RELATIONSHIP BETWEEN THE MILNOR'S $\mu$-INVARIANT AND HOMFLYPT POLYNOMIAL

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1. Introduction

For an ordered oriented link in the 3-sphere, J. Milnor [15, 16] defined a family of invariants, known as Milnor's $\mu$-invariants. For an $n$-component link $L$, Milnor invariant is determined by a sequence $I$ of elements in $\{1, 2, \ldots, n\}$ and denoted by $\overline{\mu}_L(I)$. It is known that Milnor invariants of length two are just linking numbers. In general, Milnor invariant $\overline{\mu}_L(I)$ is only well-defined modulo the greatest common divisor $\Delta_L(I)$ of all Milnor invariants $\overline{\mu}_L(J)$ such that $J$ is a subsequence of $I$ obtained by removing at least one index or its cyclic permutation. If the sequence is of distinct numbers, then this invariant is also a link-homotopy invariant and we call it Milnor's link-homotopy invariant. Here, the link-homotopy is an equivalence relation generated by ambient isotopy and self-crossing changes.

In [3], N. Habegger and X. S. Lin showed that Milnor invariants are also invariants for string links, and these invariants are called Milnor's $\mu$-invariants. For any string link $\sigma$, $\mu_{\hat{\sigma}}(I)$ coincides with $\overline{\mu}_L(I)$ modulo $\Delta_L(I)$, where $\hat{\sigma}$ is a link obtained by the closure of $\sigma$. Milnor's $\mu$-invariants of length $k$ are finite type invariants of degree $k - 1$ for any natural integer $k$, as shown by D. Bar-Natan [1] and X. S. Lin [11].

In [17], M. Polyak gave a formula expressing Milnor's $\mu$-invariant of length 3 by the Conway polynomials of knots. His idea was derived from the following relation. Both Milnor's $\mu$-invariant of length 3 for string link and the second coefficient of the Conway polynomial are finite type invariants of degree 2. He gave this relation by using Gauss diagram formulas.

Then, in [14], J-B. Meilhan and A. Yasuhara generalized it by using the clasper theory introduced by K. Habiro [4]. They showed that general Milnor's $\mu$-invariants can be represented by the HOMFLYPT polynomials of knots under some assumption. Moreover, the author and A. Yasuhara improved it in [9].

In [8], we give a formula expressing Milnor's $\mu$-invariant by the HOMFLYPT polynomials of knots under some assumption (Theorem 3.1) by using the clasper theory in [4]. The course of proof is similar to that in [14] and [9]. Moreover, Milnor's $\mu$-invariants of length 3 for any string link are given by the HOMFLYPT polynomial, which is a finite type invariant of degree 2, and the linking number. Because a finite type knot invariant of degree 2 is only the second coefficient of the Conway polynomial essentially, Milnor's $\mu$-invariants of length 3 are given by the second coefficient of the Conway polynomial and the linking number (Theorem 3.3). It is a string version of Polyak's result, and by taking modulo $\Delta(I)$, our result coincides with Polyak's result.
2. Milnor's $\mu$-invariant and HOMFLYPT polynomial

2.1. String link. Let $n$ be a positive integer and $D^2 \subset \mathbb{R}^2$ the unit disk equipped with $n$ marked points $x_1, x_2, \ldots, x_n$ in its interior, lying in the diameter on the $x$-axis of $\mathbb{R}^2$ as in Figure 1. Let $I = [0, 1]$. An $n$-string link $\sigma$ is the image of a proper embedding $\bigcup_{i=1}^n I_i \to D^2 \times I$ of the disjoint union of $n$ copies of $I$ in $D^2 \times I$, such that $\sigma|_{I_i}(0) = (x_i, 0)$ and $\sigma|_{I_i}(1) = (x_i, 1)$ for each $i$ as in Figure 1. Each string of a string link inherits an orientation from the usual orientation of $I$. The $n$-string link $\{x_1, x_2, \ldots, x_n\} \times I$ in $D^2 \times I$ is called the trivial $n$-string link and denoted by $1_n$ or $1$ simply.

![Figure 1. An n-string link](image1)

Given two $n$-string links $\sigma$ and $\sigma'$, we denote their product by $\sigma \cdot \sigma'$, which is given by stacking $\sigma'$ on the top of $\sigma$ and reparametrizing the ambient cylinder $D^2 \times I$. By this product, the set of isotopy classes of $n$-string links has a monoid structure with unit given by the trivial string link $1_n$. Moreover, the set of link-homotopy classes of $n$-string links is a group under this product.

2.2. Milnor's $\mu$-invariant for string links. Let $\sigma = \bigcup_{i=1}^n I_i$ in $D^2 \times I$ be an $n$-string link. We consider the fundamental group $\pi_1(D^2 \times I \setminus \sigma)$ of the complement of $\sigma$ in $D^2 \times I$, where we choose a point $b$ as a base point and curves $\alpha_1, \cdots, \alpha_n$ as meridians in Figure 2.

