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VOLUMES OF CONVEX CORES OF HYPERBOLIC 3–MANIFOLDS

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ABSTRACT. We discuss recent results on volume of the convex core of an incompressible hyperbolic 3 manifold. By work of Brock the volume is related to Weil-Petersson translation distance. Krasnov and Schlenker have shown that the volume is "commensurable" with another quantity – the renormalized volume. We give an exposition of the behavior of both quantities infinitesimally and on globally.

1. INTRODUCTION

For the last thirty years, since the pioneering work of Jorgensen and Thurston in the 70s, the geometry of hyperbolic 3 manifolds have been the focus of abundant research. There are three main reasons for this. Firstly, Thurston proved that unless a compact 3 manifold satisfies a certain obstruction condition, specifically it contains an incompressible torus or a 2 -sphere, then it must be hyperbolic. Secondly, by Mostow-Prasad rigidity a hyperbolic structure of finite volume in dimension 3 is unique and geometric invariants are in fact topological invariants. Thirdly, Thurston and his students have given us a variety of tools, e.g. SNAPPEA, for constructing hyperbolic 3 manifolds and determining these geometric invariants – in particular their volumes. The behavior of volume as the underlying manifold changes has been much studied. It was observed by Thurston and Jorgenson that given an upper bound on volume and a lower bound on injectivity radius there are only finitely many compact hyperbolic manifolds satisfying these bounds. A consequence of Thurston's Dehn surgery theorem is that there such are infinitely many distinct hyperbolic manifolds of volume below the volume of the Whitehead link (or any other cusped manifold) see [14] for a more detailed account. Neumann and Zagier [23] gave a formula for the volume under Dehn filling. More recently volumes have been studied in relation to the conjecture of Kashaev-Murakami [21]. Thus the structure of the set of volumes is rich and quite delicate.

One of Thurston's constructions concerns the mapping torus of a surface diffeomorphism. Let $\Sigma = \Sigma_{g,m}$ be an orientable surface of genus g with m punctures. We will suppose that $3g-3+m \geq 1$ so that Σ admits a Riemannian metric of constant curvature -1, a hyperbolic structure of finite area, which, by Gauss-Bonnet, satisfies Area $\Sigma = 2\pi|\chi(\Sigma)| = 2\pi(2g-2+m)$ with respect to the hyperbolic metric. The isotopy classes of orientation preserving automorphisms of Σ, called mapping classes, were classified into three families by Nielsen and Thurston [27], namely periodic, reducible and pseudo-Anosov. Choose a representative h of a mapping class φ, and consider its mapping

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torus,

$$\Sigma \times [0,1]/(x,1) \sim (h(x),0).$$

Since the topology of the mapping torus depends only on the mapping class φ, we denote its topological type by N_φ. No power of a pseudo- Anosov diffeomorphism preserves a homotopy class of essential closed curves on the surface. This means that the mapping torus does not contain an incompressible torus. Now a theorem of Thurston [28] asserts that N_φ admits a hyperbolic structure iff φ is pseudo-Anosov.

1.1. **Quasi-Fuchsian space the proof.** There are several proofs of this theorem in the litterature. The basic idea is to prove a fixed point theorem for the action of the diffeomorphism on the space of hyperbolic structures on $\tilde{N}_\varphi$ the infinite cyclic covering space of N_φ. The manifold $\tilde{N}_\varphi$ is homeomorphic to a product $\Sigma \times \mathbf{R}$ and to each hyperbolic structure on Σ there is a corresponding hyperbolic structure on $\tilde{N}_\varphi$. By the Uniformization Theorem the hyperbolic structure on Σ induces an isometry between $\tilde{\Sigma}$ and the hyperbolic plane $\mathbf{H}$ the group of deck transformations is identified with a discrete subgroup of the isometries of $\mathbf{H}$. Any inclusion of $\mathbf{H}$ into 3 dimensional hyperbolic space $\mathbf{H}^3$ induces an inclusion of its group of isometries into $isom(\mathbf{H}^3)$ and the quotient of the $\mathbf{H}^3$ by the Γ, image of the deck transformations, is homeomorphic to $\tilde{N}_\varphi$. Any structure obtained in this way is called a *Fuchsian.* Since there is a unique hyperbolic metric in each conformal class the Fuchsian structures are in 1-1 correspondence with the points of $\mathcal{T}$, the *Teichmuller space* of the surface. In fact, the set of all hyperbolic structures on $\tilde{N}_\varphi$ is in 1-1 correspondence with the product $\mathcal{T} \times \mathcal{T}$. One way to see this is to consider the action of the group Γ on $\partial_\infty \mathbf{H}^3$ conformal boundary of $\mathbf{H}^3$. Hyperbolic space is homeomorphic to a disc and its confromal boundary can be identified with the Riemann sphere. The action of Γ extends to $\partial_\infty \mathbf{H}^3$ and is identified with a subroup of conformal automorphisms of the sphere. The *limit set* $\Lambda(\Gamma)$ of Γ, that is the smallest, closed Γ-invariant subset of $\partial_\infty \mathbf{H}^3$, is a topological circle which separates the sphere into two discs $\mathbf{H}^+$ and $\mathbf{H}^-$. The quotient space $\mathbf{H}^\pm/\Gamma$ is a pair of Rieman surfaces $X^\pm$, each homeomorphic to Σ. The conformal structures on X^+ and X^- defines a point in $\mathcal{T} \times \mathcal{T}$. It can be proved that any such structure is *geometrically finite* that is the associated group of deck transformations admits a finite sided fundamental region. Some additional work is needed to prove the so-called Ahlfors-Bers Theorem which asserts that the geometrically finite hyperbolic structures on $\tilde{N}_\varphi$ are in 1-1 correspondence with $\mathcal{T} \times \mathcal{T}$. Historically, because of the approach adopted by Ahlfors-Bers via quasi-conformal deformations of Γ, this set is called *quasi-Fuchsian space* and there is a bijection

