UPWARD CLOSURE AND AMALGAMATION IN THE GENERIC MULTIVERSE OF A COUNTABLE MODEL OF SET THEORY

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ABSTRACT. I prove several theorems concerning upward closure and amalgamation in the generic multiverse of a countable transitive model of set theory. Every such model W has forcing extensions $W[c]$ and $W[d]$ by adding a Cohen real, which cannot be amalgamated in any further extension, but some nontrivial forcing notions have all their extensions amalgamable. An increasing chain $W[G_0] \subseteq W[G_1] \subseteq \cdots$ has an upper bound $W[H]$ if and only if the forcing had uniformly bounded essential size in W. Every chain $W \subseteq W[c_0] \subseteq W[c_1] \subseteq \cdots$ of extensions adding Cohen reals is bounded above by $W[d]$ for some W-generic Cohen real d.

Consider a countable transitive model of set theory $W \models \text{ZFC}$ in the context of all its forcing extensions. Several natural questions immediately suggest themselves concerning issues of amalgamation and upward-closure. For example, can any two such models be amalgamated into a common larger model? In other words, is this collection of models upward directed? When can we expect to find upper bounds for increasing chains? In this article, I shall resolve these and other similar

questions. In particular, theorem 4 shows that there are forcing extensions $W[c]$ and $W[d]$, each adding a Cohen real, which have no common further extension; theorem 8 generalizes this non-amalgamation phenomenon to a wide class of other forcing notions, but theorem 9 shows that some forcing notions do always admit amalgamation. For upward closure, theorem 13 shows that every chain

$$W[c_0] \subseteq W[c_1] \subseteq W[c_2] \subseteq \cdots$$

of Cohen-real extensions of W has an upper bound $W[d]$ in another Cohen-real extension, and theorem 12 shows generally that any chain of forcing extensions has an upper bound if and only if the forcing was uniformly bounded in essential size.

In order to make a self-contained presentation, this article includes several results adapted from my previous joint work with Gunter Fuchs and Jonas Reitz [FHR15, §2], as well as some joint work with Giorgio Venturi.

1. THE GENERIC MULTIVERSE

Before presenting the results, let me briefly place the work into a somewhat broader context, which furthermore has connections with issues in the philosophy of set theory. Namely, the forcing extensions of a fixed model of set theory W form an upward oriented cone in what is called the *generic multiverse* of W, which is the collection of all models that one can reach from W by iteratively moving either to a forcing extension or a ground model, in each case by set forcing in the relevant model. Thus, every model M in the generic multiverse of W is reachable by a zig-zag path of models, where at each step we take either a forcing extension or a ground. The generic multiverse of W itself can be viewed as a small part, a local neighborhood, of any of the much larger collections of models that express fuller multiverse conceptions. For example, one could look at the class-forcing multiverse, arising by closing W under class forcing extensions and grounds, or the pseudo-ground multiverse, obtained by closing under pseudo-grounds, or the multiverse arising by closing under arbitrary extensions and inner models, and so on.

These collections of models, each a toy multiverse, if you will, offer various mathematically precise contexts in which one may investigate multiverse issues. Questions that begin philosophically, perhaps concerning the nature of what one might imagine as the full actual multiverse—the multiverse in which our (current) set-theoretic universe is one amongst many other set-theoretic worlds instantiating all

the various alternative concepts of set that we might adopt—are transformed into analogous but mathematically precise questions in the toy multiverses, and we may hope to settle them. In this way, philosophical contemplation becomes mathematical investigation, and each toy multiverse serves as a proxy for the full actual multiverse.[1] This article is an instance of the process: by presenting the mathematical solutions to several natural questions about closure and amalgamation in the case of the generic multiverse of a given countable transitive model of set theory W, we hope to gain insight about what might be true in the multiverse of V.