![Figure 2. Longitude of string link](image2)
By Stallings' theorem [18], for any positive integer \( q \), the inclusion map
\[
i : D^2 \times \{0\} \setminus \{x_1, \ldots, x_n\} \to D^2 \times I \setminus \sigma
\]
induce an isomorphism of the lower central series quotients of the fundamental groups
\[
\iota_* : \pi_1(D^2 \times \{0\} \setminus \{x_1, \ldots, x_n\}) \to \pi_1(D^2 \times I \setminus \sigma)
\]
where given a group \( G, G_q \) means the \( q \)-th lower central subgroup of \( G \). The fundamental group \( \pi_1(D^2 \times \{0\} \setminus \{x_1, \ldots, x_n\}) \) is a free group generated by \( \alpha_1, \ldots, \alpha_n \). We then consider the \( j \)-th longitude \( l_j \) of \( \sigma \) in \( D^2 \times I \), where \( l_j \) is the closure of the preferred parallel curve of \( \sigma_j \), whose endpoints lie on the \( x \)-axis in \( D^2 \times \{0, 1\} \) as in Figure 2. We then consider the image of the longitude \( \iota_*^{-1}(l_j) \) by the Magnus expansion and denote \( \mu(i_1, \ldots, i_k, j) \) the coefficient of \( X_{i_1}X_{i_2} \cdots X_{i_k} \) in the Magnus expansion.

**Theorem 2.1** ([3]). For any positive integer \( q \), if \( k < q \), then \( \mu(i_1, \ldots, i_k, j) \) is invariant under isometry. Moreover, if the sequence \( i_1, \ldots, i_k, j \) is of distinct numbers, then \( \mu(i_1, \ldots, i_k, j) \) is also link-homotopy invariant.

We call this invariant Milnor's \( \mu \)-invariant.

### 2.3. HOMFLYPT polynomial

Recall the definition of the HOMFLYPT polynomial.

The HOMFLYPT polynomial \( P(L; t, z) \in \mathbb{Z}[t^{\pm 1}, z^{\pm 1}] \) of an oriented link \( L \) is defined by the following two formulas:

1. \( P(U; t, z) = 1 \), and
2. \( t^{-1}P(L_+; t, z) - tP(L_-; t, z) = zP(L_0; t, z) \),

where \( U \) denotes the trivial knot and \( L_+, L_- \) and \( L_0 \) are link diagrams which are identical everywhere except near one crossing, where they look as follows:

\[
L_+ = \begin{array}{c}
\nearrow
\end{array} \quad ; \quad L_- = \begin{array}{c}
\nwedge
\end{array} \quad ; \quad L_0 = \begin{array}{c}
\nwedge
\end{array}
\]

Recall that the HOMFLYPT polynomial of a knot \( K \) is of the form \( P(K; t, z) = \sum_{k=0}^{N} P_{2k}(K; t)z^{2k} \), where \( P_{2k}(K; t) \in \mathbb{Z}[t^{\pm 1}] \) is called the \( 2k \)-th coefficient polynomial of \( K \).

### 3. MAIN THEOREM

Given a sequence \( I \) of elements of \( \{1, 2, \ldots, n\} \), \( J < I \) will be used for any subsequence \( J \) of \( I \), possibly \( I \) itself, and \( |J| \) will denote the length of the sequence \( J \).

Let \( \sigma \) be an \( n \)-string link. Given a sequence \( I = i_1i_2 \cdots i_m \) obtained from \( 12 \cdots n \) by deleting some elements, and a subsequence \( J = j_1j_2 \cdots j_k \) of \( I \), we define a knot \( \sigma_{I,J} \) as the closure of the product \( b_I \cdot \sigma_J \). Here \( \sigma_J \) is the \( m \)-string link obtained from \( \sigma \) by deleting the \( i \)-th string, for all \( i \in \{1, 2, \ldots, n\} \setminus \{i_1, i_2, \ldots, i_m\} \) and replacing the \( i \)-th string with a trivial string underpassing all other components, for all \( i \in \{i_1, i_2, \ldots, i_m\} \setminus \{j_1, j_2, \ldots, j_k\} \) and \( b_I \) is the \( m \)-braid associated with the permutation \( b = (i_1 i_2 \cdots i_{m-1} i_m) \) and such that the arc with connecting \( (b^{k}(i_1), 0) \) with \( (b^{k+1}(i_1), 1) \) underpasses all arcs with connecting \( (b^{k}(i_1), 0) \) with \( (b^{k+1}(i_1), 1) \) in \([0, 1] \times [0, 1]\) of braid diagram for \( k < k' < n \). See Figure 3 for an example. We then have the following Theorem.
\begin{align*}
\mu_\sigma(I) &= \frac{(-1)^{m-1}}{(m-1)!2^{m-1}} \sum_{J<I} (-1)^{|J|} P_0^{(m-1)}(\sigma_{I,J};1),
\end{align*}

where $P_0^{(m-1)}(\cdot;1)$ is the $(m-1)$-th derivative of the 0-th coefficient $P_0(\cdot;t)$ of the HOMFLYPT polynomial evaluated at $t = 1$.

