<table>
<thead>
<tr>
<th>項目</th>
<th>内容</th>
</tr>
</thead>
<tbody>
<tr>
<td>タイトル</td>
<td>THE ITERATED REMAINDERS OF THE RATIONALS (Research Trends on Set-theoretic and Geometric Topology and their Prospect)</td>
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<td>著者</td>
<td>加藤 昭男</td>
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THE ITERATED REMAINDERS OF THE RATIONALS

AKIO KATO

ABSTRACT. Repeat taking remainders of Stone-Čech compactifications of the rationals
\[ Q^{(1)} = Q^* = \beta Q \setminus Q, \quad Q^{(2)} = \beta Q^{(1)} \setminus Q^{(1)}, \quad Q^{(3)} = \beta Q^{(2)} \setminus Q^{(2)}, \quad Q^{(4)} \ldots. \]

We point out that they have similar structures, but are topologically different. In particular we prove here that \( Q^{(1)} \neq Q^{(3)} \). This result will be generalized to show that \( Q^{(n)} \neq Q^{(n+2)} \) for any \( n \geq 1 \) in the forthcoming paper [4].

1. INTRODUCTION

Consider the space of rationals \( \mathbb{Q} \), and repeat taking its remainders of Stone-Čech compactifications \( \mathbb{Q}^{(n+1)} = (\mathbb{Q}^{(n)})^* = \beta \mathbb{Q}^{(n)} \setminus \mathbb{Q}^{(n)} (n \geq 0) \)
where \( \mathbb{Q}^{(0)} = \mathbb{Q} \), i.e.,
\[ Q^{(1)} = Q^*, \quad Q^{(2)} = Q^{**}, \quad Q^{(3)} = Q^{***}, \ldots. \]

Van Douwen [2] asked whether or not \( Q^{(n)} \approx Q^{(n+2)} \) for \( n \geq 1 \), remarking that \( Q^{(m)} \) for even \( m \) is never homeomorphic to \( Q^{(n)} \) for odd \( n \), because the former is \( \sigma \)-compact but the latter is not.

In this paper we point out that both \( Q^{(n)} \) and \( Q^{(n+2)} \) have a similar structure of "fiber bundle" for every \( n \geq 1 \), but they are topologically different. In particular we here show that \( Q^{(1)} \neq Q^{(3)} \), which we can generalize in the forthcoming paper [4] to show that \( Q^{(n)} \neq Q^{(n+2)} \) for any \( n \geq 1 \), answering van Douwen's question.

The precise connections of the remainders can be seen by the following construction. Viewing \( \beta \mathbb{Q} \) as a compactification of \( Q^{(1)} \), let
\[ \Phi_0 : \beta Q^{(1)} = Q^{(1)} \cup Q^{(2)} \to Q \cup Q^{(1)} = \beta Q \]
be the Stone extension of the identity map \( id : Q^{(1)} \to Q^{(1)} \). Denote by
\[ \phi_0 : Q^{(2)} \to Q^{(0)} \]
the restriction of \( \Phi_0 \). Next let
\[ \Phi_1 : \beta Q^{(2)} = Q^{(2)} \cup Q^{(3)} \to Q^{(1)} \cup Q^{(2)} = \beta Q^{(1)} \]
be the Stone extension of the identity map \( id : Q^{(2)} \to Q^{(2)} \), and let
\[ \phi_1 : Q^{(3)} \to Q^{(1)} \]

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denote the restriction of $\Phi_1$. In this way, for every $n \geq 0$ we can generally get the Stone extension

$$\Phi_n : \beta \mathbb{Q}^{(n+1)} = \mathbb{Q}^{(n+1)} \cup \mathbb{Q}^{(n+2)} \rightarrow \mathbb{Q}^{(n)} \cup \mathbb{Q}^{(n+1)} = \beta \mathbb{Q}^{(n)}$$

of the identity map $id : \mathbb{Q}^{(n+1)} \rightarrow \mathbb{Q}^{(n+1)}$, and its restriction map

$$\phi_n : \mathbb{Q}^{(n+2)} \rightarrow \mathbb{Q}^{(n)}.$$ 

Since every $\Phi_n (n \in \omega)$ is perfect, so is every $\phi_n$. Hence every $\mathbb{Q}^{(n)} (n \in \omega)$ is Lindelöf since both $\mathbb{Q}^{(0)} = \mathbb{Q}$, $\mathbb{Q}^{(1)}$ are Lindelöf. We can also see that $\mathbb{Q}^{(n)}$ is $\sigma$-compact for even $n$, but $\mathbb{Q}^{(n)}$ is not for odd $n$, because $\mathbb{Q}^{(0)}$ is $\sigma$-compact but $\mathbb{Q}^{(1)}$ is not since $\mathbb{Q}^{(1)}$ is a perfect pre-image of the irrationals $\mathbb{P}$ as we see below.