$$\begin{array}{ccc} \mathcal{T} \times \mathcal{T} & \to & \{ \text{ geometrically finite hyperbolic structures on } \tilde{N}_\varphi \} \\ (X^+, X^-) & \mapsto & QF((X^+, X^-). \end{array}$$

To avoid complicating the discussion with considerations of non-compact sets we suppose for the instant that Σ is compact. Choose a pseudo-Anosov automorphism φ on Σ and choose a marked Riemann surface $X \in \mathcal{T}$ on the Teichmüller geodesic invariant by φ. The diffeomorphism φ acts naturally on $\mathcal{T}$ by pre-composing φ^{-1} on the marking of $X \in \mathcal{T}$ and consider a family of quasi-Fuchsian manifolds $\{QF(\varphi^{-n}X, \varphi^n X)\,;\, n \in \mathbb{Z}\}$. The intuition behind Thurston's theorem is that, for n is sufficiently large :

- $QF(\varphi^{-n}X, \varphi^n X)$ is "quite close" to the infinite cyclic covering space of N_φ
- the map φ induces a map on $QF(\varphi^{-n}X, \varphi^n X)$ that is "quite close" to being an isometry on a" big subset".

By passing to the limit one should obtain a hyperbolic structure on $\tilde{N}_\varphi$ and an isometry φ_∞ induced by φ such that the quotient space is homeomorphic to N_φ so that it inherits a hyperbolic structure from the covering map.

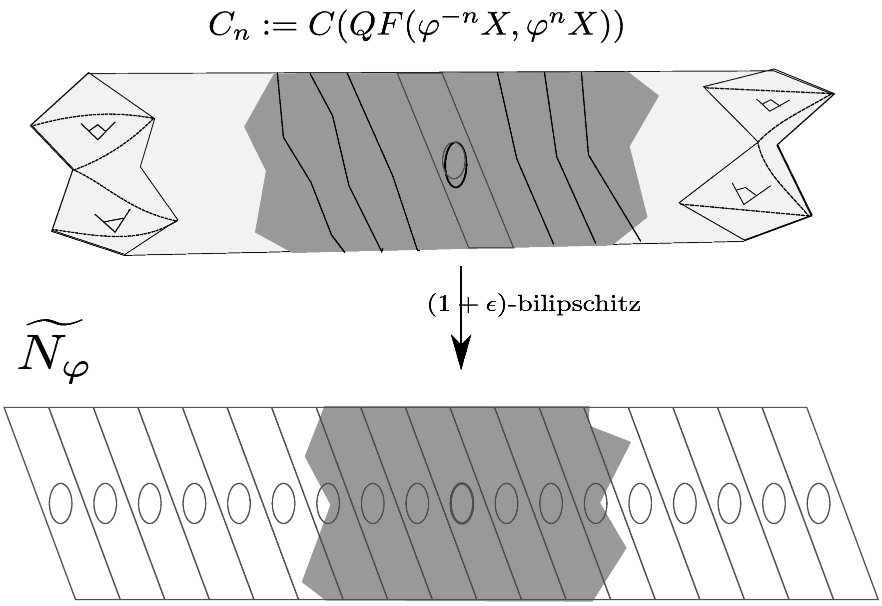

FIGURE 1. The sequence of (convex cores) quasi Fuchsian manifolds is nearly isometric to a big subset of the infinite cyclic cover $\tilde{N}_\varphi$.