One may view the generic multiverse of W as a Kripke model of possible worlds, connected by the forcing extension and ground model relations as a notion of accessibility, and this perspective leads one to consider the modal logic of forcing (see [HL08, HL13, Ham03]). An open question arising from that work is the following:

Question 1. *Does the inclusion relation coincide with the ground-model/forcing-extension relation in the generic multiverse? That is, if M is in the generic multiverse of W and $W \subseteq M$, must M be a forcing extension of W?*

A related open question concerns downward directedness:

Question 2. *If M and N have a common forcing extension, must they have a common ground model?*

In other words, if $M[G] = N[H]$ for M-generic $G \subseteq \mathbb{P} \in M$ and N-generic $H \subseteq \mathbb{Q} \in N$, then must there be model W such that $M = W[g]$ and $N = W[h]$ both arise as forcing extensions of W? The *downward directed grounds hypothesis* (DDG) is the axiom asserting that any two ground models of the universe have a common deeper ground. Although it may appear to involve a second-order quantifier, over grounds, in fact this axiom is first-order expressible in the language of set theory, using the uniform definition of the ground models (see [FHR15]). Indeed, there is an indexed parameterizaton W_r for all sets r of all the ground models of V by set forcing, and so one may also formulation the set-directed strengthening of the DDG, which asserts that for any set I, there is a ground model W_s contained in every W_r for $r \in I$.

The two questions are connected by the following fact.

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[1] I discuss this proxy idea further in the final parts of [Ham14], but see also [Ham12, Ham11, Ham09].

Theorem 3. *If the downward directed grounds hypothesis holds through-out the generic multiverse of W, then inclusion coincides with the ground-model/forcing-extension relation in that generic multiverse.*

Proof. The DDG assumption implies that whenever one has a ground model of a forcing extension, then it is also a forcing extension of a ground model. Thus, the DDG in the generic multiverse of W implies that the zig-zag paths need never go up and then down, that is, from a model M up to a forcing extension $M[G]$ and then down to a ground model $N \subseteq M[G]$, because since N is a ground of $M[G]$ there is some N-generic filter $H \subseteq \mathbb{Q} \in N$ for which $N[H] = M[G]$, and so by the DDG there is a common ground $U \subseteq M \cap N$ such that $M = U[g]$ and $N = U[h]$. So one could have gotten from M to N by going down to U, and then up to $U[h] = N$. Thus, the generic multiverse of W consists of the forcing extensions $U[g]$ of the grounds U of W. And if one such model $U[g]$ is contained in another $U[h]$, then $U \subseteq U[g] \subseteq U[h]$, so that $U[g]$ is an intermediate ZFC model between a ground model U and a forcing extension $U[h]$. It now follows by the intermediate model theorem (see [Jec03, cor. 15.43], also [FHR15, fact 11]) that $U[g]$ is a ground of $U[h]$ by a quotient of the forcing giving rise to $U \subseteq U[h]$. $\square$

Toshimichi Usuba has very recently announced a proof of the downward directed grounds DDG hypothesis, and indeed, of the strong DDG in ZFC, which is very welcome and exciting news, and this will settle question 2 as well as question 1, in light of theorem 3.

2. Non-amalgamation in the generic multiverse

Let's begin with the basic non-amalgamation result, which I first heard from W. Hugh Woodin in the 1990s.

Theorem 4 (Woodin, [FHR15, obs. 35]). *If W is any countable transitive model of set theory, then there are W-generic Cohen reals c and d, for which the corresponding forcing extensions $W[c]$ and $W[d]$ have no common extension to a model of set theory with the same ordinals.*

Proof. Let us view Cohen forcing as the partial order $2^{<\omega}$ consisting of finite binary sequences ordered by extension. Enumerate the dense subsets of this forcing in W as $\langle D_n \mid n < \omega \rangle$. Fix a real $z \in 2^\omega$ that could not possibly exist in any forcing extension of W, such as a real coding a relation on ω with order type Ord^W. We shall now build the reals $c, d \in 2^\omega$ in stages, with

