mathcal{U}_y| < \kappa$. Note that $\mathcal{U}_y \subseteq M$. Then we can find $O_y \in \mathcal{U}_y \cap M$ with $x^* \notin O_y$. The family $\{O_y : y \in X \cap M\}$ is an open cover of $X \cap M$. By our assumption, there is $\mathcal{U}' \subseteq \{O_y : y \in X \cap M\}$ such that $|\mathcal{U}'| < \kappa$ and $\mathcal{U}'$ covers $X \cap M$. Since $\mathcal{U}' \subseteq M$, we have $\mathcal{U}' \in M$. Then, by the elementarity of M, $\mathcal{U}'$ covers the whole of X. This contradicts that $x^* \notin O$ for every $O \in \mathcal{U}'$. □

Lemma 2.3. *Let κ be a regular cardinal. Let X be a T_2-space such that for every subset Y and open cover $\mathcal{U}$ of Y, there is a subcover of $\mathcal{U}$ of size $< \kappa$ which covers Y. Then $|X| \leq 2^{<\kappa}$.*

Proof. By Lemma 2.2, it is enough to see that for every $x \in X$, there is a family $\mathcal{U}$ of open sets of such that $|\mathcal{U}| < \kappa$ and $\{x\} = \bigcap \mathcal{U}$.

Let $x \in X$ and suppose to the contrary that $\{x\} \neq \bigcap \mathcal{U}$ for every $\mathcal{U}$ with size $< \kappa$. Let $\mathcal{U}_0$ be the family of all open neighborhoods of x and $\mathcal{F} = \{\overline{O} \setminus \{x\} : O \in \mathcal{U}_0\}$. By our assumption, $\bigcap \mathcal{F}' \neq \emptyset$ for every $\mathcal{F}' \subseteq \mathcal{F}$ of size $< \kappa$. Since every open cover of $X \setminus \{x\}$ has a subcover of size $< \kappa$, we have that $\bigcap \mathcal{F} \neq \emptyset$. On the other hand, since X is T_2, for each $y \in X$ with $y \neq x$, there is an open neighborhood O of x with $y \notin \overline{O}$. Thus $\bigcap \mathcal{F} = \emptyset$, a contradiction. □

Lemma 2.4. *Suppose X_M is uncountable, second countable, and T_2. Let $\delta \in M$ be the ordinal with* $\mathrm{ot}(M \cap \delta) = \omega_1$ *(note that $|M \cap |X|| \geq \omega_1$, hence such a δ must exist). Then δ is regular uncountable, $\delta \leq |X|$, and for every subset Y of X, the following hold:*

(1) Y has a subset which is dense in Y and of size $< \delta$, and
(2) every open cover of Y has a subcover of size $< \delta$.

Hence $|X| \leq 2^{<\delta}$. Moreover if X_M is T_3 and $2^{<\delta} = \delta$, then X has an open base of size $< \delta$.

Proof. First suppose to the contrary that $\mathrm{cf}(\delta) < \delta$. Then there is a cofinal map $f : \mathrm{cf}(\delta) \to \delta$ with $f \in M$. By the elementarity of M, $f \restriction (\mathrm{cf}(\delta) \cap M)$ is a cofinal map from $M \cap \mathrm{cf}(\delta)$ to $M \cap \delta$. Since $\mathrm{cf}(\delta) < \delta$, we have $\mathrm{ot}(M \cap \mathrm{cf}(\delta)) < \mathrm{ot}(M \cap \delta) = \omega_1$. Thus $f``(M \cap \mathrm{cf}(\delta))$ must be bounded in $\sup(M \cap \delta)$, this is a contradiction.

For (1), suppose X has a subset Y which has no dense subset of size $< \delta$. By Fact 2.1, we can find a left-separated sequence $\langle x_i : i < \delta \rangle$ in X. Let $\langle O_i : i < \delta \rangle$ be a sequence of open sets which witnesses the left-separatedness of $\langle x_i : i < \delta \rangle$. By the elementarity of M, we may assume $\langle x_i : i < \delta \rangle, \langle O_i : i < \delta \rangle \in M$. Then $\langle O_i \cap M : i \in M \cap \delta \rangle$ witnesses that $\langle x_i : i \in M \cap \delta \rangle$ is left-separated in X_M. Since $\mathrm{ot}(M \cap \delta) = \omega_1$, $\langle x_i : i \in M \cap \delta \rangle$ witnesses that X_M has a left separated sequence of length ω_1, so X_M is not hereditarily separable, this contradicts to the second countability of X_M. (2) follows from a similar argument.

