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# Alternate Correspondences Between Two Optimal Stopping Problems

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**Abstract**

Let $(B, \mathbf{p})$ denote the best-choice problem and $(D, \mathbf{p})$ the duration problem when the total number $N$ of objects is a bounded random variable with prior $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ for a known upper bound $n$. Gnedin (2005) discovered the correspondence relation between these two different optimal stopping problems. That is, for any given prior $\mathbf{p}$, there exists another prior $\mathbf{q}$, such that $(D, \mathbf{p})$ is equivalent to $(B, \mathbf{q})$. This paper, motivated by his discovery, attempts to find the alternate correspondence $\{\mathbf{p}^{(m)}, m \geq 0\}$ such that $(D, \mathbf{p}^{(m-1)})$ is equivalent to $(B, \mathbf{p}^{(m)})$ for all $m \geq 1$, starting with $\mathbf{p}^{(0)} = (0, \ldots, 0, 1)$. To be more precise, the duration problem is distinguished into $(D_1, \mathbf{p})$ or $(D_2, \mathbf{p})$, referred to as MODEL 1 or MODEL 2, depending on whether the planning horizon is $N$ or $n$. The aforementioned problem is MODEL 1. For MODEL 2 as well, we can find the similar alternate correspondence $\{\mathbf{p}^{[m]}, m \geq 0\}$. We treat both the no-information model and the full-information model and examine the limiting behaviors of their optimal rules and optimal values related to the alternate correspondences as $n \to \infty$.

## 1 Introduction

In the *best-choice problem*, a version of the secretary problem (see, e.g. Samuels (1991) for a survey), a fixed known number $n$ of rankable objects appear one at a time in random order with all $n!$ permutations equally likely (1 being the best and $n$ the worst). Each time an object appears, we must decide either to select it and stop observing or reject it and continue observing, based on the relative rank of the current object with respect to its predecessors. The objective is to find a stopping rule that maximizes the probability of selecting the best of all $n$ objects. Evidently we can confine our selection to a relatively best object. For ease of description, we often call an object *candidate*, if it is relatively best upon arrival.

As a different version of the secretary problem, Ferguson et al.(1992) considered the optimal stopping problem, referred to as the *duration problem*, in the same framework described above. We only select a candidate.

Define $T_k$ as the time of the first candidate after $k$ if there is one, and as $n+1$ if there is none. Then the duration of holding a candidate selected at time $k$ is $(T_k - k)/n$ (division by $n$ is for normalization) and the objective of this problem is to find a stopping rule that maximizes the expected duration of holding a candidate.

These two classical problems with fixed horizon $n$ were generalized to the problems with random horizon by introducing uncertainty about the number $N$ of the actually available objects. The selection must be made by time $N$. Throughout this paper, we simply assume the random variable $N$, independent of the arrival order of the objects, to be bounded by $n$ and have a prior distribution $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, where $p_k = P\{N = k\}$ are such that $\sum_{k=1}^n p_k = 1$ and $p_n > 0$. It is also assumed $n \geq 2$, unless otherwise specified. Define, for later use, $\boldsymbol{\pi}=(\pi_1, \pi_2, \ldots, \pi_n)$ and $\boldsymbol{\sigma}=(\sigma_1, \sigma_2, \ldots, \sigma_n)$ as functions of $\mathbf{p}$ where, for $1 \leq k \leq n$,

$$\begin{aligned} \pi_k &= p_k + p_{k+1} + \cdots + p_n \\ \sigma_k &= \pi_k + (n-k)p_k. \end{aligned}$$

When $N$ has a prior $\mathbf{p}$, we simply denote the best-choice problem by $(B, \mathbf{p})$ and the duration problem by $(D, \mathbf{p})$. Though the objective of $(B, \mathbf{p})$ is to select the best of all $N$ objects, $(D, \mathbf{p})$ can be distinguished into two problems denoted by $(D_1, \mathbf{p})$ or $(D_2, \mathbf{p})$, depending on whether the final stage of the planning horizon is $N$ or $n$. That is, the duration of a candidate selected at time $k$ is defined as $(T_k - k)/n$ as before, but if no further candidate appears by time $N$, $T_k$ is interpreted as $N+1$ for $(D_1, \mathbf{p})$ and as $n+1$ for $(D_2, \mathbf{p})$. The problem $(D_k, \mathbf{p}), k = 1, 2$, is referred to as the MODEL $k$ of the duration problem. We denote the optimal values of the problems $(B, \mathbf{p})$ and $(D_k, \mathbf{p})$ by $v_n^B(\mathbf{p})$ and $v_n^{D_k}(\mathbf{p})$ respectively to make explicit the dependence on $n$ and $\mathbf{p}$. Note that the classical problems occur if $N$ degenerates to $n$ (i.e. $\mathbf{p} = (0, \ldots, 0, 1)$), in which case there exists no difference between $(D_1, \mathbf{p})$ and $(D_2, \mathbf{p})$.