$$\beta \mathbb{Q}^{(0)} \quad \Phi_0 \quad \beta \mathbb{Q}^{(1)} \quad \Phi_1 \quad \beta \mathbb{Q}^{(2)} \quad \Phi_2 \quad \beta \mathbb{Q}^{(3)}$$

FIG. 1

A collection $\mathcal{B}$ of nonempty open sets of $X$ is called a $\pi$-base for $X$ if every nonempty open set in $X$ includes some member of $\mathcal{B}$. The minimal cardinality of such a $\pi$-base is called the $\pi$-weight of $X$. Note that any dense subspace of $X$ has the same $\pi$-weight as $X$, and any space of countable $\pi$-weight is separable. Consequently, any dense subset of a space of countable $\pi$-weight is also of countable $\pi$-weight, and hence separable. So, all of $\beta \mathbb{Q}^{(n)}$, $\mathbb{Q}^{(n)} (n \in \omega)$ are of countable $\pi$-weight, and hence separable.

Recall that an onto map $g : X \rightarrow Y$ is called irreducible if every nonempty open subset $U$ of $X$ includes some fiber $g^{-1}(y)$, and it is well known and easy to see that

(1) every extension of a homeomorphism is irreducible, and
(2) the restriction of a closed irreducible map to any dense subset is irreducible.
Therefore we can see that all of the maps $\Phi_n, \phi_n$ ($n \in \omega$) are perfect irreducible. Consider the partition of the closed interval $[0,1] = Q \cup P$ where

$$Q = [0,1] \cap Q \cong Q \text{ and } P = [0,1] \setminus Q \cong P,$$

and let $f : \beta Q \to [0,1]$ be the Stone extension of the homeomorphism $Q \cong Q$. Then the restriction $f_0 = f \restriction Q^{(1) : Q^{(1)}} \to P \cong P$ is perfect irreducible. Thus we get the following sequence of perfect irreducible maps:

$$Q \leftarrow Q^{(2)} \leftarrow Q^{(4)} \leftarrow \cdots ; \ P \leftarrow Q^{(1)} \leftarrow Q^{(3)} \leftarrow Q^{(5)} \leftarrow \cdots.$$

All spaces are assumed to be completely regular and Hausdorff, and maps are always continuous, unless otherwise stated. "Partition" is synonymous with "disjoint union." For a subset $A$ of some compact space $K$ we use the notation $A^*$ to denote the remainder $\text{cl}_K A \setminus A$ when $K$ is clear from the context. Our terminologies are based upon [3].

2. SIMILAR STRUCTURES

We first show that both $Q^{(n)}$ and $Q^{(n+2)}$ have a similar structure for every $n \geq 1$. In general, for any space $Y$ let us denote by $H(Y)$ the collection of all homeomorphisms $h : Y \approx Y$. Let $X$ be a nowhere compact, dense-in-itself space, where nowhere compact (or nowhere locally compact) means that $X$ contains no compact neighborhood, or equivalently, that $X$ is a dense subset of some/any compact space $K$ such that the remainder $K \setminus X$ is also dense in $K$. Let $cX$ be some compactification of $X$ and let $H_* \subseteq H(X)$ denote the collection of all $h \in H(X)$ such that

$$(\ast) \quad h \text{ is extendable to } c(h) \in H(cX).$$

(Of course, $H_* = H(cX)$ if $cX = \beta X$.) Let $X^{(1)} = cX \setminus X$ be the remainder, and for every $h \in H_*$ define $h^{(1)} \in H(X^{(1)})$ to be the restriction of $c(h)$ to $X^{(1)}$. Next consider the Stone-Čech compactification $\beta X^{(1)}$ of $X^{(1)}$ and the Stone extension $\beta h^{(1)} \in H(\beta X^{(1)})$ of $h^{(1)}$. Let $X^{(2)} = \beta X^{(1)} \setminus X^{(1)}$ be the remainder, and define $h^{(2)} \in H(X^{(2)})$ to be the restriction of $\beta h^{(1)}$ to the remainder $X^{(2)}$; hence

$$h : X \approx X, \quad h^{(1)} : X^{(1)} \approx X^{(1)}, \quad h^{(2)} : X^{(2)} \approx X^{(2)}.$$