$$c = \bigcup_n c_n \qquad d = \bigcup_n d_n,$$

where c_n and d_n are the finite binary initial segments of c and d, respectively, that have been specified by stage n. We undertake the construction in such a way that c_n and d_n are each in D_n, so that the reals individually are W-generic, but from c and d together, we can compute z. To begin, let c_0 be any element of D_0, and let d_0 consist of $|c_0|$ many 0s, followed by a 1 and then the 0th bit $z(0)$ of z, and then extended so that $d_0 \in D_0$. Next, we extend c_0 by padding with 0s until it has the length of d_0, and then a 1, and then the next bit $z(1)$, followed by an extension to c_1 that is in D_1. Now form d_1 by padding d_0 with 0s until the length of c_1, followed by a 1, followed by $z(2)$, and then extended to an element $d_1 \in D_1$. And so on in this same pattern. Since the sequences c_n and d_n are in D_n, it follows that both c and d will be W-generic Cohen reals. But notice that if we have both c and d together, then because the padding with 0s exactly identifies the coding points, we can therefore reconstruct the construction history c_n and d_n and therefore compute the real z. So there can be no common extension $W[c], W[d] \subseteq U$ to a model of ZFC with the same ordinals, as if both $c, d \in U$, then z would also be in U, contrary to our assumption on z. $\qquad\square$

The same argument generalizes to construct three W-generic Cohen reals c, d, e such that any two of them are mutually W-generic, but the three models $W[c], W[d], W[e]$ have no common extension with the same ordinals. And more generally:

Theorem 5 ([FHR15, obs. 36]). *If W is any countable transitive model of set theory, then for any finite n there are distinct W-generic Cohen reals $c_0, \ldots, c_n$, any proper subset of which is mutually W-generic, but the models $W[c_i]$ altogether have no common extension to a model of set theory with the same ordinals as W.*

Proof. Build the reals $c_k = \bigcup_s c_{k,s}$ in stages. Fix a bad real z, which cannot exist in any extension of W with the same ordinals. Enumerate the dense sets D_s of W for the forcing $\text{Add}(\omega, n)$ to add n many Cohen reals. At a given stage, consider each $i \leq n$ in turn and extend all the other $c_{j,s}$ for $j \neq i$ in such a way so as to ensure that $\langle c_j \rangle_{j \neq i}$ is in D_s, and then pad them all with 0s to make them all have the same length; pad $c_{j,s}$ with 0s also to this length, followed by a 1, followed by the next digit of z. In this way, $\langle c_j \rangle_{j \neq i}$ is W-generic for adding n many Cohen reals, so they are mutually W-generic, but the whole collection $\langle c_j \rangle_j$ computes the construction history and also the forbidden real z, and therefore cannot exist in any extension of W with the same ordinals. $\qquad\square$

Let us consider whether this pattern continues into the infinite.

Question 6. *If W is a countable transitive model of set theory, must there be W-generic Cohen reals $\langle c_n \mid n < \omega \rangle$, such that any finitely many of them are mutually W-generic, but the models $W[c_n]$ for all $n < \omega$ have no common extension to a model of set theory with the same ordinals?*

The answer, provided by theorem 12 and more forcefully by theorem 13, is that no, in this infinite case we have amalgamation: every increasing chain of Cohen-real extensions $W[c_n]$ is bounded above by $W[d]$ for some W-generic Cohen real d, so that $W[c_n] \subseteq W[d]$ for all n.

Question 7. *Does the nonamalgamation result of theorem 4 hold for other forcing notions? Does every nontrivial forcing notion exhibit non-amalgamation?*

In other words, if W is a countable transitive model of set theory and $\mathbb{Q} \in W$ is a nontrivial notion of forcing, are there W-generic filters $g, h \subseteq \mathbb{Q}$ such that $W[g]$ and $W[h]$ have no common forcing extension?

The first thing to say about question 7 is that there is a large class of forcing notions $\mathbb{Q}$ for which the non-amalgamation phenomenon occurs. In particular, the reader may observe as an exercise that the proof of theorem 4 directly generalizes to many other forcing notions, such as adding Cohen subsets to higher cardinals, or collapsing cardinals to ω or to another cardinal. Let us push this a bit further, however, by defining that a notion of forcing $\mathbb{Q}$ is *wide*, if it is not $|\mathbb{Q}|$-c.c. below any condition. In other words, $\mathbb{Q}$ is wide, if below every condition $q \in \mathbb{Q}$, there is an antichain in $\mathbb{Q} \restriction q$ of the same size as $\mathbb{Q}$. Many commonly considered forcing notions are wide, and these all exhibit the non-amalgamation phenomenon.