A stopping rule is said to be *simple* if, for a given positive integer $s_n (\leq n)$, it passes over the first $s_n - 1$ objects and stops with the first candidate if any. The value $s_n$ is referred to as the *critical* number of the simple rule. It is well known that the optimal rules of the classical problems are simple. However, the form of the optimal rule depends on $\mathbf{p}$, implying that it is not necessarily simple. Define

$$\mathbf{p}^{(0)} = (0, \ldots, 0, 1), \quad \mathbf{p}^{(1)} = (\frac{1}{n}, \ldots, \frac{1}{n}, \frac{1}{n})$$

as two special priors. $\mathbf{p}^{(0)}$ corresponds to the fixed horizon and $\mathbf{p}^{(1)}$ to the random horizon with $N$ uniform on $\{1, 2, \ldots, n\}$. Then Ferguson et al. (1992) recognized the equivalence between $(D_1, \mathbf{p}^{(0)})$ and $(B, \mathbf{p}^{(1)})$. Extending this equivalence, Gnedin (2005) discovered the further correspondences between the MODEL 1 of the duration problem and the best-choice problem . According to the former half of Proposition 4.1 and Corollary 4.1 of Gnedin (2005), this discovery can be stated as Proposition 1.1 for our framework.

**Proposition 1.1.** (Equivalence between $(D_1, \mathbf{p})$ and $(B, \mathbf{q})$.) *For any given prior* $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ *on* $N$*, there exists another prior* $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ *defined from* $\mathbf{p}$ *as*

$$\mathbf{q} = \frac{\boldsymbol{\pi}}{E[N]}$$

*such that* $(D_1, \mathbf{p})$ *is equivalent to* $(B, \mathbf{q})$ *in the sense that these two problems have the same optimal rules. Moreover, their optimal values only differ by the factor* $\frac{E[N]}{n}$*, namely,*

$$v_n^{D_1}(\mathbf{p}) = \frac{E[N]}{n} v_n^B(\mathbf{q}).$$

**Proposition 1.2.** (Equivalence between $(D_2, \mathbf{p})$ and $(B, \mathbf{q})$.) *For any given prior* $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ *on* $N$*, there exists another prior* $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ *defined from* $\mathbf{p}$ *as*

$$\mathbf{q} = \frac{\boldsymbol{\sigma}}{n}$$

*such that* $(D_2, \mathbf{p})$ *is equivalent to* $(B, \mathbf{q})$ *in the sense that these two problems have the same optimal rules and the same optimal values. Thus,*

$$v_n^{D_2}(\mathbf{p}) = v_n^B(\mathbf{q}).$$

We omit the proof of Proposition 1.2, because it goes quite parallel to that of Proposition 4.1 of Gnedin (2005).

The set $\left\{\mathbf{p}^{[m]} = \left(p_1^{[m]}, p_2^{[m]}, \ldots, p_n^{[m]}\right), m \geq 0\right\}$ with $\mathbf{p}^{[0]} = (0, \ldots, 0, 1)$ is referred to as the *alternate correspondence of type 2*, if $(D_2, \mathbf{p}^{[m-1]})$ is equivalent to $(B, \mathbf{p}^{[m]})$ for all $m \geq 1$.

Our main concerns are to find the explicit expressions of the two alternate correspondences (the type $k$ corresponds to the MODEL $k$ of the duration problem, where $k = 1, 2$). We also examine the optimal rules and the optimal values related to these alternate correspondences. Of further interest is to derive the limiting values of $v_n^B(\mathbf{p}^{(m)})$ and $v_n^B(\mathbf{p}^{[m]})$ as $n \to \infty$. These are discussed in Section 2.1.

