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THE ITERATED REMAINDERS OF THE RATIONALS

AKIO KATO

ABSTRACT. Repeat taking remainders of Stone-Čech compactifications of the rationals

\[ Q^{(1)} = Q^* = \beta Q \setminus Q, \quad Q^{(2)} = \beta Q^{(1)} \setminus Q^{(1)}, \quad Q^{(3)} = \beta Q^{(2)} \setminus Q^{(2)}, \quad Q^{(4)} \ldots . \]

We point out that they have similar structures, but, are topologically different. In particular we prove here that \( Q^{(1)} \not\approx Q^{(3)} \). This result will be generalized to show that \( Q^{(n)} \not\approx Q^{(n+2)} \) for any \( n \geq 1 \) in the forthcoming paper [4].

1. INTRODUCTION

Consider the space of rationals \( Q \), and repeat taking its remainders of Stone-Čech compactifications \( Q^{(n+1)} = (Q^{(n)})^* = \beta Q^{(n)} \setminus Q^{(n)} (n \geq 0) \) where \( Q^{(0)} = Q \), i.e.,

\[ Q^{(1)} = Q^*, \quad Q^{(2)} = Q^{**}, \quad Q^{(3)} = Q^{***}, \ldots . \]

Van Douwen [2] asked whether or not \( Q^{(n)} \approx Q^{(n+2)} \) for \( n \geq 1 \), remarking that \( Q^{(m)} \) for even \( m \) is never homeomorphic to \( Q^{(n)} \) for odd \( n \), because the former is \( \sigma \)-compact but the latter is not.

In this paper we point out that both \( Q^{(n)} \) and \( Q^{(n+2)} \) have a similar structure of “fiber bundle” for every \( n \geq 1 \), but they are topologically different. In particular we here show that \( Q^{(1)} \not\approx Q^{(3)} \), which we can generalize in the forthcoming paper [4] to show that \( Q^{(n)} \not\approx Q^{(n+2)} \) for any \( n \geq 1 \), answering van Douwen’s question.

The precise connections of the remainders can be seen by the following construction. Viewing \( \beta Q \) as a compactification of \( Q^{(1)} \), let

\[ \Phi_0 : \beta Q^{(1)} = Q^{(1)} \cup Q^{(2)} \to Q \cup Q^{(1)} = \beta Q \]

be the Stone extension of the identity map \( id : Q^{(1)} \to Q^{(1)} \). Denote by

\[ \phi_0 : Q^{(2)} \to Q^{(0)} \]

the restriction of \( \Phi_0 \). Next let

\[ \Phi_1 : \beta Q^{(2)} = Q^{(2)} \cup Q^{(3)} \to Q^{(1)} \cup Q^{(2)} = \beta Q^{(1)} \]

be the Stone extension of the identity map \( id : Q^{(2)} \to Q^{(2)} \), and let

\[ \phi_1 : Q^{(3)} \to Q^{(1)} \]

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denote the restriction of $\Phi_1$. In this way, for every $n \geq 0$ we can generally get the Stone extension

$$\Phi_n : \beta \mathbb{Q}^{(n+1)} = \mathbb{Q}^{(n+1)} \cup \mathbb{Q}^{(n+2)} \rightarrow \mathbb{Q}^{(n)} \cup \mathbb{Q}^{(n+1)} = \beta \mathbb{Q}^{(n)}$$

of the identity map $id : \mathbb{Q}^{(n+1)} \rightarrow \mathbb{Q}^{(n+1)}$, and its restriction map

$$\phi_n : \mathbb{Q}^{(n+2)} \rightarrow \mathbb{Q}^{(n)}.$$ 

Since every $\Phi_n$ $(n \in \omega)$ is perfect, so is every $\phi_n$. Hence every $\mathbb{Q}^{(n)}$ $(n \in \omega)$ is Lindelöf since both $\mathbb{Q}^{(0)} = \mathbb{Q}$, $\mathbb{Q}^{(1)}$ are Lindelöf. We can also see that $\mathbb{Q}^{(n)}$ is $\sigma$-compact for even $n$, but $\mathbb{Q}^{(n)}$ is not for odd $n$, because $\mathbb{Q}^{(0)}$ is $\sigma$-compact but $\mathbb{Q}^{(1)}$ is not since $\mathbb{Q}^{(1)}$ is a perfect pre-image of the irrationals $\mathbb{P}$ as we see below.