Theorem 8. *If W is a countable transitive model of* ZFC *and $\mathbb{Q}$ is a nontrivial notion of forcing that is wide in W, then:*

(1) *There are W-generic filters $g, h \subseteq \mathbb{Q}$, such that the corresponding forcing extensions $W[g]$ and $W[h]$ have no common extension to a model of set theory with the same ordinals as W.*

(2) *Indeed, for any finite number n, there are W-generic filters $g_0, \ldots, g_n \subseteq \mathbb{Q}$, such that any proper subset of them are mutually W-generic, but there is no common extension of all the $W[g_k]$ to a model of set theory with the same ordinals as W.*

(3) *Furthermore, it suffices for these conclusions that $\mathbb{Q}$ should have merely a nontrivial subforcing notion that is wide.*

Proof. Consider first just the first case, where we have two generic filters. Enumerate the dense subsets of $\mathbb{Q}$ in W as $\langle D_n \mid n < \omega \rangle$, and using the wideness of $\mathbb{Q}$, fix in W an assignment to each condition $q \in \mathbb{Q}$ a maximal antichain $A_q \subseteq \mathbb{Q} \restriction q$ and an enumeration of it as $\langle q^{(\alpha)} \mid \alpha < |\mathbb{Q}| \rangle$. Fix also an enumeration of $\mathbb{Q}$ in order type $|\mathbb{Q}|$, which we may assume is an infinite cardinal in W. Outside of W, fix a bad real z, which cannot exist in any extension of W to a model of set theory with the same ordinals, such as a real coding the ordinals of W. We shall construct g and h to be the respective filters generated by the descending sequences $p_0 \geq p_1 \geq \cdots$ and $q_0 \geq q_1 \geq \cdots$, choosing $p_n, q_n \in D_n$. Begin with any $p_0, q_0 \in D_0$. If p_n and q_n are defined, then let $\alpha < |\mathbb{Q}|$ be the ordinal for which q_n is the αth element of $\mathbb{Q}$. We first extend p_n to the $(2 \cdot \alpha + z(n))^{\text{th}}$ element of A_{p_n}, thereby coding α and the value of $z(n)$, and then extend further to a condition $p_{n+1} \in D_{n+1}$. Next, on the other side, we extend q_n by picking the βth element of A_{q_n}, where p_{n+1} is the βth element of $\mathbb{Q}$, and then extend further to $q_{n+1} \in D_{n+1}$. In this way, the filters g and h generated respectively by the p_n and q_n will each be W-generic, but in any extension of W that has both g and h, we will be able to recover the map $n \mapsto \langle p_n, q_n, z(n) \rangle$, because if we know p_n, then the way that g meets A_{p_n} determines both q_n and $z(n)$, and the way that h meets A_{q_n} determines p_{n+1}. So any extension of W with both g and h also has z, which by assumption cannot exist in any extension of W with the same ordinals. So $W[g]$ and $W[h]$ are non-amalgamable, as desired.

Just as in theorem 5, the argument generalizes to the case of adding any finite number of W-generic filters $g_0, \ldots, g_n \subseteq \mathbb{Q}$, such that if one omits any one of them, the result is W-generic for $\mathbb{Q} \times \cdots \times \mathbb{Q}$, but the full sequence cannot exist in any extension of W with the same ordinals. One fixes a bad real z, and then enumerates the dense sets for the n-fold product $\mathbb{Q}^n$, extending all but one so as to meet the relevant dense set, extending the excluded condition into its antichain so as to code the information that was just added by extending the other conditions, and then also coding one more bit of z. Omitting any one filter will result in a W-generic product of filters, but if one has all of them, then one can reconstruct the entire construction history and therefore also z.

Finally, let us suppose merely that $\mathbb{Q}$ has a subforcing notion $\mathbb{Q}_0 \subseteq \mathbb{Q}$ that is wide. By what we have proved already, we may find $g_0, h_0 \subseteq \mathbb{Q}_0$ which are W-generic for $\mathbb{Q}_0$, but are non-amalgamable over W. Next, we may find $W[g_0]$-generic and $W[h_0]$-generic filters g/g_0 and h/h_0, respectively, for the quotient forcing. It follows that $W[g]$ and $W[h]$ are

non-amalgamable, since any extension of them would also amalgamate $W[g_0]$ and $W[h_0]$. $\qquad\square$

The third claim of the theorem is relevant, for example, in the case of the Lévy collapse of an inaccessible cardinal κ. This forcing is not wide, because it has size κ and is κ-c.c., but the Lévy collapse does have numerous small wide forcing factors—for example, it adds a Cohen real—and these are sufficient to cause the non-amalgamation phenomenon.