Note that $X^{(1)}$ is dense in $\beta X$, and $X^{(2)}$ is dense in $\beta X^{(1)}$, since we assume that $X$ is nowhere compact. Viewing that $\beta X$ is a compactification of $X^{(1)}$, we can consider the Stone extension $\Phi : \beta X^{(1)} \to \beta X$ of the identity map $id_{X^{(1)}} : X^{(1)} = X^{(1)}$. Let $\phi : X^{(2)} \to X$ be the restriction of $\Phi$. Then both $\Phi$ and $\phi$ are perfect irreducible maps. We can show that the correspondence $H(X) \supset H_* \ni h \mapsto h^{(2)} \in H(X^{(2)})$ is compatible with the perfect irreducible map $\phi$, i.e.,

**Lemma 2.1.** $h \circ \phi = \phi \circ h^{(2)} : X^{(2)} \to X.$
\textbf{Proof.} To show this equality, it suffices to prove the equality
\[ c(h) \circ \Phi = \Phi \circ \beta h^{(1)} : \beta X^{(1)} \rightarrow c X, \]
which follows from the obvious equality
\[ h^{(1)} \circ id_{X^{(1)}} = id_{X^{(1)}} \circ h^{(1)} : X^{(1)} \rightarrow X^{(1)} \]
on the dense subset \( X^{(1)} \) of \( \beta X^{(1)} \).
\[ \square \]

\textbf{Corollary 2.2.} If \( h(x) = y \) for \( x, y \in X \), then \( h^{(2)}(\phi^{-1}(x)) = \phi^{-1}(y) \).

\textbf{Proof.} The inclusion \( h^{(2)}(\phi^{-1}(x)) \subseteq \phi^{-1}(y) \) follows from 2.1. Since \( h \) is a homeomorphism, we can replace \( h \) by \( h^{-1} \) to get the reverse inclusion. \[ \square \]

Taking \( X = \mathbb{Q} \), \( cX = \beta \mathbb{Q} \), \( \mathcal{H}_{\star} = H(\mathbb{Q}) \), we can deduce from 2.1 that
\[ h \circ \phi_{0} = \phi_{0} \circ h^{(2)} : \mathbb{Q}^{(2)} \rightarrow \mathbb{Q} \text{ for every } h \in H(\mathbb{Q}). \]

Let \( [0,1] = Q \cup P \), \( Q \approx \mathbb{Q} \), \( P \approx \mathbb{P} \) be as at the end of §1, and take \( X = P \), \( cX = [0,1] \); then \( X^{(1)} = Q \), \( X^{(2)} = Q^{(1)} \), and the corresponding map \( \phi \) in Fig. 2 is identical to the map \( f_{0} : Q^{(1)} \rightarrow P \) at the end of §1. Note that \( \mathcal{H}_{\star} \subseteq H(P) \) is the collection of all homeomorphisms of \( P \) extendable to homeomorphisms of \([0,1]\). Then we can deduce from 2.1 that
\[ h \circ f_{0} = f_{0} \circ h^{(2)} : Q^{(1)} \rightarrow P \text{ for every } h \in \mathcal{H}_{\star}. \]

Note that for every pair of irrationals \( p_{1} < p_{2} \) in \( P = [0,1]\setminus\mathbb{Q} \) we can find an \( h \in \mathcal{H}_{\star} \) such that \( h(p_{1}) = p_{2} \); for example, we can take as \( c(h) \) in \((\ast)\) a strictly increasing function \( c(h) : [0,1] \rightarrow [0,1] \) such that \( c(h)(Q) = Q \), \( c(h)(0) = 0 \), \( c(h)(p_{1}) = p_{2} \), \( c(h)(1) = 1 \). For \( m \geq 1 \) define \( g_{2m} \) and \( f_{2m-1} \) by
\[ g_{2m} = \phi_{0} \circ \phi_{2} \circ \cdots \circ \phi_{2m-2} : Q(2m) \rightarrow \mathbb{Q}, \]
\[ f_{2m-1} = f_{0} \circ \phi_{1} \circ \phi_{3} \circ \cdots \circ \phi_{2m-3} : Q(2m-1) \rightarrow P. \]
Then, using 2.1 we can extend the above (2-1), (2-2) to the followings, respectively, for $m \geq 1$.