In contrast to the above no-information model in which the observations are the relative ranks of the objects, the *full-information* model is the problem in which the observations are the true values of $N$ objects $X_1, X_2, \ldots, X_N$, assumed to be i.i.d. random variables from a known continuous distribution, taken without loss of generality to be the uniform distribution on the interval $[0, 1]$. $N$ is also assumed to be independent of

$X_1, X_2, \ldots$. Let $L_k = \max\{X_1, X_2, \ldots, X_k\}$ and call the *kth* object (or $X_k$) a candidate if it is a relative maximum, i.e. $X_k = L_k$. Consider a class of stopping rules of the form

$$\tau_N(\mathbf{a}) = \min\{k : X_k = L_k \geq a_k\} \wedge N,$$

where $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ is a given sequence of thresholds satisfying the monotone condition $1 \geq a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$. This rule is said to be a *monotone rule* (with thresholds $\mathbf{a}$). It is well known that the optimal rules of the classical problems, i.e. problems with fixed horizon, are monotone (cf. Gilbert and Mosteller (1966) and Ferguson et al. (1992)). This full-information model is considered in Section 3.

## 2 Alternate correspondences

### 2.1 No-information model

If the optimal rule of $(B, \mathbf{p})$ $((D_k, \mathbf{p}), k = 1, 2)$ is simple, we denote its (optimal) critical number by $s_n^B(\mathbf{p})$ $(s_n^{D_k}(\mathbf{p}))$. The following theorem gives the main results concerning the alternate correspondences.

**Theorem 2.1.**
(a) (Alternate correspondence of type 1.) *Let* $\mathbf{p}^{(0)} = (0, \ldots, 0, 1)$ *and*

$$p_k^{(m)} = \frac{\binom{n+m-1-k}{m-1}}{\binom{n+m-1}{m}}, \quad 1 \leq k \leq n,$$

*for* $m \geq 1$. *Then* $\{\mathbf{p}^{(m)}, m \geq 0\}$ *is the alternate correspondence of type 1. The optimal rule is simple for both* $(D_1, \mathbf{p}^{(m)})$ *and* $(B, \mathbf{p}^{(m)})$, *and we have the following relations for* $m \geq 0$.

$$\begin{aligned} s_n^{D_1}(\mathbf{p}^{(m)}) &= s_n^B(\mathbf{p}^{(m+1)}), \\ v_n^{D_1}(\mathbf{p}^{(m)}) &= \frac{m+n}{(m+1)n} v_n^B(\mathbf{p}^{(m+1)}). \end{aligned}$$

(b) (Alternate correspondence of type 2.) *Let* $\mathbf{p}^{[0]} = (0, \ldots, 0, 1)$ *and*

$$p_k^{[m]} = \left(\frac{n-k+1}{n}\right)^m - \left(\frac{n-k}{n}\right)^m, \quad 1 \leq k \leq n,$$

*for* $m \geq 1$. *Then* $\{\mathbf{p}^{[m]}, m \geq 0\}$ *is the alternate correspondence of type 2. The optimal rule is simple for both* $(D_2, \mathbf{p}^{[m]})$ *and* $(B, \mathbf{p}^{[m]})$, *and we have the following relations for* $m \geq 0$.

$$\begin{aligned} s_n^{D_2}(\mathbf{p}^{[m]}) &= s_n^B(\mathbf{p}^{[m+1]}), \\ v_n^{D_2}(\mathbf{p}^{[m]}) &= v_n^B(\mathbf{p}^{[m+1]}). \end{aligned}$$

**Remark 2.1.** The two random variables $N^{(m)}$ and $N^{[m]}$ can be related to sampling balls from an urn without replacement and with replacement respectively. Suppose that there exists an urn containing $n+m-1$ balls numbered $1, 2, \ldots, n+m-1$. We draw $m$ balls randomly from the urn without replacement. Then $N^{(m)}$ denotes the smallest of the $m$ numbers drawn. Suppose that there exists an urn containing $n$ balls numbered $1, 2, \ldots, n$. We draw $m$ balls one at a time randomly from the urn with replacement. Then $N^{[m]}$ denotes the smallest of the $m$ numbers drawn.