Meanwhile, the answer to the second part of question 7 is negative, because some forcing notions can always amalgamate their generic filters. Specifically, let us define that a forcing notion $\mathbb{Q}$ exhibits *automatic mutual genericity over* W, if whenever $g, h \subseteq \mathbb{Q}$ are distinct W-generic filters, then they are mutually generic, so that $g \times h$ is W-generic for $\mathbb{Q} \times \mathbb{Q}$. In this case, both $W[g]$ and $W[h]$ would be contained in $W[g \times h]$, which would be a forcing extension of W amalgamating them. Internalizing the concept to ZFC, let us define officially that a forcing notion $\mathbb{Q}$ exhibits automatic mutual genericity, if in every forcing extension of V, any two distinct V-generic filters $G, H \subseteq \mathbb{Q}$ are mutually V-generic for $\mathbb{Q}$. (This is first-order expressible in the language of set theory.) It is easy to see that if $\mathbb{Q}$ has the property that whenever $p \perp q$ and p forces that $\dot{D}$ is dense below $\check{q}$, then there is a set D in the ground model that is dense below q, and a strengthening $p' \leq p$ such that p' forces $\check{D} \subseteq \dot{D}$, then $\mathbb{Q}$ exhibits automatic mutual genericity over the ground model. This is a rigidity concept, since if $\mathbb{Q}$ has nontrivial automorphisms, or even if two distinct cones in $\mathbb{Q}$ are forcing equivalent, then clearly it cannot exhibit automatic mutual genericity, since mutually generic filters are never isomorphic by a ground-model isomorphism.

Theorem 9. *If $\Diamond$ holds, then there is a notion of forcing that exhibits automatic mutual genericity. If there is a transitive model of* ZFC, *then there is one W with a notion of forcing $\mathbb{Q}$, such that any two distinct W-generic filters $g, h \subseteq \mathbb{Q}$ are mutually generic and hence amalgamable by $W[g \times h]$.*

Proof. By [FH09, thm. 2.6], it follows that $\Diamond$ implies that there is a Suslin tree T on ω_1 that is *Suslin off the generic branch*, in the sense of [FH09, def. 2.2], which means that after forcing with T, which adds a generic branch $b \subseteq [T]$, the tree remains Suslin below any node that is not on b. (A generic Suslin tree also has this property; see [FH09, thm. 2.3].) If a tree is Suslin off the generic branch, then it must also have the unique branch property—forcing with it adds exactly one

branch—since a second branch would contradict the Suslinity of that part of the tree, and thus, this property is a strong form of rigidity. But more, such a tree used as a forcing notion exhibits automatic mutual genericity. To see this, suppose that $g, h \subseteq T$ are distinct V-generic filters for this forcing, individually. Let $p \in T$ be a node of the tree that lies on h, but not g. Since the tree was Suslin off the generic branch in V, it follows that T_p, the part of T consisting of nodes comparable with p, is a Suslin tree in $V[g]$. Thus, every antichain of T_p in $V[g]$ is refined by a level of the tree. Since h is a cofinal branch through T_p, it follows that h meets every level of the tree and hence also every antichain in $V[g]$. So h is $V[g]$-generic and thus they are mutually generic.

For the second claim, if there is a countable transitive model of ZFC, then there is one W satisfying $\Diamond$, which therefore has a tree that is Suslin off the generic branch. Thus, any two distinct W-generic filters $g, h \subseteq \mathbb{Q}$ are mutually generic and so $W[g]$ and $W[h]$ are amalgamated by $W[g \times h]$, which is a forcing extension of W. $\square$

The proof of theorem 9 shows that it is relatively consistent with ZFC that there is a forcing notion exhibiting automatic mutual genericity and hence supporting amalgamation, but the argument doesn't settle the question of whether such kind of forcing exists in every model of set theory.