\[\begin{array}{cccc}
\beta \mathbb{Q}^{(0)} & \Phi_0 & \beta \mathbb{Q}^{(1)} & \Phi_1 \\
Q^{(1)} & \Phi_1 & Q^{(1)} & Q^{(2)} \Phi_2 & Q^{(3)} & \ldots \\
Q^{(0)} & \phi_0 & Q^{(2)} & \ldots & \phi_2 & Q^{(4)}
\end{array}\]

FIG. 1

A collection $B$ of nonempty open sets of $X$ is called a $\pi$-base for $X$ if every nonempty open set in $X$ includes some member of $B$. The minimal cardinality of such a $\pi$-base is called the $\pi$-weight of $X$. Note that any dense subspace of $X$ has the same $\pi$-weight as $X$, and any space of countable $\pi$-weight is separable. Consequently, any dense subset of a space of countable $\pi$-weight is also of countable $\pi$-weight, and hence separable. So, all of $\beta \mathbb{Q}^{(n)}$, $\mathbb{Q}^{(n)}$ $(n \in \omega)$ are of countable $\pi$-weight, and hence separable.

Recall that an onto map $g : X \rightarrow Y$ is called irreducible if every nonempty open subset $U$ of $X$ includes some fiber $g^{-1}(y)$, and it is well known and easy to see that

(1) every extension of a homeomorphism is irreducible, and

(2) the restriction of a closed irreducible map to any dense subset is irreducible.
Therefore we can see that all of the maps $\Phi_n, \phi_n \ (n \in \omega)$ are perfect irreducible. Consider the partition of the closed interval $[0, 1] = Q \cup P$ where

$$Q = [0, 1] \cap \mathbb{Q} \approx \mathbb{Q} \quad \text{and} \quad P = [0, 1] \setminus \mathbb{Q} \approx \mathbb{P},$$

and let $f : \beta \mathbb{Q} \rightarrow [0, 1]$ be the Stone extension of the homeomorphism $\mathbb{Q} \approx \mathbb{Q}$. Then the restriction $f_0 = f \mid \mathbb{Q}(1) : \mathbb{Q}(1) \rightarrow P \approx \mathbb{P}$ is perfect irreducible. Thus we get the following sequence of perfect irreducible maps:

$$Q \leftarrow \mathbb{Q}(2) \leftarrow \mathbb{Q}(4) \leftarrow \cdots ; \mathbb{P} \leftarrow \mathbb{Q}(1) \leftarrow \mathbb{Q}(3) \leftarrow \mathbb{Q}(5) \leftarrow \cdots.$$ 

All spaces are assumed to be completely regular and Hausdorff, and maps are always continuous, unless otherwise stated. “Partition” is synonymous with “disjoint union.” For a subset $A$ of some compact space $K$ we use the notation $A^*$ to denote the remainder $\text{cl}_K A \setminus A$ when $K$ is clear from the context. Our terminologies are based upon [3].

2. Similar Structures

We first show that both $\mathbb{Q}^{(n)}$ and $\mathbb{Q}^{(n+2)}$ have a similar structure for every $n \geq 1$. In general, for any space $Y$ let us denote by $H(Y)$ the collection of all homeomorphisms $h : Y \approx Y$. Let $X$ be a nowhere compact, dense-in-itself space, where nowhere compact (or nowhere locally compact) means that $X$ contains no compact neighborhood, or equivalently, that $X$ is a dense subset of some/any compact space $K$ such that the remainder $K \setminus X$ is also dense in $K$. Let $cX$ be some compactification of $X$ and let $\mathcal{H}_* \subseteq H(X)$ denote the collection of all $h \in H(X)$ such that

\[(*) \quad h \text{ is extendable to } c(h) \in H(cX).\]