1.2. **Geometry of** $QF((X^+, X^-)$. We see from the above that, even if we are only really interested in compact hyperbolic 3 manifolds, it is natural to study hyperbolic structures on non-compact manifolds. The topology of $QF((X^+, X^-)$ is very simple since it is just the product of a surface with $\mathbf{R}$. However, the geometry of $QF((X^+, X^-)$ varies subtly with the conformal structures (X^+, X^-). When Thurston and Jorgensen began their work little was known and progress was slow until several breakthroughs in the early 2000s and since then our understanding was greatly improved. The main result to be cited here is Brock, Canary and Minsky's solution of the Ending Lamination Conjecture. The ending lamination conjecture is, in a sense, a generalization of the Mostow Rigidity Theorem to hyperbolic manifolds of infinite volume. Mostow rigidity theorem asserts that the fundamental group determines the manifold up to isometry when it is of finite volume As we have seen above, for $QF((X^+, X^-)$ the fundamental group is not enough to determine the manifold: one also needs to know the conformal structures on the surfaces (X^+, X^-). Another, more subtle, case is that of non geometrically finite structures the limit set of Γ does not separate $\partial_\infty \mathbf{H}^3$ into two discs so that we do not have a pair of conformal structures $X^\pm$ but just one or perhaps none if Γ is a space filling curve – for example for the group Γ of deck transformations of $\tilde{N}_\varphi$. Thurston introduced the notation of *ending* *laminations* for such groups and conjectured that these would determine the hyperbolic structure.

Very roughly an ending lamination encodes which closed curves are "short" in the an end of a non-geometrically finite 3 manifold. A geodesic lamination on φ can be thought of as the limit of closed simple curves γ_n. For example, consider the hyperbolic structure on $\tilde{N}_\varphi$ and pick any closed simple curve γ_0, the systole for example, and define

$$\gamma_n := \varphi_\infty^n(\gamma_0).$$

Now recall that φ_∞ acts by isometry so that all the curves γ_n have the same length. If we consider the sequences of closed geodesics on Σ $\gamma_n, n > 0$ and $\gamma_n, n < 0$ then they converge respectively to the stable and unstable laminations for φ. It is not hard to see that $\tilde{N}_\varphi$ has exactly two ends and that the curves $\gamma_n, n > 0$ and $\gamma_n, n < 0$ "exit" the manifold via different ends. In fact if $\alpha_n, n > 0$ is any sequence of geodesics in $\tilde{N}_\varphi$ of uniformly bounded lenght exiting by the same end as $\gamma_n, n > 0$ then it will converge to the stable lamination too.

2. Geometry of the convex core

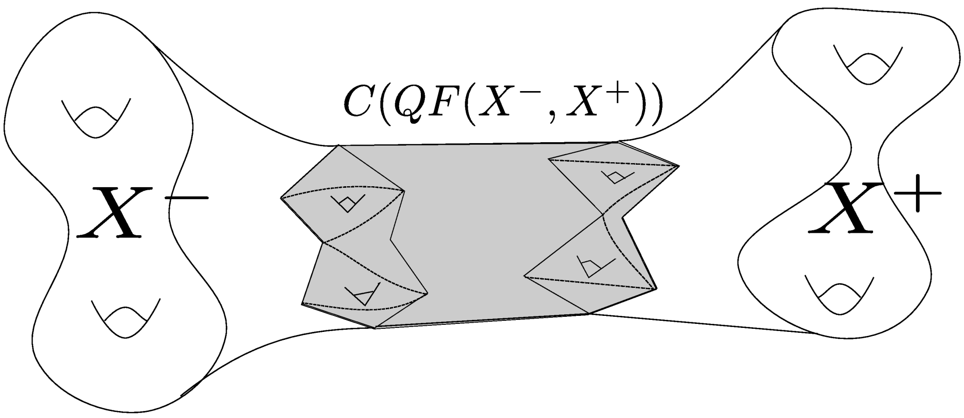

FIGURE 2. The quasi-Fuchsian manifold, the surfaces at infinity and the convex core.

The convex core $C(QF((X^+, X^-))$ of a $QF((X^+, X^-)$ is defined to be the smallest (non empty) closed, geodesically convex subset. There is a natural construction for the convex core as follows: if $QF((X^+, X^-)$ is the quotient of **H**3 by a discrete group Γ then $C(QF((X^+, X^-))$ is the quotient of $C(\Lambda(\Gamma))$, the *convex hull of the limit set*, by Γ. If Σ is compact then $C(QF((X^+, X^-))$ is compact and is in fact a *compact core* in the sense of Scott, that is the inclusion is a homotopy equivalence, so that it carries the topology of $C(QF((X^+, X^-))$. It is clear that there are many advantages to working with compact manifolds especially when considering convergence of sequences.

According to a theorem of Thurston the boundary of the convex core $\partial C(QF((X^+, X^-))$ consists of a pair of surfaces $\Sigma^\pm$ each homeomorphic to Σ. An important case is when $X^\pm$ corresponds to a Fuchsian structure and then $\partial C(QF((X^+, X^-))$ is totally geodesic (and the convex hull of the limit set is just a copy of **H**.) Otherwise $\Sigma^\pm$ are not smoothly embedded in $QF((X^+, X^-)$, the set of singular points is a lamination consists of a geodesic lamination called the *pleating lamination.* The simplest case of a pleating lamination is a single simple closed curve and it is not hard to see that the surface must be "bent" along this curve and that their is a dihedral angle between the support planes that intersect in this curve. In general the bending occurs along a geodesic lamination λ with no closed leaves and the "dihedral angle" is replaced by a *transverse bending measure.* This associates to an arc $\alpha \subset \Sigma^{\pm}$ a number $i(\alpha, \lambda)$ which measures how much the arc deviates from a geodesic segment in $QF((X^+, X^-)$.