\begin{equation}
(h) \quad h \circ g_{2m} = g_{2m} \circ h^{(2m)} : Q^{(2m)} \to Q \quad \text{for every } h \in H(Q),
\end{equation}

\begin{equation}
(g) \quad h \circ f_{2m-1} = f_{2m-1} \circ h^{(2m-1)} : Q^{(2m-1)} \to P \quad \text{for every } h \in H_{*},
\end{equation}

where $h^{(n)} \in H(Q^{(n)})$. Combining these results with 2.2 we can summarize that

\textbf{Theorem 2.3.} Let $m \geq 1$. Then every $Q^{(2m)}$ admits a perfect irreducible projection $g_{2m}$ onto $Q$, and every $Q^{(2m-1)}$ admits a perfect irreducible projection $f_{2m-1}$ onto $P \approx \mathbb{P}$, with the additional property that they are "fiberwise" homogeneous in the following sense:

1. For any $q_{1} < q_{2} \in Q$ there exists a homeomorphism of $Q^{(2m)}$, induced by a homeomorphism of $Q$, carrying the fiber $g_{2m}^{-1}(q_{1})$ to $g_{2m}^{-1}(q_{2})$.
2. For any $p_{1} < p_{2} \in P$ there exists a homeomorphism of $Q^{(2m-1)}$, induced by a homeomorphism of $P$, carrying the fiber $f_{2m-1}^{-1}(p_{1})$ to $f_{2m-1}^{-1}(p_{2})$.

Moreover, under CH (=the Continuum Hypothesis) every fiber $g_{2m}^{-1}(q)$ of $q \in Q$ as well as every fiber $f_{2m-1}^{-1}(p)$ of $p \in P$ is homeomorphic to $\omega^{*} = \beta \omega \setminus \omega$.

This last assertion follows from the well-known

\textbf{Fact 2.4.} (see 1.2.6 in [8] or 3.37 in [9]) (CH) Let $Y$ be a 0-dimensional, locally compact, $\sigma$-compact, non-compact space of weight at most $c$. Then $Y^{*} = \beta Y \setminus Y$ and $\omega^{*}$ are homeomorphic.

Indeed, put $Z = g_{2m}^{-1}(q)$ and $Y = \beta Q^{(2m-1)} \setminus Z$. Then $Z$ is a zero-set of the 0-dimensional $\beta Q^{(2m-1)}$ included in the remainder $Q^{(2m)} = \beta Q^{(2m-1)} \setminus Q^{(2m-1)}$, so that $Y^{*} = \beta Y \setminus Y = Z$. Since $Y$ is a cozero-set and separable, $Y$ satisfies the condition in 2.4. Hence $Z \approx \omega^{*}$. Similarly we can prove that $f_{2m-1}^{-1}(p) \approx \omega^{*}$.

3. Remote Points and Extremally Disconnected Points

To analyze further the structure of $Q^{(n)}$'s, we need the notion of remote points and extremally disconnected points. A point $p \in \beta X \setminus X$ is called a remote point of $X$ if $p \notin \text{cl}_{\beta X} F$ for every nowhere dense closed subset $F$ of $X$. Van Douwen [2], Chae, Smith [1], showed

\textbf{Fact 3.1.} Every non-pseudocompact space of countable $\pi$-weight has $2^{c}$ many remote points.

An easy consequence of this fact is

\textbf{Fact 3.2.} Let $X$ be a non-compact, Lindelöf space of countable $\pi$-weight. Then remote points of $X$ form a $G_{\delta}$-dense subset of $X^{*} = \beta X \setminus X$. 

Proof. Choose any point \( p \in X^* \) and a zero-set \( Z \) of \( \beta X \) containing \( p \). Since \( X \) is Lindelöf, we can suppose that \( Z \) misses \( X \). Put \( Y = \beta X \setminus Z \); then \( \beta Y = \beta X \), and \( Y \) is of countable \( \pi \)-weight since \( X \) is. Hence 3.1 implies that \( Y^* = Z \) contains remote points of \( Y \), which are also remote points of \( X \).