The following corollary gives the additional limiting relations if we define, for $m \geq 0$,

$$s^{[B,m]} = \lim_{n\to\infty} \frac{s_n^B(\mathbf{p}^{[m]})}{n}, \qquad v^{[B,m]} = \lim_{n\to\infty} v_n^B(\mathbf{p}^{[m]}),$$

$$s^{(D_1,m)} = \lim_{n\to\infty} \frac{s_n^{D_1}(\mathbf{p}^{(m)})}{n}, \qquad v^{(D_1,m)} = \lim_{n\to\infty} v_n^{D_1}(\mathbf{p}^{(m)}),$$

$$s^{[D_2,m]} = \lim_{n\to\infty} \frac{s_n^{D_2}(\mathbf{p}^{[m]})}{n}, \qquad v^{[D_2,m]} = \lim_{n\to\infty} v_n^{D_2}(\mathbf{p}^{[m]}).$$

**Corollary 2.1.** *We have the following relations for $m \geq 0$.*

(*a*) $s^{[B,m]} = s^{(B,m)}, \quad v^{[B,m]} = v^{(B,m)}$

(*b*) $s^{(D_1,m)} = s^{(B,m+1)}, \quad v^{(D_1,m)} = \frac{1}{m+1} v^{(B,m+1)}$

(*c*) $s^{[D_2,m]} = s^{[B,m+1]}, \quad v^{[D_2,m]} = v^{[B,m+1]}$

(*d*) $s^{[D_2,m]} = s^{(D_1,m)}, \quad v^{[D_2,m]} = (m+1) v^{(D_1,m)}$

The following theorem gives the limiting results. Table 1 gives some numerical values.

**Theorem 2.2.** *For $m \geq 2$, $s^{(B,m)}$ and $v^{(B,m)}$ are calculated as follows.*
(i) *The value of $s^{(B,m)}$ is given as a solution $x \in (0,1)$ to the equation*

$$\frac{1}{2}\log^2 x + (1 + h_{m-1}) \log x + \sum_{j=1}^{m-1} \left( \frac{1 + h_{m-1} - h_{j-1}}{j} \right) (1-x)^j = 0,$$

*where $h_k = \sum_{j=1}^{k} 1/j, k \geq 1$ and $h_0 = 0$.*
(ii) *The value of $v^{(B,m)}$ is given by*

$$v^{(B,m)} = ms \sum_{j=m}^{\infty} \frac{(1-s)^j}{j} = -ms \left( \log s + \sum_{j=1}^{m-1} \frac{(1-s)^j}{j} \right),$$

*where, for easier reading,* $s^{(B,m)}$ *is abbreviated to* $s$.

**Table 1**
Values of $s^{(B,m)}$ and $v^{(B,m)}$ for several $m$.

| | $m$ | | | | | | |
|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 10 |
| $s^{(B,m)}$ | 0.3679 | 0.1353 | 0.0775 | 0.0539 | 0.0412 | 0.0334 | 0.0171 |
| $v^{(B,m)}$ | 0.3679 | 0.2707 | 0.2535 | 0.2469 | 0.2435 | 0.2414 | 0.2372 |

## 3 Full-information model

For the full-information model, the following results give a sufficient condition for the optimal rule to be monotone for each of the three problems and also give, when this condition is met, the explicit expressions for the optimal monotone thresholds and the optimal values.

**Theorem 3.1.**
(a) (Alternate correspondence of type 1.) *The optimal rule is monotone for both* $(D_1, \mathbf{p}^{(m)})$ *and* $(B, \mathbf{p}^{(m)})$, *and we have the following relations for* $m \geq 0$.

$$\begin{aligned} \mathbf{a}^{D_1}(\mathbf{p}^{(m)}) &= \mathbf{a}^B(\mathbf{p}^{(m+1)}), \\ v_n^{D_1}(\mathbf{p}^{(m)}) &= \frac{m+n}{(m+1)n} v_n^B(\mathbf{p}^{(m+1)}), \end{aligned}$$

where $\mathbf{a}^B(\mathbf{p})$ denotes the monotone thresholds corresponding to $\mathbf{p}$.
(b) (Alternate correspondence of type 2.) *The optimal rule is monotone for both* $(D_2, \mathbf{p}^{[m]})$ *and* $(B, \mathbf{p}^{[m]})$, *and we have the following relations for* $m \geq 0$.

$$\begin{aligned} \mathbf{a}^{D_2}(\mathbf{p}^{[m]}) &= \mathbf{a}^B(\mathbf{p}^{[m+1]}), \\ v_n^{D_2}(\mathbf{p}^{[m]}) &= v_n^B(\mathbf{p}^{[m+1]}). \end{aligned}$$

Note that we use the same notations $v_n^B(\mathbf{p})$ and $v_n^{D_k}(\mathbf{p})$ to denote the optimal values for the full-information model if no confusion occurs.

