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RELATIONSHIP BETWEEN THE MILNOR'S $\mu$-INVARIANT AND HOMFLYPT POLYNOMIAL (Topology, Geometry and Algebra of low-dimensional manifolds)

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RELATIONSHIP BETWEEN THE MILNOR’S $\mu$-INVARIANT AND HOMFLYPT POLYNOMIAL

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1. INTRODUCTION

For an ordered oriented link in the 3-sphere, J. Milnor [15, 16] defined a family of invariants, known as Milnor’s $\overline{\mu}$-invariants. For an $n$-component link $L$, Milnor invariant is determined by a sequence $I$ of elements in $\{1, 2, \ldots, n\}$ and denoted by $\overline{\mu}_L(I)$. It is known that Milnor invariants of length two are just linking numbers. In general, Milnor invariant $\overline{\mu}_L(I)$ is only well-defined modulo the greatest common divisor $\Delta_L(I)$ of all Milnor invariants $\overline{\mu}_L(J)$ such that $J$ is a subsequence of $I$ obtained by removing at least one index or its cyclic permutation. If the sequence is of distinct numbers, then this invariant is also a link-homotopy invariant and we call it Milnor’s link-homotopy invariant. Here, the link-homotopy is an equivalence relation generated by ambient isotopy and self-crossing changes.

In [3], N. Habegger and X. S. Lin showed that Milnor invariants are also invariants for string links, and these invariants are called Milnor’s $\mu$-invariants. For any string link $\sigma$, $\mu_{\hat{\sigma}}(I)$ coincides with $\overline{\mu}_I(I)$ modulo $\Delta_{\hat{\sigma}}(I)$, where $\sigma$ is a link obtained by the closure of $\hat{\sigma}$. Milnor’s $\mu$-invariants of length $k$ are finite type invariants of degree $k - 1$ for any natural integer $k$, as shown by D. Bar-Natan [1] and X. S. Lin [11].

In [17], M. Polyak gave a formula expressing Milnor’s $\overline{\mu}$-invariant of length 3 by the Conway polynomials of knots. His idea was derived from the following relation. Both Milnor’s $\mu$-invariant of length 3 for string link and the second coefficient of the Conway polynomial are finite type invariants of degree 2. He gave this relation by using Gauss diagram formulas.

Then, in [14], J-B. Meilhan and A. Yasuhara generalized it by using the clasper theory introduced by K. Habiro [4]. They showed that general Milnor’s $\overline{\mu}$-invariants can be represented by the HOMFLYPT polynomials of knots under some assumption. Moreover the author and A. Yasuhara improved it in [9].

In [8], we give a formula expressing Milnor’s $\mu$-invariant by the HOMFLYPT polynomials of knots under some assumption (Theorem 3.1) by using the clasper theory in [4]. The course of proof is similar to that in [14] and [9]. Moreover, Milnor’s $\mu$-invariants of length 3 for any string link are given by the HOMFLYPT polynomial, which is a finite type invariant of degree 2, and the linking number. Because a finite type knot invariant of degree 2 is only the second coefficient of the Conway polynomial essentially, Milnor’s $\mu$-invariants of length 3 are given by the second coefficient of the Conway polynomial and the linking number (Theorem 3.3). It is a string version of Polyak’s result, and by taking modulo $\triangle(I)$, our result coincides with Polyak’s result.

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2. Milnor’s $\mu$-invariant and HOMFLYPT polynomial

2.1. String link. Let $n$ be a positive integer and $D^2 \subset \mathbb{R}^2$ the unit disk equipped with $n$ marked points $x_1, x_2, \ldots, x_n$ in its interior, lying on the $x$-axis of $\mathbb{R}^2$ as in Figure 1. Let $I = [0,1]$. An $n$-string link $\sigma$ is the image of a proper embedding $\bigcup_{i=1}^{n} I_i \to D^2 \times I$ of the disjoint union of $n$ copies of $I$ in $D^2 \times I$, such that $\sigma|_{I_i}(0) = (x_i,0)$ and $\sigma|_{I_i}(1) = (x_i,1)$ for each $i$ as in Figure 1. Each string of a string link inherits an orientation from the usual orientation of $I$. The $n$-string link $\{x_1, x_2, \ldots, x_n\} \times I$ in $D^2 \times I$ is called the trivial $n$-string link and denoted by $1_n$ or $1$ simply.