A space \( T \) is said to be extremally disconnected at a point \( p \in T \) (see [2]) if \( p \notin \text{cl}_T U_1 \cap \text{cl}_T U_2 \) for every pair of disjoint open sets \( U_1, U_2 \) in \( T \). Let us call such a point \( p \) as an extremally disconnected point of \( T \), or simply, an e.d. point of \( T \), and denote the set of all such e.d. points by \( \text{Ed}(T) \). A space \( T \) is extremally disconnected if every point of \( T \) is an e.d. point, i.e., \( \text{Ed}(T) = T \). If \( S \) is dense in \( T \), we always have \( \text{cl}_T U = \text{cl}_T (U \cap S) \) for every open set \( U \) of \( T \); hence a point \( p \in S \) is an e.d. point of \( S \) if and only if it is an e.d. point of \( T \), i.e., \( \text{Ed}(S) = S \cap \text{Ed}(T) \).

Fact 3.3. ([2]) (1) Any remote point of \( X \) is an e.d. point of \( \beta X \).
(2) Suppose \( X \) is first countable and hereditarily separable, and \( p \in \beta X \setminus X \). Then \( p \) is a remote point of \( X \) if and only if \( p \) is an e.d. point of \( \beta X \).

Let us call a point \( p \in T \) a common boundary point of \( T \) if \( p \) is not an e.d. point of \( T \), i.e., if \( p \notin \text{cl}_T U_1 \cap \text{cl}_T U_2 \) for some pair of disjoint open sets \( U_1, U_2 \) in \( T \). Similarly, we call a subset \( A \subseteq T \) a common boundary set in \( T \) if \( A \subseteq \text{cl}_T U_1 \cap \text{cl}_T U_2 \) for some pair of disjoint open sets \( U_1, U_2 \) in \( T \). We abbreviate “common boundary” to “co-boundary.” (Such \( p \), \( A \) are called “2-point,” “2-set,” respectively, in [2].) Note that any co-boundary set in \( T \) is nowhere dense in \( T \), but the converse need not be true. Let \( \text{Cob}(T) = T \setminus \text{Ed}(T) \) denote the set of all co-boundary points of \( T \). Note also that if \( A \) is a co-boundary set, then every point of \( A \) is obviously a co-boundary point, but the converse need not be true except the case \( A \) is a countable discrete subset:

Lemma 3.4. Suppose \( A \) is a countable discrete subset consisting of co-boundary points of \( T \). Then \( A \), and hence also \( \text{cl}_T A \), is a co-boundary set in \( T \). Therefore, if \( T \) is compact, \( \text{Cob}(T) \) is always countably compact in the strong sense that every countable discrete subset has compact closure in \( \text{Cob}(T) \).

Proof. Let \( A = \{ a_n \}_{n \in \omega} \subseteq \text{Cob}(T) \) be discrete in \( T \), and choose disjoint open sets \( \{ W_n \}_{n \in \omega} \) in \( T \) such that \( a_n \in W_n \). In each \( W_n \), choose disjoint open sets \( U_n, V_n \) with \( a_n \in \text{cl}_T U_n \cap \text{cl}_T V_n \). Put \( U = \bigcup_{n \in \omega} U_n \) and \( V = \bigcup_{n \in \omega} V_n \). Then these disjoint open sets \( U, V \) satisfy \( A \subseteq \text{cl}_T U \cap \text{cl}_T V \), and hence \( \text{cl}_T A \subseteq \text{cl}_T U \cap \text{cl}_T V \).

For an open set \( U \subseteq X \) its maximal open extension \( \text{Ex}(U) \subseteq \beta X \) is defined by

\[
\text{Ex}(U) = \beta X \setminus \text{cl}_{\beta X}(X \setminus U).
\]

Suppose \( W \) is an open set in \( \beta X \); then

\[
\text{cl}_{\beta X} W = \text{cl}_{\beta X}(W \cap X) = \text{cl}_{\beta X} \text{Ex}(W \cap X).
\]
Therefore we see

**Fact 3.5.** Suppose $p \in \beta X \setminus X$. Then $p$ is a co-boundary point of $\beta X$ if and only if $p \in \text{cl}_{\beta X} \text{Ex}(U) \cap \text{cl}_{\beta X} \text{Ex}(V)$ for some disjoint open sets $U, V$ in $X$.