(Of course, $\mathcal{H}_* = H(cX)$ if $cX = \beta X$.) Let $X^{(1)} = cX \setminus X$ be the remainder, and for every $h \in \mathcal{H}_*$ define $h^{(1)} \in H(X^{(1)})$ to be the restriction of $c(h)$ to $X^{(1)}$. Next consider the Stone-Čech compactification $\beta X^{(1)}$ of $X^{(1)}$ and the Stone extension $\beta h^{(1)} \in H(\beta X^{(1)})$ of $h^{(1)}$. Let $X^{(2)} = \beta X^{(1)} \setminus X^{(1)}$ be the remainder, and define $h^{(2)} \in H(X^{(2)})$ to be the restriction of $\beta h^{(1)}$ to the remainder $X^{(2)}$; hence

$$h : X \approx X, \quad h^{(1)} : X^{(1)} \approx X^{(1)}, \quad h^{(2)} : X^{(2)} \approx X^{(2)}.$$ 

Note that $X^{(1)}$ is dense in $\beta X$, and $X^{(2)}$ is dense in $\beta X^{(1)}$, since we assume that $X$ is nowhere compact. Viewing that $\beta X$ is a compactification of $X^{(1)}$, we can consider the Stone extension $\Phi : \beta X^{(1)} \rightarrow \beta X$ of the identity map $id_{X^{(1)}} : X^{(1)} \approx X^{(1)}$. Let $\phi : X^{(2)} \rightarrow X$ be the restriction of $\Phi$. Then both $\Phi$ and $\phi$ are perfect irreducible maps. We can show that the correspondence $H(X) \supseteq \mathcal{H}_* \ni h \mapsto h^{(2)} \in H(X^{(2)})$ is compatible with the perfect irreducible map $\phi$, i.e.,

Lemma 2.1. \quad $h \circ \phi = \phi \circ h^{(2)} : X^{(2)} \rightarrow X$. 

Proof. To show this equality, it suffices to prove the equality
\[
c(h) \circ \Phi = \Phi \circ \beta h^{(1)} : \beta X^{(1)} \to c X,
\]
which follows from the obvious equality
\[
h^{(1)} \circ \text{id}_{X^{(1)}} = \text{id}_{X^{(1)}} \circ h^{(1)} : X^{(1)} \to X^{(1)}
\]
on the dense subset $X^{(1)}$ of $\beta X^{(1)}$. \qed

Corollary 2.2. If $h(x) = y$ for $x, y \in X$, then $h^{(2)}(\phi^{-1}(x)) = \phi^{-1}(y)$.

Proof. The inclusion $h^{(2)}(\phi^{-1}(x)) \subseteq \phi^{-1}(y)$ follows from 2.1. Since $h$ is a homeomorphism, we can replace $h$ by $h^{-1}$ to get the reverse inclusion. \qed

Taking $X = \mathbb{Q}$, $c X = \beta \mathbb{Q}$, $\mathcal{H}_* = H(\mathbb{Q})$, we can deduce from 2.1 that
\[
(2-1) \quad h \circ \phi_0 = \phi_0 \circ h^{(2)} : \mathbb{Q} \to \mathbb{Q} \quad \text{for every} \ h \in \mathcal{H}(\mathbb{Q}).
\]
Let $[0, 1] = Q \cup P$, $Q \approx \mathbb{Q}$, $P \approx \mathbb{P}$ be as at the end of §1, and take $X = P$, $c X = [0, 1]$; then $X^{(1)} = Q$, $X^{(2)} = Q^{(1)}$, and the corresponding map $\phi$ in Fig. 2 is identical to the map $f_0 : Q^{(1)} \to P$ at the end of §1. Note that $\mathcal{H}_* \subseteq H(P)$ is the collection of all homeomorphisms of $P$ extendable to homeomorphisms of $[0, 1]$. Then we can deduce from 2.1 that
\[
(2-2) \quad h \circ f_0 = f_0 \circ h^{(2)} : Q^{(1)} \to P \quad \text{for every} \ h \in \mathcal{H}_*.
\]
Note that for every pair of irrationals $p_1 < p_2$ in $P = [0, 1]\setminus \mathbb{Q}$ we can find an $h \in \mathcal{H}_*$ such that $h(p_1) = p_2$; for example, we can take as $c(h)$ in (*) a strictly increasing function $c(h) : [0, 1] \to [0, 1]$ such that $c(h)(Q) = Q$, $c(h)(0) = 0$, $c(h)(p_1) = p_2$, $c(h)(1) = 1$. For $m \geq 1$ define $g_{2m}$ and $f_{2m-1}$ by
\[
g_{2m} = \phi_0 \circ \phi_2 \circ \cdots \circ \phi_{2m-2} : \mathbb{Q}^{(2m)} \to \mathbb{Q},
\]
\[
f_{2m-1} = f_0 \circ \phi_1 \circ \phi_3 \circ \cdots \circ \phi_{2m-3} : \mathbb{Q}^{(2m-1)} \to P.
\]
Then, using 2.1 we can extend the above (2-1), (2-2) to the followings, respectively, for \( m \geq 1 \).