Now suppose $2^{<\delta} = \delta$. We see that X has an open base of size $< \delta$. Since X_M is T_3, so is X. Since X has a dense subset of size $< \delta$, X has an open base of size $\leq 2^{<\delta} = \delta$. Now fix an open base $\mathcal{B} = \{O_i : i < \delta\}$ with $\mathcal{B} \in M$. We see that $\{O_i : i < \gamma\}$ is a base for some $\gamma < \delta$. By the definition of the topology of X_M, $\mathcal{B}_M = \{O_i \cap M : i \in M \cap \delta\}$ is an open base for X_M. Since X_M is second countable, $\mathcal{B}_M$ has a countable subset $\mathcal{B}'$ which is a base for X_M. On the other hand, since $\mathrm{ot}(M \cap \delta) = \omega_1$, there is $\gamma \in M \cap \delta$ such that $\mathcal{B}' \subseteq \{O_i \cap M : i \in M \cap \gamma\}$. We see that $\{O_i : i < \gamma\}$ is an open base for X. Suppose otherwise. Since $\{O_i : i < \gamma\} \in M$, there is $x \in X \cap M$ and an open $O \in M$ such that there is no $i < \gamma$ with $x \in O_i \subseteq O$.

$\mathcal{B}' \subseteq \{O_i \cap M : i \in M \cap \gamma\}$, hence there is $i \in M \cap \gamma$ with $x \in O_i \cap M \subseteq O \cap M$. Then $x \in O_i \subseteq O$ by the elementarity of M, this is a contradiction. □

We use the following fact:

Fact 2.5 (Tall [3]). *Suppose X_M is completely metrizable, separable, and uncountable. If $|X| \leq 2^\omega$ or $|\mathbb{R} \cap M|$ is uncountable, then $X_M = X$.*

Proposition 2.6. *Suppose $2^\omega > \omega_1$ and $2^{<\kappa} = \kappa$ for every regular cardinal κ with $2^\omega < \kappa < |X|$. If X_M is completely metrizable, separable, and uncountable, then $X = X_M$.*

Proof. If $|X| \leq 2^\omega$, we are done by Fact 2.5. Suppose $|X| > 2^\omega$. Let $\kappa \in M$ be such that $\mathrm{ot}(M \cap \kappa) = \omega_1$. κ is regular uncountable by Lemma 2.4, and since $|M \cap |X|| = |X_M| = 2^\omega > \omega_1 = |M \cap \kappa|$, we have $\kappa < |X|$.

If $\kappa > 2^\omega$, then $2^{<\kappa} = \kappa$ by our assumption. So $|X| \leq 2^{<\kappa} = \kappa$ by Lemma 2.4, but this is a contradiction. Thus we have $\kappa \leq 2^\omega$. Then $|M \cap 2^\omega| \geq |M \cap \kappa| = \omega_1$. Thus $|M \cap 2^\omega| = |M \cap \mathbb{R}|$ is uncountable, so we have $X_M = X$ by Fact 2.5 again. □

Now we have the following conclusion.

Corollary 2.7. *(1) Suppose $2^\omega > \omega_1$ and $2^\kappa = \kappa^+$ for every cardinal $\kappa \geq 2^\omega$. Then for every X, if X_M is completely metrizable, separable, and uncountable, then $X_M = X$.*

(2) Suppose $2^\omega > \omega_1$. For every X, if X_M is completely metrizable, separable, and uncountable, but $X_M \neq X$, then $|X| > (2^\omega)^+$.

This corollary shows that the affirmative answer of an irrational problem is consistent with Martin's Maximum: A standard argument shows that we can construct a model of ZFC in which Martin's Maximum holds, $2^\omega = \omega_2$, and $2^\kappa = \kappa^+$ for every cardinal $\kappa \geq \omega_2$. In this model, we have that for every space X, if X_M is completely metrizable, separable, and uncountable, then $X = X_M$.

As mentioned before, Tall showed that if $0^\#$ does not exist, then the affirmative answer of an irrational problem holds ([2]). The previous observation also shows that the affirmative answer of an irrational problem can hold even if $0^\#$ exists (or more strong large cardinal properties hold).

Under GCH, we also have the following:

Corollary 2.8. *Suppose GCH. Suppose X_M is completely metrizable, separable, and uncountable. Let $\kappa \in M$ be such that $\mathrm{ot}(M \cap \kappa) = \omega_1$. Then $w(X) < |X| = \kappa$. In particular, $|X|$ must be regular.*

Proof. We have $\kappa \leq |X| \leq 2^{<\kappa} = \kappa$ by Lemma 2.4, and $w(X) < \kappa$ by Lemma 2.4 again. □

References

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