We denote the boundary of a subset $W$ in $Y$ by $\text{Bd}_Y W$ so that $\text{Bd}_Y W = \text{cl}_Y W \setminus W$ if $W$ is open in $Y$. Van Douwen [2] proved the equality

$$(*) \quad \text{Bd}_{\beta X} \text{Ex}(U) = \text{cl}_{\beta X} \text{Bd}_X (U)$$

for every open set $U$ of $X$. (Note that 3.3 (1) follows from this equality since $\text{Bd}_X (U)$ is a nowhere dense subset of $X$.) Using this $(*)$ and 3.5 we get an "inner" characterization of co-boundary points, hence of e.d. points also, of $\beta X$ for a normal space $X$:

**Lemma 3.6.** Assume $X$ is normal, and $p \in \beta X \setminus X$. Then $p$ is a co-boundary point of $\beta X$ if and only if $p \in \text{cl}_{\beta X} F$ for some co-boundary set $F$ in $X$. In other words, $p$ is an e.d. point of $\beta X$ if and only if

$p \notin \text{cl}_{\beta X} F$ for every co-boundary set $F$ in $X$.

**Proof.** By 3.5 it suffices to show the equality

$$\text{cl}_{\beta X} \text{Ex}(U) \cap \text{cl}_{\beta X} \text{Ex}(V) = \text{cl}_{\beta X} (\text{cl}_X U \cap \text{cl}_X V)$$

for disjoint open sets $U, V$ in $X$, since $\text{cl}_X U \cap \text{cl}_X V$ is a co-boundary set in $X$. Using $(*)$ we get

$$\text{cl}_{\beta X} \text{Ex}(U) \cap \text{cl}_{\beta X} \text{Ex}(V) = \text{Bd}_{\beta X} \text{Ex}(U) \cap \text{Bd}_{\beta X} \text{Ex}(V)$$

$$= (\text{cl}_{\beta X} \text{Bd}_X U) \cap (\text{cl}_{\beta X} \text{Bd}_X V).$$

Since $X$ is normal, this set is equal to $\text{cl}_{\beta X} (\text{Bd}_X U \cap \text{Bd}_X V)$, where $\text{Bd}_X U \cap \text{Bd}_X V = \text{cl}_X U \cap \text{cl}_X V$. \qed

**Lemma 3.7.** Suppose $A$ is a closed subset of a normal space $X$. Then $A \subseteq \text{Ed}(X)$ implies $\text{cl}_{\beta X} A \subseteq \text{Ed}(\beta X)$.

**Proof.** Let $A$ be a closed subset of a normal space $X$, and that $A \subseteq \text{Ed}(X)$. Let $F$ be any co-boundary closed set in $X$. By 3.6 it suffices to show that $\text{cl}_{\beta X} F \cap \text{cl}_{\beta X} A = \emptyset$. Since $F \subseteq \text{Cob}(X)$ and $A \subseteq \text{Ed}(X)$, we know that $F, A$ are disjoint closed subsets of $X$. Hence the normality of $X$ implies that $\text{cl}_{\beta X} F \cap \text{cl}_{\beta X} A = \emptyset$. \qed

The next lemma shows how co-boundary points or e.d. points behave w.r.t. closed irreducible maps. Let $g$ be a map from $X$ onto $Y$. For a subset $U \subseteq X$ define $g^o (U) \subseteq Y$, a small image of $U$, by

$$y \in g^o (U) \quad \text{if and only if} \quad g^{-1} (y) \subseteq U,$$

i.e., $g^o (U) = Y \setminus g (X \setminus U) \subseteq g(U)$; so, $g$ is irreducible if $g^o (U) \neq \emptyset$ for every non-empty open set $U$. Note an obvious useful formula

$$g^o (U \cap V) = g^o (U) \cap g^o (V)$$
for any sets $U, V \subseteq X$, which especially implies that $g^{o}(U) \cap g^{o}(V) = \emptyset$ whenever $U \cap V = \emptyset$. Suppose $g$ is closed irreducible. Then it is well known that $g^{o}(U)$ is non-empty and open whenever $U$ is, and 
$$\text{cl}_{Y} g^{o}(U) = \text{cl}_{Y} g(U) = g(\text{cl}_{X} U)$$
for every open subset $U \subseteq X$.