Note that the above vanishing assumption for string link is equivalent to that any $(m - 2)$-substring link is link-homotopic to the trivial string link.

\begin{remark}
Theorem 1.1 remains valid if we use one of the following two alternative definitions of $b_I$. One is that we use "overpasses" instead of "underpasses". The other is that we use "any $i \in \{i_1, i_2, \cdots, i_m\}" instead of "i_1".

We also give the case of $\mu$-invariants of length 3 without the assumption.

\begin{theorem}
Let $\sigma$ be an $n$-string link and $I = i_1i_2i_3$ be a length 3 sequence with distinct numbers in $\{1, 2, \cdots, n\}$. We then have
\begin{align*}
\mu_\sigma(I) &= -\sum_{J<i} (-1)^{|J|} a_2(\overline{\sigma_{I,J}}) - lk_\sigma(i_1i_2)lk_\sigma(i_2i_3) + A_I,
\end{align*}
where $a_2$ is the second coefficient of the Conway polynomial, $lk_\sigma(ij)$ is the linking number of the $i$-th component and $j$-th component of $\sigma$, and
\begin{align*}
A_I &= \begin{cases}
\quad lk_\sigma(i_1i_2) & (i_2 < i_3 < i_1) \\
-kk_\sigma(i_1i_2) & (i_1 < i_3 < i_2) \\
\quad 0 & (\text{otherwise}).
\end{cases}
\end{align*}
\end{remark}

\begin{remark}
This operation from a string link to a knot corresponds to $Y$-graph sum of links defined by M. Polyak. By taking this formula modulo $\Delta_{\overline{\sigma_{I,J}}}(I)$, we get Polyak's relation between Milnor's $\overline{\mu}$-invariants and Conway polynomials [17].

\begin{remark}
In [19], K. Taniyama gave a formula expressing Milnor's $\overline{\mu}$-invariants of length 3 for links by the second coefficient of the Conway polynomial assuming that all linking numbers vanish.

\begin{remark}
In [12], J.B. Meilhan showed that all finite type invariants of degree 2 for string link was given by some invariants (Theorem 2.8). So the formula in Theorem 3.3 could also be derived from [12].

\section{Examples}

\begin{example}
Let $\sigma$ be a 3-string link showed by Figure 3. Then $\mu_{123}(\sigma) = -1$, $\mu_{132}(\sigma) = \mu_{213}(\sigma) = 1$ and $\mu_{231}(\sigma) = \mu_{312}(\sigma) = \mu_{321}(\sigma) = 0$. And $lk_\sigma(12) = lk_\sigma(23) = 1$ and $\Delta_{\overline{\sigma_{123}}}(13) = 0$.

On the other hand, $\overline{\sigma_{123,123}}$ and $\overline{\sigma_{123,23}}$ are the figure-eight knot, and $\overline{\sigma_{123,J}}$ ($J \neq 123, 23$) is the trivial knot. Therefore we obtain
\begin{align*}
-\sum_{J<123} (-1)^{|J|} a_2(\overline{\sigma_{123,J}}) - lk_\sigma(12)lk_\sigma(23) = a_2(4_1) - a_2(4_1) - 1 \cdot 1 = -1.
\end{align*}
\end{example}
Similarly, we have
\[ - \sum_{J<231} (-1)^{|J|} a_2(\overline{\sigma_{231,J}}) - lk_\sigma(23)lk_\sigma(31) = a_2(3_1\sharp 4_1) - a_2(3_1) - a_2(4_1) - 1 \cdot 0 = 0, \]
\[ - \sum_{J<312} (-1)^{|J|} a_2(\overline{\sigma_{312,J}}) - lk_\sigma(31)lk_\sigma(12) + lk_\sigma(13) = a_2(3_1) - a_2(3_1) - 0 \cdot 1 + 0 = 0. \]

Moreover, $\overline{\sigma_{132,32}}$ is the figure-eight knot and $\overline{\sigma_{132,J}} (J \neq 32)$ is the trivial knot. Therefore we obtain
\[ - \sum_{J<132} (-1)^{|J|} a_2(\overline{\sigma_{132,J}}) - lk_\sigma(13)lk_\sigma(32) - lk_\sigma(13) = -a_2(4_1) - 0 \cdot 1 - 0 = 1. \]

Similarly, we have
\[ - \sum_{J<213} (-1)^{|J|} a_2(\overline{\sigma_{213,J}}) - tk_\sigma(21)lk_\sigma(13) = a_2(7_6) - a_2(3_1) - a_2(4_1) - 1 \cdot 0 = 1, \]
\[ - \sum_{J<321} (-1)^{|J|} a_2(\overline{\sigma_{321,J}}) - lk_\sigma(32)lk_\sigma(21) = a_2(5_2) - a_2(3_1) - 1 \cdot 1 = 0. \]

**Figure 3**

REFERENCES


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