Question 10. *Is it consistent with* ZFC *that there is no forcing notion with automatic mutual genericity?*

Note that there are other weaker kinds of necessary amalgamation. For example, if c is a W-generic Cohen real and $A \subseteq \omega_1^W$ is W-generic for the forcing to add a Cohen subset of ω_1, then c and A are mutually generic, because the forcing to add A is countably closed in W and therefore does not add new antichains for the forcing $\mathrm{Add}(\omega, 1)$ to add c. This phenomenon extends to many other pairs of forcing notions $\mathbb{P}$ and $\mathbb{Q}$, such that any W-generic filters $g \subseteq \mathbb{P}$ and $h \subseteq \mathbb{Q}$ are necessarily mutually generic.

3. UPWARD CLOSURE IN THE GENERIC MULTIVERSE

Let us turn now to the question of upward closure. Suppose that we have a countable increasing chain of forcing extensions

$$W \subseteq W[G_0] \subseteq W[G_1] \subseteq W[G_2] \subseteq \cdots,$$

where W is a countable transitive model of set theory.

Question 11. *Under which circumstances may we find an upper bound, a forcing extension $W[H]$ for which $W[G_n] \subseteq W[H]$ for all $n < \omega$?*

The question is answered by theorem 12, which provides a necessary and sufficient criterion. It is easy to see several circumstances where there can be no such upper bound. For example, if the extensions $W[G_n]$ collapse increasingly large initial segments of W, in such a way that every cardinal of W is collapsed in some $W[G_n]$, then obviously we cannot find an extension of W to a model of ZFC with the same ordinals as W. It is also easy to see that in general, we cannot require that $\langle G_n \mid n < \omega \rangle \in W[H]$; this is simply too much to ask. For example, if every G_n is a W-generic Cohen real, then we could flip the initial bits of each G_n in such a way that the resulting infinite sequence $\langle G_n \mid n < \omega \rangle$ was coding an arbitrary real z, even though such a change would not affect the models $W[G_n]$, since each G_n individually was changed only finitely. This issue is discussed at length in [FHR15].

Following ideas in [HLL15], let us define that the *forcing degree* of a forcing extension $W \subseteq W[H]$, where $H \subseteq \mathbb{P} \in W$ is W-generic, is the smallest size in W of the Boolean completion of a forcing notion $\mathbb{Q} \in W$ for which there is a W-generic filter $G \subseteq \mathbb{Q}$ for which $W[G] = W[H]$. Thus, the forcing degree of a forcing extension is the smallest size of a complete Boolean algebra realizing that extension as a forcing extension.

Theorem 12. *Suppose that W is a countable transitive model of* ZFC *and that*

$$W \subseteq W[G_0] \subseteq W[G_1] \subseteq W[G_2] \subseteq \cdots \subseteq W[G_n] \subseteq \cdots$$

is an increasing chain of forcing extensions $W[G_n]$, where $G_n \subseteq \mathbb{Q}_n$ is W-generic. Then the following are equivalent:

(1) *The chain is bounded above by a forcing extension $W[H]$, for some forcing notion $\mathbb{Q} \in W$ and W-generic filter $H \subseteq \mathbb{Q}$.*

(2) *The forcing degrees of the extensions $W \subseteq W[G_n]$ are bounded in W*

Proof. $(2 \to 1)$. This direction is essentially [FHR15, thm. 39], but I shall sketch the argument. Let us first handle the case of product forcing, rather than iterated forcing, the case for which we have a tower with the form

$$W \subseteq W[g_0] \subseteq W[g_0][g_1] \subseteq W[g_0][g_1][g_2] \subseteq \cdots,$$

where the $g_n \subseteq \mathbb{P}_n \in W$ are finitely mutually generic over W, and the $\mathbb{P}_n$ are uniformly bounded in size by a cardinal γ in W. Let $\theta > \gamma$ be a sufficiently large regular cardinal in W so that we may enumerate $\langle \mathbb{R}_\alpha \mid \alpha < \theta \rangle$ in W all the possible forcing notions in W of size at most γ, up to isomorphism, with unbounded repetition. Let $\mathbb{R} = \prod_\alpha \mathbb{R}_\alpha$ be