\begin{align}
(2-3) \quad h \circ g_{2m} &= g_{2m} \circ h^{(2m)} : \mathbb{Q}^{(2m)} \to \mathbb{Q} \quad \text{for every } h \in \mathcal{H}(\mathbb{Q}), \\
(2-4) \quad h \circ f_{2m-1} &= f_{2m-1} \circ h^{(2m-1)} : \mathbb{Q}^{(2m-1)} \to P \quad \text{for every } h \in \mathcal{H}_*,
\end{align}

where \( h^{(n)} \in \mathcal{H}(\mathbb{Q}^{(n)}) \). Combining these results with 2.2 we can summarize that

**Theorem 2.3.** Let \( m \geq 1 \). Then every \( \mathbb{Q}^{(2m)} \) admits a perfect irreducible projection \( g_{2m} \) onto \( \mathbb{Q} \), and every \( \mathbb{Q}^{(2m-1)} \) admits a perfect irreducible projection \( f_{2m-1} \) onto \( P \approx \mathbb{P} \), with the additional property that they are "fiber-wise" homogeneous in the following sense:

1. For any \( q_1 < q_2 \in \mathbb{Q} \) there exists a homeomorphism of \( \mathbb{Q}^{(2m)} \), induced by a homeomorphism of \( \mathbb{Q} \), carrying the fiber \( g_{2m}^{-1}(q_1) \) to \( g_{2m}^{-1}(q_2) \).
2. For any \( p_1 < p_2 \in P \) there exists a homeomorphism of \( \mathbb{Q}^{(2m-1)} \), induced by a homeomorphism of \( P \), carrying the fiber \( f_{2m-1}^{-1}(p_1) \) to \( f_{2m-1}^{-1}(p_2) \).

Moreover, under CH (=the Continuum Hypothesis) every fiber \( g_{2m}^{-1}(q) \) of \( q \in \mathbb{Q} \) as well as every fiber \( f_{2m-1}^{-1}(p) \) of \( p \in P \) is homeomorphic to \( \omega^* = \beta \omega \setminus \omega \).

This last assertion follows from the well-known

**Fact 2.4.** (see 1.2.6 in [8] or 3.37 in [9]) (CH) Let \( Y \) be a 0-dimensional, locally compact, \( \sigma \)-compact, non-compact space of weight at most \( c \). Then \( Y^* = \beta Y \setminus Y \) and \( \omega^* \) are homeomorphic.

Indeed, put \( Z = g_{2m}^{-1}(q) \) and \( Y = \beta \mathbb{Q}^{(2m-1)} \setminus Z \). Then \( Z \) is a zero-set of the 0-dimensional \( \beta \mathbb{Q}^{(2m-1)} \) included in the remainder \( \mathbb{Q}^{(2m)} = \beta \mathbb{Q}^{(2m-1)} \setminus \mathbb{Q}^{(2m-1)} \), so that \( Y^* = \beta Y \setminus Y = Z \). Since \( Y \) is a cozero-set and separable, \( Y \) satisfies the condition in 2.4. Hence \( Z \approx \omega^* \). Similarly we can prove that \( f_{2m-1}^{-1}(p) \approx \omega^* \).