2.1. **Comparing boundaries.** A very difficult question is to understand how the geometry of $\partial C(QF((X^+, X^-))$ varies with $(X^+, X^-) \in \mathcal{T} \times \mathcal{T}$. Sullivan began these investigations, conjecturing that the nearest point retraction $r : X^{\pm} \to \Sigma^{\pm}$ from the conformal boundary equipped with its Poincaré metric to the boundary of the convex core with its induced metric was 2-bi-Lipschitz. Many people worked on this question, notably Epstein-Marden, Epstein-Marden-Markovic and Bridgeman. Although the conjecture is false, in a very elegant treatment, Bridgeman shows that there is a $1 + K$-Lipschitz homotopy inverse for r where $K = \pi/\sinh^{-1}(1)$. To prove this he introduces the notiuon of *average bending* for the pleating lamination $\lambda^{\pm} \subset \Sigma^{\pm}$. If α is a geodesic arc in $\partial C(QF((X^+, X^-))$ then the average bending $B(\alpha)$ is defined to be the bending per unit length, or specifically

$$B(\alpha) := \frac{i(\alpha, \lambda)}{l(\alpha)},$$

where $i(\alpha, \lambda)$ is the intersection number and $l(\alpha)$ is the length of the arc. Bridgeman's approach is based on bounding average bending and he proves the following:

Theorem 2.1. *Let $K = \pi/\sinh^{-1}(1)$ then for any closed geodesic $\alpha \subset \partial C(QF((X^+, X^-))$*

$$B(\alpha) \leq K.$$

3. Volume of the convex core

As we noted above, if Σ is compact then the convex core of $QF((X^+, X^-)$ is compact and so its volume well defined. In fact, the notion of geometrical finiteness can be defined in terms of the finiteness of the volume of an ϵ-neighborhood of $C(QF((X^+, X^-))$.

There are two questions which come to mind immediately:

- How does the volume vary on a small scale i.e. infinitesimally?
- How does the volume vary on a large scale ?

The first of these was studied by Bonahon and the second by Brock and we describe there approaches below.

3.1. **Variational formula.** In 3-dimensional hyperbolic geometry, the classical *Schlafli formula* expresses the variation of the volume of a hyperbolic polyhedron in terms of the length of its edges and of the variation of its dihedral angles. Bonahon [4] proves an analogous formula for the variation of the volume $C(QF((X^+, X^-))$ and more generally for the convex core of a geometrically finite hyperbolic 3–manifold M, as we vary the hyperbolic metric of M. What is difficult is taking account of the way in which the pleating lamination varies. Bonahon does this by showing that the variation of the bending of the boundary of the convex core is described by a geodesic lamination with a certain transverse distribution. He proves that the variation of the volume of the convex core is then equal to 1/2 the length of this transverse distribution.

Bonahon's approach is elegant but it seems difficult to extract "large scale" estimates for volume from his formula.

3.2. **Comparison with Weil-Petersson distance.** There are two natural metrics on Teichmuller space $\mathcal{T}$:

- The Teichmuller metric $d_{\mathcal{T}}$, which is the solution to an optimisation problem: it is the log of the minimal quasiconformal dilation of maps $f : X \to Y$.
- The Weil-Peterson metric d_{wp}. This is a Riemanian metric and to define it one must first discuss the tangent space to $\mathcal{T}$ which we do in the Appendix following the approach of Bers'.

3.2.1. *Comparing the two metrics.* Both theses metrics are natural in the sense that every mapping class is an isometry for the metric. The Teichmuller metric is complete whereas the Weil-Petersson metric though geodesically convex but not complete. In fact there is a Finsler metric on the tangent space to $\mathcal{T}$ which induces the Teichmuller distance. Using this Linch proved in her thesis that there is an inequality relating the two distances are related:

Theorem 3.1.

$$d_{wp} \leq |2\pi\chi(\Sigma)|^{\frac{1}{2}} d_{\mathcal{T}}.$$

3.2.2. *Comparing with the pants complex.* Much later, Brock showed that $\mathcal{T}$ equipped with the Weil-Petersson metric is quasi-isometric to the pants complex. The *pants complex* $\mathcal{P}$ is just a graph the edges of which are pants decompositions of the surface Σ and the edges are pairs of vertices which are related by an elementary move (see figure). Since it is a graph the pants complex comes equipped with a simplicial metric $d_{\mathcal{P}}$. Recall that the Ber's constant of a surface Σ is a number $L > 0$ such that for any hyperbolic metric on Σ there is a pair of pants of length less than L. Buiding on the ideas of Minsky, Brock defines a "rough projection"

$$\pi : \mathcal{T} \to \mathcal{P},$$

which takes a conformal structure X to a pair of pants P of length less than the Bers' constant for the Poincaré metric in the class of X.