We now consider the limiting optimal values as $n \to \infty$. Let, for $m \geq 0$,

$$\begin{aligned} v^{(B,m)} &= \lim_{n\to\infty} v_n^B(\mathbf{p}^{(m)}), \\ v^{[B,m]} &= \lim_{n\to\infty} v_n^B(\mathbf{p}^{[m]}), \\ v^{(D_1,m)} &= \lim_{n\to\infty} v_n^{D_1}(\mathbf{p}^{(m)}), \\ v^{[D_2,m]} &= \lim_{n\to\infty} v_n^{D_2}(\mathbf{p}^{[m]}). \end{aligned}$$

Then, from Theorem 3.1, we obviously have the following results analogous to Corollary 2.1.

**Corollary 3.1.** *We have the following limiting relations for $m \geq 0$.*

(a) $v^{[B,m]} = v^{(B,m)}$

(b) $v^{(D_1,m)} = \frac{1}{m+1} v^{(B,m+1)}$

(c) $v^{[D_2,m]} = v^{[B,m+1]}$

(d) $v^{[D_2,m]} = (m+1) v^{(D_1,m)}$.

The explicit expressions of $v^{(D_1,m)}$ and $v^{[D_2,m]}$ have been already obtained in Tamaki (2016), i.e. $v^{(D_1,m)}$ appears as $v_m^{(1)}$ in his Theorem 3.2 and $v^{[D_2,m]}$ as $v_m^{(2)}$ in his Theorem 4.2 (the derivation is based on a planar Poisson process approach developed by Gnedin (1996, 2004) or Samuels (2004)). Because of $v^{(B,m)} = m v^{(D_1,m-1)}$ from Corollary 3.1 (b) for $m \geq 1$, if we let

$$I(c) = \int_c^\infty \frac{e^{-x}}{x} dx, \quad J(c) = \int_0^c \frac{e^x - 1}{x} dx$$

and introduce the additional functions

$$\begin{aligned} I_m(c) &= \int_c^\infty \frac{m! e^{-x}}{x^{m+1}} dx, \\ K_m(c) &= \int_0^c \frac{x^m e^x}{m!} dx, \\ L_m(c) &= \int_0^c \frac{m! e^{-x}}{x^{m+1}} K_m(x) dx \end{aligned}$$

for $m \geq 0$, we can give $v^{(B,m)}$ as follows.

**Theorem 3.2.** *For $m \geq 1$, let $c_m$ be a unique root $c$ of the equation*

$$\sum_{k=0}^{m-1} \frac{(-c)^k}{k!} (1 - L_k(c)) = e^{-c} (1 - J(c)).$$

*Then*

$$\begin{aligned} v^{(B,m)} &= m \left( \frac{(m-1)! K_{m-1}(c)}{c^{m-1}} - \frac{c e^c L_{m-1}(c)}{m+c} \right) \left( \frac{c^{m-1} I_{m-1}(c)}{(m-1)!} - \frac{c^m I_m(c)}{m!} \right) \\ &\quad + \frac{m}{m+c} L_{m-1}(c), \end{aligned}$$

*where $c_m$ is abbreviated to $c$. For $m = 0$,*

$$v^{(B,0)} = e^{-c_0} + (e^{c_0} - c_0 - 1) I(c_0) \approx 0.58016,$$

*where* $c_0 \approx 0.80435$ *is a unique root* $c$ *of the equation* $J(c) = 1$.
Table 2 presents some numerical values of $c_m$ and $v^{(B,m)}$.

**Table 2**
Values of $c_m$ and $v^{(B,m)}$ for several $m$.

| | $m$ | | | | | | |
|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 10 |
| $c_m$ | 0.8044 | 2.1198 | 3.6925 | 5.3520 | 7.0411 | 8.7423 | 17.3014 |
| $v^{(B,m)}$ | 0.5802 | 0.4352 | 0.4045 | 0.3926 | 0.3865 | 0.3827 | 0.3753 |

See Tamaki(2016b) for more detail of this paper.

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