**Lemma 3.8.** Let $g : X \to Y$ be any closed irreducible map. Then $g$ maps co-boundary points to co-boundary points, i.e., $g(\text{Cob}(X)) \subseteq \text{Cob}(Y)$. Furthermore, for every $x \in X$
$g(x) \in \text{Cob}(Y)$ if and only if $x \in \text{Cob}(X)$ or $|g^{-1}(g(x))| > 1$, i.e.,
$g(x) \in \text{Ed}(Y)$ if and only if $x \in \text{Ed}(X)$ and $g^{-1}(g(x)) = \{x\}$.

Consequently, $g^{-1}(\text{Ed}(Y)) \subseteq \text{Ed}(X)$, and the restriction of $g$ to $g^{-1}(\text{Ed}(Y)) \to \text{Ed}(Y)$ is a homeomorphism.

**Proof.** Let $U_{1}, U_{2}$ be any disjoint open sets in $X$. Then
$g(\text{cl}_{X} U_{1} \cap \text{cl}_{X} U_{2}) \subseteq g(\text{cl}_{X} U_{1}) \cap g(\text{cl}_{X} U_{2}) = \text{cl}_{Y} g^{o}(U_{1}) \cap \text{cl}_{Y} g^{o}(U_{2}),$
and $g^{o}(U_{1}), g^{o}(U_{2})$ are disjoint open. Hence $g$ maps co-boundary points to co-boundary points. Similarly, we can show that
$|g^{-1}(g(x))| > 1$ implies $g(x) \in \text{Cob}(Y)$.

Indeed, if we take two points $x_{1} \neq x_{2}$ in $g^{-1}(g(x))$, we can choose disjoint open sets $U_{1}, U_{2}$ in $X$ such that $x_{1} \in U_{1}$ and $x_{2} \in U_{2}$ (using the Hausdorffness of $X$), getting $g(x) \in g(\text{cl}_{X} U_{1}) \cap g(\text{cl}_{X} U_{2}) = \text{cl}_{Y} g^{o}(U_{1}) \cap \text{cl}_{Y} g^{o}(U_{2}).$

So, to complete our proof, assume $g(x) \in \text{Cob}(Y)$ and $|g^{-1}(g(x))| = 1$; then we need to show $x \in \text{Cob}(X)$. The condition $g(x) \in \text{Cob}(Y)$ implies that $g(x) \in \text{cl}_{Y} V_{1} \cap \text{cl}_{Y} V_{2}$ for some disjoint open sets $V_{1}, V_{2}$ in $Y$. Since $g$ is a closed map, $g(x) \in \text{cl}_{Y} V_{i}$ implies $g^{-1}(g(x)) \cap \text{cl}_{X} g^{-1}(V_{i}) \neq \emptyset$ for $i = 1, 2$. Hence the condition $g^{-1}(g(x)) = \{x\}$ implies $x \in \text{cl}_{X} g^{-1}(V_{1}) \cap \text{cl}_{X} g^{-1}(V_{2})$, showing $x \in \text{Cob}(X)$. \hfill \Box

4. **Topological Difference of $\mathbb{Q}^{(1)}$ and $\mathbb{Q}^{(3)}$**

Now let us apply the general theory in §3 to our spaces
$$\beta \mathbb{Q}^{(n)} = \mathbb{Q}^{(n)} \cup \mathbb{Q}^{(n+1)} (n \geq 0).$$

Recall that every $\mathbb{Q}^{(n)}$ is of countable $\pi$-weight and Lindelöf, hence normal. Put $C_{n} = \text{Cob}(\mathbb{Q}^{(n)})$ and $E_{n} = \text{Ed}(\mathbb{Q}^{(n)})$; then this gives a partition of $\mathbb{Q}^{(n)}$
$$\mathbb{Q}^{(n)} = C_{n} \cup E_{n}.$$ 