Theorem 3.2 (Brock). *Let Σ be a compact surface then here exists $K_1 > 1,\ K_2 > 0$ such that for all $(X^+, X^-) \in \mathcal{T} \times \mathcal{T}$*

$$\frac{1}{K_1} d_{wp}(X^+, X^-) - K_2 \leq d_{\mathcal{P}}(\pi(X^+), \pi(X^-)) \leq K_1 d_{wp}(X^+, X^-) + K_2$$

So, on a large scale, Teichmueller space $\mathcal{T}$ is modelled on the pants complex $\mathcal{P}$. With some additional work he proves the following comparison theorem for distance and volume.

Theorem 3.3 (Brock). *Let Σ be a compact surface then here exists $K_1 > 1,\ K_2 > 0$ such that for all $(X^+, X^-) \in \mathcal{T} \times \mathcal{T}$*

$$\frac{1}{K_1} d_{wp}(X^+, X^-) - K_2 \leq \mathrm{vol} C(QF((X^+, X^-)) \leq K_1 d_{wp}(X^+, X^-) + K_2$$

It is important to note that though, by applying Linch's Theorem above, we get upper bound for volume in terms of Teichmuller distance no lower bound is possible.

It is well known that there is a family of pseudo-Anasov automorphisms φ_k such that $||\varphi_k||_{\mathrm{WP}}$ are bounded whilst the entropy of φ_k, which is just the translation distance for the Teichmüller metric, diverges. For example, if Σ is the once punctured torus the mapping class group is isomorphic to $PSL(2, \mathbb{Z}$ Consider the sequence of diffeomorphisms defined as follows:

$$\varphi_k := \begin{pmatrix} 1+k & 1 \\ k & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$$

It is easy to see that for $k > 0$ the mapping class φ_k is pseudo-Anosov and that the sequence of dilatations tends to ∞ as $k \to \infty$. However, the volume of the sequence of mapping tori N_{φ_k} is bounded. In fact, the mapping tori N_{φ_k} converge to a cusped hyperbolic 3-manifold.

In light of this example and Brock's Theorem above one sees that the relationship between volume and Weil-Petersson distance is stronger than that with the Teichmüller distance.

4. W-volume and Renormalized Volume

The renormalized volume Rvol is a numerical invariant associated to an infinite-volume Riemannian manifold with some special structure near infinity, extracted from the divergent integral of the volume form. Early instances of renormalized volumes appear in Henningson–Skenderis for asymptotically hyperbolic Einstein metrics, and in Krasnov for Schottky hyperbolic 3-manifolds. Renormalized volume of convex cocompact hyperbolic 3-manifolds were studied extensively by Krasnov and Schlenker in [17] using a geometric construction due to C. Epstein for studying ends.

From our point of view the interest of renormalized volume is that it is "commensurable" with the usual hyperbolic volume, that is they differ by a bounded quantity only depending on the topology of the surface, and that it has a very nice Schaffli formula. Both these observations are due essentially to Krasnov and Schlenker.

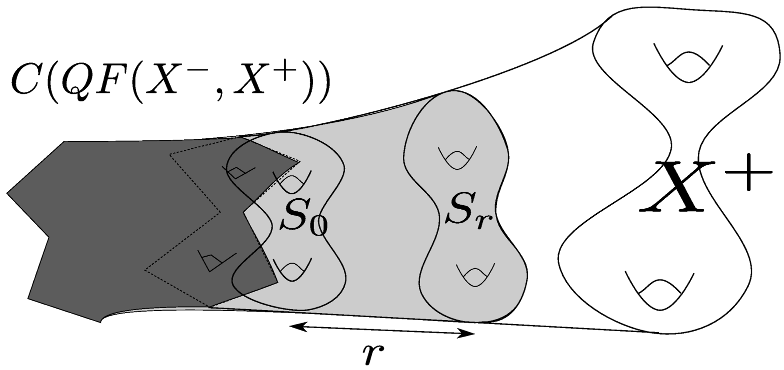

FIGURE 3. The ends of a quasi-Fuchsian manifold can be foliated by $C^{1,1}$ surfaces equidistant to the boundary of the convex core.