### 3. Remote Points and Extremally Disconnected Points

To analyze further the structure of \( \mathbb{Q}^{(n)} \)'s, we need the notion of remote points and extremally disconnected points. A point \( p \in \beta X \setminus X \) is called a remote point of \( X \) if \( p \notin \text{cl}_{\beta X} F \) for every nowhere dense closed subset \( F \) of \( X \). Van Douwen [2], Chae, Smith [1], showed

**Fact 3.1.** Every non-pseudocompact space of countable \( \pi \)-weight has \( 2^c \) many remote points.

An easy consequence of this fact is

**Fact 3.2.** Let \( X \) be a non-compact, Lindelöf space of countable \( \pi \)-weight. Then remote points of \( X \) form a \( G_\delta \)-dense subset of \( X^* = \beta X \setminus X \).
Proof. Choose any point $p \in X^*$ and a zero-set $Z$ of $\beta X$ containing $p$. Since $X$ is Lindelöf, we can suppose that $Z$ misses $X$. Put $Y = \beta X \setminus Z$; then $\beta Y = \beta X$, and $Y$ is of countable $\pi$-weight since $X$ is. Hence 3.1 implies that $Y^* = Z$ contains remote points of $Y$, which are also remote points of $X$. \hfill \Box

A space $T$ is said to be extremally disconnected at a point $p \in T$ (see [2]) if $p \notin \text{cl}_T U_1 \cap \text{cl}_T U_2$ for every pair of disjoint open sets $U_1$, $U_2$ in $T$. Let us call such a point $p$ an extremally disconnected point of $T$, or simply, an e.d. point of $T$, and denote the set of all such e.d. points by $\text{Ed}(T)$. A space $T$ is extremally disconnected if every point of $T$ is an e.d. point, i.e., $\text{Ed}(T) = T$. If $S$ is dense in $T$, we always have $\text{cl}_T U = \text{cl}_T (U \cap S)$ for every open set $U$ of $T$; hence a point $p \in S$ is an e.d. point of $S$ if and only if it is an e.d. point of $T$, i.e., $\text{Ed}(S) = S \cap \text{Ed}(T)$.

Fact 3.3. ([2])

1. Any remote point of $X$ is an e.d. point of $\beta X$.
2. Suppose $X$ is first countable and hereditarily separable, and $p \in \beta X \setminus X$. Then $p$ is a remote point of $X$ if and only if $p$ is an e.d. point of $\beta X$.

Let us call a point $p \in T$ a common boundary point of $T$ if $p$ is not an e.d. point of $T$, i.e., if $p \notin \text{cl}_T U_1 \cap \text{cl}_T U_2$ for some pair of disjoint open sets $U_1$, $U_2$ in $T$. Similarly, we call a subset $A \subseteq T$ a common boundary set in $T$ if $A \subseteq \text{cl}_T U_1 \cap \text{cl}_T U_2$ for some pair of disjoint open sets $U_1$, $U_2$ in $T$. We abbreviate “common boundary” to “co-boundary.” (Such $p$, $A$ are called “2-point,” “2-set,” respectively, in [2].) Note that any co-boundary set in $T$ is nowhere dense in $T$, but the converse need not be true. Let $\text{Cob}(T) = T \setminus \text{Ed}(T)$ denote the set of all co-boundary points of $T$. Note also that if $A$ is a co-boundary set, then every point of $A$ is obviously a co-boundary point, but the converse need not be true except the case $A$ is a countable discrete subset:

Lemma 3.4. Suppose $A$ is a countable discrete subset consisting of co-boundary points of $T$. Then $A$, and hence also $\text{cl}_T A$, is a co-boundary set in $T$. Therefore, if $T$ is compact, $\text{Cob}(T)$ is always countably compact in the strong sense that every countable discrete subset has compact closure in $\text{Cob}(T)$.

Proof. Let $A = \{a_n\}_{n \in \omega} \subseteq \text{Cob}(T)$ be discrete in $T$, and choose disjoint open sets $\{W_n\}_{n \in \omega}$ in $T$ such that $a_n \in W_n$. In each $W_n$, choose disjoint open sets $U_n$, $V_n$ with $a_n \in \text{cl}_T U_n \cap \text{cl}_T V_n$. Put $U = \bigcup_{n \in \omega} U_n$ and $V = \bigcup_{n \in \omega} V_n$. Then these disjoint open sets $U$, $V$ satisfy $A \subseteq \text{cl}_T U \cap \text{cl}_T V$, and hence $\text{cl}_T A \subseteq \text{cl}_T U \cap \text{cl}_T V$. \hfill \Box