the finite support product. This forcing has the γ^+-chain condition. Let $H \subseteq \mathbb{R}$ be any $\bigcup_n W[g_0 \times \cdots \times g_n]$-generic filter. Select a cofinal sequence $\langle \theta_n \mid n < \omega \rangle$ converging to θ, for which $\mathbb{R}_{\theta_n} = \mathbb{P}_n$, and modify the filter H to use g_n at coordinate θ_n instead of what H had there. If H^* is the new filter, then $H(\theta_n) = g_n$, but at all other coordinates it agrees with H. Since $\mathbb{R}$ is γ^+-c.c., it follows that any maximal antichain for $\mathbb{R}$ in W has bounded support, and thus interacts with only finitely many of the coordinates θ_n upon which we performed surgery. But H is mutually generic with those finitely many g_n, and so that finite amount of surgery will preserve genericity. So $W[H^*]$ is a forcing extension of W, and every $g_n \in W[H^*]$ by construction. So $W[g_0 \times \cdots \times g_n] \subseteq W[H^*]$, as desired. For the general case, where we have iterated forcing rather than product forcing, consider a tower $W \subseteq W[G_0] \subseteq W[G_1] \subseteq \cdots$, where each $G_n \subseteq \mathbb{Q}_n \in W$ is W-generic and the $\mathbb{Q}_n$ are bounded in size. By collapsing the bound, and furthermore using a filter g for the collapse that is not only W-generic, but also $W[G_n]$-generic for every n—this is possible because there are still only countably many dense sets altogether in $\bigcup_n W[G_n]$—we produce a larger tower $W \subseteq W[g] \subseteq W[g][G_0] \subseteq W[g][G_1] \subseteq \cdots$, where now the forcing $\mathbb{Q}_n$ is countable in $W[g]$ and thus isomorphic to the forcing to add a Cohen real there. By quotient forcing, we may therefore view this larger tower as $W \subseteq W[g] \subseteq W[g][c_0] \subseteq W[g][c_0][c_1] \subseteq W[g][c_0][c_1][c_2] \subseteq \cdots$, where $W[g][G_n] = W[g][c_0 \times \cdots \times c_n]$. Thus, we have reduced to the case of product forcing, for which we have already explained how to find an upper bound $W[H]$ containing every $W[g][c_o \times \cdots \times c_n]$ and hence also every $W[G_n]$ in the original tower.

$(1 \to 2)$. This direction is similar to [HLL15, lemma 23], which was used in the context of the modal logic of forcing to show that the value of the forcing degree of a model over a fixed ground model is a *ratchet*, which is to say, that it can be made larger, but never smaller, with further forcing. Suppose that we have a tower $W \subseteq W[G_0] \subseteq W[G_1] \subseteq \cdots$, which is bounded above by the forcing extension $W[H]$, where $H \subseteq \mathbb{Q} \in W$ is W-generic. Since $W \subseteq W[G_n] \subseteq W[H]$, it follows by the intermediate model result of [Jec03, lemma 15.43] that there is a complete subalgebra $\mathbb{C} \subseteq \mathbb{B}$, where $\mathbb{B}$ is the Boolean completion of $\mathbb{Q}$ in W, such that $W[G_n] = W[G']$ for some W-generic filter $G' \subseteq \mathbb{C}$. Thus, the forcing degree of the extension $W \subseteq W[G_n]$ is bounded by the size of $|\mathbb{B}|^W$, and this does not depend on n. So the extensions have uniformly bounded forcing degrees over W. $\qquad\qquad\square$

In the result of theorem 12, the upper bound of $W[H]$ provided by the proof involves possibly collapsing a lot of cardinals, but we might

not want to do that. For example, in question 6 we have a tower of extensions

$$W \subseteq W[c_0] \subseteq W[c_1] \subseteq W[c_2] \subseteq \cdots ,$$

where each c_n is a W-generic Cohen real, and we'd like to know whether we can find an upper bound also of this form. A close inspection of the proof of $(2 \to 1)$ in theorem 12 shows that we can dispense with the collapse forcing in this case, but the rest of the argument involves an uncountable product $\mathbb{R}$ of Cohen-real forcing; we can actually use $\mathrm{Add}(\omega, \omega_1)$ in that argument for this case. So the proof does not directly produce an upper bound in the form $W[d]$ of adding a single Cohen real.