4.0.1. *Definition.* To simplify the exposition we assume that Σ is compact. Let M be a quasi-Fuchsian manifold $\mathbb{H}^3/\Gamma$ homeomorphic to $\Sigma \times \mathbb{R}$. Following [17] we say that a codimension-zero smooth compact convex submanifold $N \subset M$ is *strongly convex* if the normal hyperbolic Gauss map from $\partial N = \partial_+ N \sqcup \partial_- N$ to the boundary at infinity Ω_Γ/Γ is a homeomorphism. For example, a closed ε-neighborhood of the convex core of a quasi-Fuchsian manifold is strongly convex. Let $S_0 = \partial N$ then there is a family of surfaces

$\{S_r\}_{r\geq 0}$ equidistant to S_0 foliating the ends of M. If g_r denotes the induced metric on S_r, then define a metric at infinity associated to the family $\{S_r\}_{r\geq 0}$ by

$$g = \lim_{r\to\infty} 2e^{-2r} g_r.$$

The resulting metric g in fact belongs to the *conformal class at infinity* that is the conformal structure determined by the complex structure on Ω_Γ/Γ. It is easy to see that if we start with a strongly convex submanifold bounded by S_{r_0} for some $r_0 > 0$, then the limiting metric is $e^{2r_0}g$. Namely, if we shift the parametrization of an equidistant foliation by r_0, then the limiting metric changes only by scaling e^{2r_0}.

Conversely, if g is a Riemannian metric in the conformal class at infinity, then Theorem 5.8 in [17] shows that there is a unique foliation of the ends of M by equidistant surfaces with compatible parametrization of leaves starting $r_0 \geq 0$ so that the associated metric at infinity is equal to g. Notice that the parametrization may have to start with a positive r_0. The construction of a foliation is due to Epstein [11]. Then, a natural functional in the context of strongly convex submanifolds $N \subset M$ is the *W-volume* defined by

$$W(M, g) := \operatorname{vol} N_r - \frac{1}{4}\int_{S_r} H_r\, da_r + \pi r \chi(\partial M), \tag{4.1}$$

where the parametrization r is induced by g, N_r is a strongly convex submanifold bounded by the associated leaf S_r, H_r is the mean curvature of S_r and da_r is the induced area form of S_r. Note that, strictly speaking, H_r is only well defined on a set of full measure as the surfaces S_r are not C^2 but only $C^{1,1}$ see [6] for a discussion. A simple computation which shows that the W-volume depends only on the metric at infinity g, justifying the notation.

The renormalized volume of M is now defined by

$$\operatorname{Rvol}(M) := \sup_g W(M, g),$$

where the supremum is taken over all metrics g in the conformal class at infinity such that the area of each surface at infinity $X^{\pm}$ with respect to g is $2\pi|\chi(\Sigma)|$. Section 7 in [17] presents an argument, based on the variational formula, that the metric of constant curvature -1 is a critical point of the functional $W(M, g)$ and that this is a local maximum. Guillarmou–Moroianu–Schlenker study the change of $W(M, g)$ under conformal transformation and prove that there is a unique maximum.

4.0.2. *Commensurability.* When Σ is compact Schlenker, using Bridgeman's bound on average bending proves that the volume and the renormalized volume are "commensurable" that is they differ by a bounded quantity only depending (in the compact case) on the topology of the surface. Bridgeman and Canary using a slightly modified approach reprove this result and we have:

Theorem 4.1. *Let Σ be a compact surface then Then,*

$$\operatorname{vol} C(QF((X^+, X^-)) - D \leq \operatorname{Rvol} QF((X^+, X^-) \leq \operatorname{vol} C(QF((X^+, X^-)), \tag{4.2}$$

with $D = 9.185|\chi(\Sigma)|$.

The proof of this theorem is interesting and shows why the sup in the definition is important. It hinges on the comparison of two metrics defined on the $X^{\pm}$. The first of these is the Poncaré metric $\rho(z)|dz|$ and the second the so-called the Thurston metric

$\tau(z)|dz|$. According to a result of Herron, Ma and Minda thes metrics are 2 bi-Lipschitz and satisfy

$$\frac{1}{2}\tau(z) \leq \rho(z) \leq \tau(z).$$

When one computes the corresponding W-volumes of $\rho(z)$ and $\tau(z)$ one obtains respectively the renormalised volume and $\operatorname{vol} C(QF((X^+, X^-)) - \frac{1}{4}L(\lambda)$ where λ is the pleating lamination, see Schlenker [24] for details.

4.0.3. *Schaffli formula.* The tangent space of $\mathcal{T} \times \mathcal{T}$ can be identified with the space of Beltrami differentials μ on $X^\pm$ (see appendix for details).

Theorem 4.2. *Under an infinitesimal deformation of the complex structure (X^+, X^-) represented by the Beltrami differential μ*

$$d\,\mathrm{Rvol}_{(X^+,X^-)}(\mu) = -\frac{1}{2}\,\mathrm{Re}\,(q,\,\mu) = -\frac{1}{2}\int_{X^\pm}\mathrm{Re}\,\mu q$$

holds where q is the Schwarzian associated to the projective structure.