For an open set $U \subseteq X$ its maximal open extension $\text{Ex}(U) \subseteq \beta X$ is defined by

$$\text{Ex}(U) = \beta X \setminus \text{cl}_{\beta X} (X \setminus U).$$

Suppose $W$ is an open set in $\beta X$; then

$$\text{cl}_{\beta X} W = \text{cl}_{\beta X} (W \cap X) = \text{cl}_{\beta X} \text{Ex}(W \cap X).$$
Therefore we see

**Fact 3.5.** Suppose \( p \in \beta X \setminus X \). Then \( p \) is a co-boundary point of \( \beta X \)
if and only if \( p \in cl_{\beta X}Ex(U) \cap cl_{\beta X}Ex(V) \) for some disjoint open sets \( U, V \)
in \( X \).

We denote the boundary of a subset \( W \) in \( Y \) by Bd\( Y \) so that Bd\( Y \) =
\( \text{cl}_{Y}W \setminus W \) if \( W \) is open in \( Y \). Van Douwen [2] proved the equality
\[(*)\quad \text{Bd}_{\beta X}Ex(U) = cl_{\beta X}\text{Bd}_{X}(U)\]
for every open set \( U \) of \( X \). (Note that 3.3 (1) follows from this equality since
\( \text{Bd}_{X}(U) \) is a nowhere dense subset of \( X \).) Using this (*) and 3.5 we get an
"inner" characterization of co-boundary points, hence of e.d. points also, of \( \beta X \) for a normal space \( X \):

**Lemma 3.6.** Assume \( X \) is normal, and \( p \in \beta X \setminus X \). Then \( p \) is a co-boundary point of \( \beta X \)
if and only if \( p \in \text{cl}_{\beta X}F \) for some co-boundary set \( F \)
in \( X \). In other words, \( p \) is an e.d. point of \( \beta X \) if and only if
\[ p \notin \text{cl}_{\beta X}F \text{ for every co-boundary set } F \text{ in } X. \]

**Proof.** By 3.5 it suffices to show the equality
\[ \text{cl}_{\beta X}Ex(U) \cap \text{cl}_{\beta X}Ex(V) = \text{cl}_{\beta X}(\text{cl}_{X}U \cap \text{cl}_{X}V) \]
for disjoint open sets \( U, V \) in \( X \), since \( \text{cl}_{X}U \cap \text{cl}_{X}V \) is a co-boundary set in \( X \). Using (*) we get
\[ \text{cl}_{\beta X}Ex(U) \cap \text{cl}_{\beta X}Ex(V) = \text{Bd}_{\beta X}Ex(U) \cap \text{Bd}_{\beta X}Ex(V) \]
\[ = (\text{cl}_{\beta X}\text{Bd}_{X}U) \cap (\text{cl}_{\beta X}\text{Bd}_{X}V). \]
Since \( X \) is normal, this set is equal to \( \text{cl}_{\beta X}(\text{Bd}_{X}U \cap \text{Bd}_{X}V) \), where
\( \text{Bd}_{X}U \cap \text{Bd}_{X}V = \text{cl}_{X}U \cap \text{cl}_{X}V \).

**Lemma 3.7.** Suppose \( A \) is a closed subset of a normal space \( X \). Then \( A \subseteq \text{Ed}(X) \) implies \( \text{cl}_{\beta X}A \subseteq \text{Ed}(\beta X) \).

**Proof.** Let \( A \) be a closed subset of a normal space \( X \), and that \( A \subseteq \text{Ed}(X) \).
Let \( F \) be any co-boundary closed set in \( X \). By 3.6 it suffices to show that
\( \text{cl}_{\beta X}F \cap \text{cl}_{\beta X}A = \emptyset \). Since \( F \subseteq \text{Cob}(X) \) and \( A \subseteq \text{Ed}(X) \), we know that
\( F, A \) are disjoint closed subsets of \( X \). Hence the normality of \( X \) implies that
\( \text{cl}_{\beta X}F \cap \text{cl}_{\beta X}A = \emptyset \).