Nevertheless, it is true that we can find an upper bound of this form, and this is what I shall now prove in theorem 13. Specifically, I claim that the collection of models $M[c]$ obtained by adding an M-generic Cohen real c over a fixed countable transitive model of set theory M is upwardly countably closed, in the sense that every increasing countable chain has an upper bound. I proved this theorem with Giorgio Venturi back in 2011 in a series of conversations at the Young Set Theory Workshop in Bonn and continuing at the London summer school on set theory.

Theorem 13. *For any countable transitive model $W \models \mathrm{ZFC}$, the collection of all forcing extensions $W[c]$ by adding a W-generic Cohen real is upward-countably closed. That is, for any countable tower of such forcing extensions*

$$W \subseteq W[c_0] \subseteq W[c_1] \subseteq \cdots \subseteq W[c_n] \subseteq \cdots ,$$

we may find a W-generic Cohen real d such that $W[c_n] \subseteq W[d]$ for every natural number n.

Proof. Suppose that we have such a tower of forcing extensions $W[c_0] \subseteq W[c_1] \subseteq W[c_2]$, and so on. Note that if $W[b] \subseteq W[c]$ for W-generic Cohen reals b and c, then $W[c]$ is a forcing extension of $W[b]$ by a quotient of the Cohen-real forcing. But since the Cohen forcing itself has a countable dense set, it follows that all such quotients also have a countable dense set, and so $W[c] = W[b][b_1]$ for some $W[b]$-generic Cohen real b_1. Thus, we may view the tower as having the form:

$$W[b_0] \subseteq W[b_0 \times b_1] \subseteq \cdots \subseteq W[b_0 \times b_1 \times \cdots \times b_n] \subseteq \cdots ,$$

where now it follows that any finite collection of the reals b_i are mutually W-generic.

Of course, we cannot expect in general that the real $\langle b_n \mid n < \omega \rangle$ is W-generic for $\mathrm{Add}(\omega, \omega)$, since this real may be very badly behaved.

For example, the sequence of first-bits of the b_n's may code a very naughty real z, which cannot be added by forcing over W at all. So in general, we cannot allow that this sequence is added to the limit model $W[d]$. (See further discussion in my blog post [Ham15b].) We shall instead undertake a construction by making finitely many changes to each real b_n, resulting in a real d_n, in such a way that the resulting combined real $d = \oplus_n d_n$ is W-generic for the forcing to add ω-many Cohen reals, which is of course isomorphic to adding just one. To do this, let's get a little more clear with our notation. We regard each b_n as an element of Cantor space 2^ω, that is, an infinite binary sequence, and the corresponding filter associated with this real is the collection of finite initial segments of b_n, which will be a W-generic filter through the partial order of finite binary sequences $2^{<\omega}$, which is one of the standard isomorphic copies of Cohen forcing. We will think of d as a binary function on the plane $d : \omega \times \omega \to 2$, where the nth slice d_n is the corresponding function $\omega \to 2$ obtained by fixing the first coordinate to be n.

Now, we enumerate the countably many open dense subsets of W for the forcing to add a Cohen real $\omega \times \omega \to 2$ as D_0, D_1, and so on. Now, we construct d in stages. Before stage n, we will have completely specified d_k for $k < n$, and we also may be committed to a finite condition p_{n-1} in the forcing to add ω many Cohen reals. We consider the dense set D_n. We may factor $\text{Add}(\omega, \omega)$ as $\text{Add}(\omega, n) \times \text{Add}(\omega, [n, \omega))$. Since $d_0 \times \cdots \times d_{n-1}$ is actually W-generic (since these are finite modifications of the corresponding b_k's, which are mutually W-generic, it follows that there is some finite extension of our condition p_{n-1} to a condition $p_n \in D_n$, which is compatible with $d_0 \times \cdots \times d_{n-1}$. Let d_n be the same as b_n, except finitely modified to be compatible with p_n. In this way, our final real $\oplus_n d_n$ will contain all the conditions p_n, and therefore be W-generic for $\text{Add}(\omega, \omega)$, yet every b_n will differ only finitely from d_n and hence be an element of $W[d]$. So we have $W[b_0] \cdots [b_n] \subseteq W[d]$, and we have found our upper bound. $\qquad\square$

Notice that the real d we construct is not only W-generic, but also $W[c_n]$-generic for every n.

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