5. More volume estimates

5.1. **Variational proof of inequalities.** Brock's Theorem above provides a comparison theorem between volume of the convex core and distance in the pants complex. Distance in the pants complex is defined in a combinatorial manner and seems difficult to compute in general. Using Brock's quasi-isometry between the pants complex and $\mathcal{T}$ equipped with d_{wp} does not seem to yield much information in particular since the constants in Theorem 3.3 are not explicit. Schlenker applies his Schaffli formula to prove the following:

Theorem 5.1. *Let Σ be a compact surface then*

$$\mathrm{vol} C(QF(X^+, X^-)) \leq \frac{3}{2}|2\pi\chi(\Sigma)|^{\frac{1}{2}} d_{wp}(X^+, X-) + 9.185|\chi(\Sigma)|.$$

Each of the numbers on the right hand side has a geometric interpretation:

- The term $9.185|\chi(\Sigma)|$ comes from Bridgeman's bound on average bending.
- $\frac{3}{2}$ comes from Nehari's bound on the Schwarzian of a univalent map.
- $2\pi\chi(\Sigma)$ is the hyperbolic area of the surface Σ.

It is instructive to see how Schlenker proves this formula

Proof. It suffices to prove

$$\mathrm{Rvol}\, QF(X, Y) \leq \frac{3}{2}|2\pi|\chi(\Sigma)|^{\frac{1}{2}} d_w p(X, Y)\,, \tag{5.1}$$

Let $Y : [0, d] \to \mathcal{T}$ be the unit speed Teichmüller geodesic joining X and Y, so that, in particular, $Y(0) = X$, $Y(d) = Y$ and $d = d_{\mathrm{T}}(X, Y)$. If $\{QF(X, Y(t))\}_{0 \leq t \leq d}$ denotes the associated one-parameter family of quasi-Fuchsian manifolds then, applying Shlenker's Schaffli formula one has:

$$d\,\mathrm{Rvol}(\dot{X}, \dot{Y}(t)) = -\frac{1}{2}\,\mathrm{Re}\left((q_X(t), \dot{X}) + (q_{Y(t)}(t), \dot{Y}(t))\right),$$

where $\dot{X}$, $\dot{Y}(t)$ are the tangent vectors to the deformations of the complex structures on each boundary. Integrating the variation of Rvol along the path $Y(t)$ ($t \in [0, d]$) and using the fact that $X(t) = Y(0)$ is constant, we obtain

$$\operatorname{Rvol} QF(X,Y) = -\frac{1}{2}\operatorname{Re}\int_{t=0}^{d}(q_{Y(t)}(t), \dot{Y}(t))dt.$$

The renormalised volume is a real number and so it suffices to bound the module of the right hand side of this equation. Let ρ denote the hyperbolic metric on R and consider,

$$\begin{aligned}|(q_{Y(t)}(t)), \dot{Y}(t))|^2 &\leq \int_R |q_{Y(t)}(t)|^2\rho^-2\int_R|\dot{Y}(t)|^2\rho^2\\ &= \left(\int_R \frac{|q_{Y(t)}(t)|^2}{\rho^4}\rho^2\right)||\dot{Y}(t)||^2_{wp}\\ &\leq ||q||^2_\infty\left(\int_R\rho^2\right)||\dot{Y}(t)||^2_{wp}\end{aligned}$$

Nehari's Theorem [22] allows us to bound the factor $||q_{Y(t)}(t)||_\infty$ by $\frac{3}{2}$ and since our path $Y(t)$ just a Weil Petersson geodesic $||\dot{Y}(t)||_{wp} = 1$.

The statement follows easily from these observation. □

5.2. **Estimates from 3 manifolds.** A more restricted problem is to fix a pseudo-Anosov and compare how the volume of the convex core of $C(QF(\varphi^{-n}X, \varphi^n X))$ varies with n. Kojima-McShane and independantly Brock-Bromberg prove the following:

Theorem 5.2. *Suppose Σ is compact, then*

$$|\operatorname{vol} C(QF(\varphi^{-n}X, \varphi^n X)) - 2n \operatorname{vol} N_\varphi| \tag{5.2}$$

is uniformly bounded.

As mentioned in the introduction $QF(\varphi^{-n}X, \varphi^n X)$ should be "quite close" to $\tilde{N}_\varphi$ on a large compact subset K. The proof of the theorem is to estimate the size, an control the geometry of such a compact set which is $(1+\epsilon)$ bi-Lipschitz to a subset of $\tilde{N}_\varphi$.

Combining this with Schlenker's estimate above yields :

Corollary 5.3. *If Σ is compact, then:*

$$\sqrt{2\pi|\chi(\Sigma)|}\,||\varphi||_{\mathrm{WP}} \geq \frac{4}{3}\operatorname{vol} N_\varphi$$

holds for any pseudo-Anosov φ, where $||\cdot||_{\mathrm{WP}}$ is the Weil-Petersson translation distance of φ.