The next lemma shows how co-boundary points or e.d. points behave w.r.t. closed irreducible maps. Let \( g \) be a map from \( X \) onto \( Y \). For a subset \( U \subseteq X \) define \( g^{o}(U) \subseteq Y \), a small image of \( U \), by
\[ y \in g^{o}(U) \text{ if and only if } g^{-1}(y) \subseteq U, \]
i.e., \( g^{o}(U) = Y \setminus g(X \setminus U) \subseteq g(U) \); so, \( g \) is irreducible if \( g^{o}(U) \neq \emptyset \) for every non-empty open set \( U \). Note an obvious useful formula
\[ g^{o}(U \cap V) = g^{o}(U) \cap g^{o}(V) \]
for any sets $U, V \subseteq X$, which especially implies that $g^o(U) \cap g^o(V) = \emptyset$ whenever $U \cap V = \emptyset$. Suppose $g$ is closed irreducible. Then it is well known that $g^o(U)$ is non-empty and open whenever $U$ is, and
\[
\text{cl}_Y g^o(U) = \text{cl}_Y g(U) = g(\text{cl}_X U)
\]
for every open subset $U \subseteq X$.

**Lemma 3.8.** Let $g : X \to Y$ be any closed irreducible map. Then $g$ maps co-boundary points to co-boundary points, i.e., $g(\text{Cob}(X)) \subseteq \text{Cob}(Y)$. Furthermore, for every $x \in X$
\[
g(x) \in \text{Cob}(Y) \iff x \in \text{Cob}(X) \text{ or } |g^{-1}(g(x))| > 1,
\]
and
\[
g(x) \in \text{Ed}(Y) \iff x \in \text{Ed}(X) \text{ and } g^{-1}(g(x)) = \{x\}.
\]
Consequently, $g^{-1}(\text{Ed}(Y)) \subseteq \text{Ed}(X)$, and the restriction of $g$ to
\[
g^{-1}(\text{Ed}(Y)) \to \text{Ed}(Y)
\]
is a homeomorphism.

**Proof.** Let $U_1, U_2$ be any disjoint open sets in $X$. Then
\[
g(\text{cl}_X U_1 \cap \text{cl}_X U_2) \subseteq g(\text{cl}_X U_1) \cap g(\text{cl}_X U_2) = \text{cl}_Y g^o(U_1) \cap \text{cl}_Y g^o(U_2),
\]
and $g^o(U_1), g^o(U_2)$ are disjoint open. Hence $g$ maps co-boundary points to co-boundary points. Similarly, we can show that
\[
|g^{-1}(g(x))| > 1 \text{ implies } g(x) \in \text{Cob}(Y).
\]
Indeed, if we take two points $x_1 \neq x_2$ in $g^{-1}(g(x))$, we can choose disjoint open sets $U_1, U_2$ in $X$ such that $x_1 \in U_1$ and $x_2 \in U_2$ (using the Hausdorff-ness of $X$), getting $g(x) \in g(\text{cl}_X U_1) \cap g(\text{cl}_X U_2) = \text{cl}_Y g^o(U_1) \cap \text{cl}_Y g^o(U_2).
\]
So, to complete our proof, assume $g(x) \in \text{Cob}(Y)$ and $|g^{-1}(g(x))| = 1$; then we need to show $x \in \text{Cob}(X)$. The condition $g(x) \in \text{Cob}(Y)$ implies that
\[
g(x) \in \text{cl}_Y V_1 \cap \text{cl}_Y V_2
\]
for some disjoint open sets $V_1, V_2$ in $Y$. Since $g$ is a closed map, $g(x) \in \text{cl}_Y V_i$ implies $g^{-1}(g(x)) \cap \text{cl}_X g^{-1}(V_i) \neq \emptyset$ for $i = 1, 2$. Hence the condition $g^{-1}(g(x)) = \{x\}$ implies $x \in \text{cl}_X g^{-1}(V_1) \cap \text{cl}_X g^{-1}(V_2)$, showing $x \in \text{Cob}(X)$.

\[
\square
\]