It is obvious that $E_{0} = \emptyset$, i.e., $\mathbb{Q}^{(0)} = C_{0}$, Lemma 3.4 implies that each $C_{n} (n \geq 1)$ is dense in $\mathbb{Q}^{(n)}$, and Fact 3.2 with 3.3 (1) implies that each
$E_n \ (n \geq 1)$ is dense in $Q^{(n)}$. Note in particular that $E_1$ coincides with the set of all remote points of $Q$, by 3.3 (2).

\[ \begin{array}{cccc}
\beta Q^{(0)} & \overset{\Phi_0}{=} & \beta Q^{(1)} & \overset{\Phi_1}{=} \beta Q^{(2)} & \overset{\Phi_2}{=} \beta Q^{(3)} \\
Q^{(1)} & | & E_1 : C_1 & | & E_1 : C_1 & | & E_1 : C_1 & | & \ldots \\
Q^{(0)} & \overset{\phi_0}{\leftarrow} & \beta Q^{(2)} & \overset{\phi_1}{\leftarrow} \beta Q^{(3)} & \overset{\phi_2}{\leftarrow} \beta Q^{(4)} \\
C_0 & | & \beta Q^{(2)} & | & \beta Q^{(4)} & | & \ldots
\end{array} \]

**FIG. 3**

**Property 4.1.** Let $A$ be any countable discrete subset of $E_2$ which is closed in $Q^{(2)}$. Then
(1) $\text{cl } A \subseteq E_2 \cup C_1$ in $\beta Q^{(1)}$, while (2) $\text{cl } A \subseteq E_2 \cup E_3$ in $\beta Q^{(2)}$.

**Proof.** (2) follows from 3.7. To prove (1), let $A$ be as above. Then, since $\phi_0 : Q^{(2)} \to Q^{(0)}$ is perfect, $\phi_0(A)$ is also a countable discrete closed subset of $Q^{(0)} = C_0$. Since $C_0 \cup C_1 = \text{Cob}(\beta Q^{(0)})$ is countably compact in the strong sense as stated in 3.4, we have $\text{cl } \phi_0(A) \subseteq C_0 \cup C_1$ in $\beta Q^{(0)}$. Pulling back by the map $\Phi_0$, we get $\text{cl } A \subseteq Q^{(2)} \cup C_1$ in $\beta Q^{(1)}$. This is the same as the assertion (1) since $A \subseteq E_2$.

Now we can prove the following strong assertion which in particular implies that $Q^{(1)} \not\simeq Q^{(3)}$.

**Theorem 4.2.** $Q^{(1)}$ admits no perfect irreducible map onto $Q^{(3)}$.

**Proof.** Suppose there existed a perfect irreducible map $\psi : Q^{(1)} \to Q^{(3)}$. Then, since $\beta Q^{(2)}$ can be seen as a compactification of $Q^{(3)}$, $\psi$ extends to a perfect irreducible map

$\Psi : \beta Q^{(1)} = Q^{(1)} \cup Q^{(2)} \to \beta Q^{(2)} = Q^{(3)} \cup Q^{(2)}$.

Lemma 3.8 implies then that

$E_2 \cup E_1 \supseteq \Psi^{-1}(E_2 \cup E_3) \approx E_2 \cup E_3$.

Choose any countable discrete subset $B \subseteq E_2 \subseteq Q^{(2)} \subseteq \beta Q^{(2)}$ which is closed in $Q^{(2)}$. (We can do this because $E_2$ is dense in $Q^{(2)}$, and $Q^{(2)}$ is
Lindelöf.) Put $A = \Psi^{-1}(B)$, then this $A$ is also a countable discrete subset of $E_2$ which is closed in $\mathbb{Q}^{(2)}$. Property 4.1 (2) shows $\text{cl } B \subseteq E_2 \cup E_3$ in $\beta \mathbb{Q}^{(2)}$, and so, pulling back by $\Psi$, we get
\[
\text{cl } A \subseteq \Psi^{-1}(E_2 \cup E_3) \subseteq E_2 \cup E_1
\]
in $\beta \mathbb{Q}^{(1)}$. But this contradicts 4.1 (1). $\Box$

We will be able to show in [4] that for any $n \geq 1$, $\mathbb{Q}^{(n)}$ admits no perfect irreducible map onto $\mathbb{Q}^{(n+2)}$ by analyzing further the behavior of limit points of countable discrete subsets in $\mathbb{Q}^{(m)}$. Some of the basic techniques in this paper can be found also in [5, 6, 7].

REFERENCES