Brock and Bromberg [9] observed that one can apply this inequality to get a lower bound for the diameter of the moduli space of the once punctured torus

Corollary 5.4. *The diameter of the moduli space of the once punctured torus is bounded below by*

$$\frac{1}{6}\sqrt{\frac{2}{\pi}}\mathcal{V}_8$$

where $\mathcal{V}_8$ is the volume of the regular ideal octahedron in $\mathbf{H}^3$.

The idea is to bound the Weil-Petersson distance between two points in frontier of Teichmuller space. As explained in Paragraph 3.2.2 the Teichmuller metric and the Weil-Petersson metric are not equivalent. Brock and Bromberg consider a sequence of pseudo-Anosovs on the punctured torus:

$$\varphi_k := \begin{pmatrix} 1+k^2 & k \\ k & 1 \end{pmatrix} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}.$$

Note that the normaliser of $\{N_{\varphi_k}\}$ contains

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

They observe that the volume of N_{φ_k} converges to $2\mathcal{V}_8$. Each of the φ_k admits an *axis*, that is an invariant Weil-Petersson geodesic on which it acts by translation, $\mathcal{T}$ is a proper space the sequence. Since $\mathcal{T}$ is a proper space, after making an appropriate choice of basepoint say the fixed point of J, the sequence of these axes converges to a Weil-Petersson geodesic I joining two points in the frontier. One applies Corollary 5.3 to the sequence N_{φ_k} to get a lower bound on the length of I We note that Wolpert had previously shown that the length of I was at most $2\sqrt{30}\pi^{\frac{3}{4}}$. Observe now that I is invariant under J so it double covers a (simple) geodesic in the moduli space and the so diameter is bounded below $\frac{1}{2}$ the length of I.

6. The Hessian of the renormalized volume.

Ciobotaru-Moroianu [20] and Pallete [25] study the Hessian of the renormalized volume in relation to the Fuchsian locus. Recall that the Fuchsian locus in quasi-Fuchsian space is a submanifold such that the boundary of the convex core is totally geodesic. It is easy to see that the convex core $C(QF((X^+, X^-))$is a totally geodesic embedding of the surface and so its hyperbolic volume is zero. From general considerations, we have the following characterisation of Fuchsian: $QF((X^+, X^-)$ is Fuchsian if and only if the hyperbolic volume of $C(QF((X^+, X^-))$ is zero. The boundary of the convex core of $QF((X^+, X^-)$ is pleated or bent along a lamination λ and it is totally geodesic if and only if the length $L(\lambda)$ of this lamination is zero. Combining this with Schlenker's inequality

$$\text{(6.1)} \qquad \operatorname{vol} C(QF((X^+, X^-)) - L(\lambda) \leq \operatorname{Rvol} QF((X^+, X^-)) \leq \operatorname{vol} C(QF((X^+, X^-)),$$

we see that if renormalized volume of $QF((X^+, X^-)$ is zero when it is Fuchsian. It seems natural to conjecture that the renormalized volume:

- is non negative for all convex co-compact hyperbolic 3-manifolds
- is minimal exactly when the boundary of the convex hull is totally geodesic.

Given the context, where the renormalized volume arises as a functional in physical models, it is tempting to think of as being analogous to the mass of asymptotically Euclidean manifolds thus the first question is an analogue of the positive mass conjecture.

The second question is the analogue of a question of Bonahon for hyperbolizable 3-manifolds with incompressible boundary. He conjectured that the volume of a convex core is at least half the simplicial volume of the doubled manifold. Storm, using ideas of Souto and Besson–Courtois–Gallot, solved the conjecture:

Theorem 6.1 (Storm). *If N is a hyperbolic 3-manifold homotopy equivalent to M then*

$$\operatorname{vol} C(N) \geq \frac{1}{2} \operatorname{SimpVol} DM,$$

where DM *is the double of* M.

Moreover, if $\mathrm{vol}(CN) = \frac{1}{2}\mathrm{SimpVol}(DM) > 0$ *then* M *is acylindrical,* N *is convex cocompact, and* $\partial C(N)$ *is totally geodesic.*

Some progress has been made for *almost Fuchsian manifolds*, that is for hyperbolic structures that are small deformations of a manifold with totally geodesic boundary. Fuchsian manifolds embedd as the Morianu computed the Hessian (see also Palette [25]) and shown that it is positive definite on the "normal bundle" to the Fuchsian locus in quasi-Fuchsian space. This means that for manifolds that are nearly Fuchian i.e. small deformations of Fuchsian structures the renormalised volume is positive. Uhlenbeck gave a more formal definition: An almost-Fuchsian hyperbolic 3–manifold (X, g) is a quasi-Fuchsian hyperbolic 3–manifold containing a closed minimal surface whose principal curvatures belong to $(-1, 1)$. Ciobotaru-Moroianu prove the following

Theorem 6.2 (Ciobotaru-Moroianu). *The renormalized volume of an almost Fuchsian hyperbolic 3–manifold is non-negative. Further it is zero only at the Fuchsian locus.*

It seems likely that the renormalized volume is *always* positive.

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