### 4. Topological Difference of $\mathbb{Q}^{(1)}$ and $\mathbb{Q}^{(3)}$

Now let us apply the general theory in §3 to our spaces
\[
\beta \mathbb{Q}^{(n)} = \mathbb{Q}^{(n)} \cup \mathbb{Q}^{(n+1)} (n \geq 0).
\]
Recall that every $\mathbb{Q}^{(n)}$ is of countable $\pi$-weight and Lindelöf, hence normal. Put $C_n = \text{Cob}(\mathbb{Q}^{(n)})$ and $E_n = \text{Ed}(\mathbb{Q}^{(n)})$; then this gives a partition of $\mathbb{Q}^{(n)}$
\[
\mathbb{Q}^{(n)} = C_n \cup E_n.
\]
It is obvious that $E_0 = \emptyset$, i.e., $\mathbb{Q}^{(0)} = C_0$. Lemma 3.4 implies that each $C_n$ ($n \geq 1$) is dense in $\mathbb{Q}^{(n)}$, and Fact 3.2 with 3.3 (1) implies that each
$E_n$ ($n \geq 1$) is dense in $Q^{(n)}$. Note in particular that $E_1$ coincides with the set of all remote points of $Q$, by 3.3 (2).

**Property 4.1.** Let $A$ be any countable discrete subset of $E_2$ which is closed in $Q^{(2)}$. Then
(1) $\text{cl} \ A \subseteq E_2 \cup C_1$ in $\beta Q^{(1)}$, while (2) $\text{cl} \ A \subseteq E_2 \cup E_3$ in $\beta Q^{(2)}$.

**Proof.** (2) follows from 3.7. To prove (1), let $A$ be as above. Then, since $\phi_0 : Q^{(2)} \to Q^{(0)}$ is perfect, $\phi_0(A)$ is also a countable discrete closed subset of $Q^{(0)} = C_0$. Since $C_0 \cup C_1 = \text{Cob}(\beta Q^{(0)})$ is countably compact in the strong sense as stated in 3.4, we have $\text{cl} \ \phi_0(A) \subseteq C_0 \cup C_1$ in $\beta Q^{(0)}$. Pulling back by the map $\Phi_0$, we get $\text{cl} \ A \subseteq Q^{(2)} \cup C_1$ in $\beta Q^{(1)}$. This is the same as the assertion (1) since $A \subseteq E_2$. \hfill $\square$

Now we can prove the following strong assertion which in particular implies that $Q^{(1)} \not\simeq Q^{(3)}$.

**Theorem 4.2.** $Q^{(1)}$ admits no perfect irreducible map onto $Q^{(3)}$.

**Proof.** Suppose there existed a perfect irreducible map $\psi : Q^{(1)} \to Q^{(3)}$. Then, since $\beta Q^{(2)}$ can be seen as a compactification of $Q^{(3)}$, $\psi$ extends to a perfect irreducible map

$$\Psi : \beta Q^{(1)} = Q^{(1)} \cup Q^{(2)} \to \beta Q^{(2)} = Q^{(3)} \cup Q^{(2)}.$$  

Lemma 3.8 implies then that

$$E_2 \cup E_1 \supseteq \Psi^{-1}(E_2 \cup E_3) \approx E_2 \cup E_3.$$  

Choose any countable discrete subset $B \subseteq E_2 \subseteq Q^{(2)} \subseteq \beta Q^{(2)}$ which is closed in $Q^{(2)}$. (We can do this because $E_2$ is dense in $Q^{(2)}$, and $Q^{(2)}$ is
Lindelöf.) Put $A = \Psi^{-1}(B)$, then this $A$ is also a countable discrete subset of $E_2$ which is closed in $\mathbb{Q}^{(2)}$. Property 4.1 (2) shows $\text{cl } B \subseteq E_2 \cup E_3$ in $\beta\mathbb{Q}^{(2)}$, and so, pulling back by $\Psi$, we get
\[
\text{cl } A \subseteq \Psi^{-1}(E_2 \cup E_3) \subseteq E_2 \cup E_1
\]
in $\beta\mathbb{Q}^{(1)}$. But this contradicts 4.1 (1).

We will be able to show in [4] that for any $n \geq 1$, $\mathbb{Q}^{(n)}$ admits no perfect irreducible map onto $\mathbb{Q}^{(n+2)}$ by analyzing further the behavior of limit points of countable discrete subsets in $\mathbb{Q}^{(m)}$. Some of the basic techniques in this paper can be found also in [5, 6, 7].

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