![Figure 1. An n-string link](image)

Given two $n$-string links $\sigma$ and $\sigma'$, we denote their product by $\sigma \cdot \sigma'$, which is given by stacking $\sigma'$ on the top of $\sigma$ and reparametrizing the ambient cylinder $D^2 \times I$. By this product, the set of isotopy classes of $n$-string links has a monoid structure with unit given by the trivial string link $1_n$. Moreover, the set of link-homotopy classes of $n$-string links is a group under this product.

2.2. Milnor’s $\mu$-invariant for string links. Let $\sigma = \bigcup_{i=1}^{n} I_i$ in $D^2 \times I$ be an $n$-string link. We consider the fundamental group $\pi_1(D^2 \times I \setminus \sigma)$ of the complement of $\sigma$ in $D^2 \times I$, where we choose a point $b$ as a base point and curves $\alpha_1, \cdots, \alpha_n$ as meridians in Figure 2.

![Figure 2. Longitude of string link](image)
By Stallings' theorem [18], for any positive integer $q$, the inclusion map
\[ \iota : D^2 \times \{0\} \setminus \{x_1, \ldots, x_n\} \to D^2 \times I \setminus \sigma \]
induce an isomorphism of the lower central series quotients of the fundamental groups
\[ \iota_* : \pi_1(D^2 \times \{0\} \setminus \{x_1, \ldots, x_n\}) \to \pi_1(D^2 \times I \setminus \sigma) \]
where given a group $G$, $G_q$ means the $q$-th lower central subgroup of $G$. The fundamental group $\pi_1(D^2 \times \{0\} \setminus \{x_1, \ldots, x_n\})$ is a free group generated by $\alpha_1, \ldots, \alpha_n$. We then consider the $j$-th longitude $l_j$ of $\sigma$ in $D^2 \times I$, where $l_j$ is the closure of the preferred parallel curve of $\sigma_j$, whose endpoints lie on the $x$-axis in $D^2 \times \{0,1\}$ as in Figure 2. We then consider the image of the longitude $\iota_*^{-1}(l_j)$ by the Magnus expansion and denote $\mu(i_1, \cdots, i_k, j)$ the coefficient of $X_{i_1}X_{i_2}\cdots X_{i_k}$ in the Magnus expansion.

**Theorem 2.1** ([3]). For any positive integer $q$, if $k < q$, then $\mu(i_1, \cdots, i_k, j)$ is invariant under isotopy. Moreover, if the sequence $i_1, \cdots, i_k, j$ is of distinct numbers, then $\mu(i_1, \cdots, i_k, j)$ is also link-homotopy invariant.

We call this invariant Milnor's $\mu$-invariant.

### 2.3. HOMFLYPT polynomial.

Recall the definition of the HOMFLYPT polynomial.

The **HOMFLYPT polynomial** $P(L; t, z) \in \mathbb{Z}[t^{\pm 1}, z^{\pm 1}]$ of an oriented link $L$ is defined by the following two formulas:

(1) $P(U; t, z) = 1$, and

(2) $t^{-1}P(L_{+}; t, z) - tP(L_{-}; t, z) = zP(L_{0}; t, z)$,

where $U$ denotes the trivial knot and $L_{+}$, $L_{-}$ and $L_{0}$ are link diagrams which are identical everywhere except near one crossing, where they look as follows:

\[ L_{+} = \quad ; \quad L_{-} = \quad ; \quad L_{0} = \quad . \]

Recall that the HOMFLYPT polynomial of a knot $K$ is of the form $P(K; t, z) = \sum_{k=0}^{N} P_{2k}(K; t)z^{2k}$, where $P_{2k}(K; t) \in \mathbb{Z}[t^{\pm 1}]$ is called the $2k$-th coefficient polynomial of $K$.

### 3. MAIN THEOREM

Given a sequence $J$ of elements of \(\{1, 2, \ldots, n\}\), $J < I$ will be used for any subsequence $J$ of $I$, possibly $I$ itself, and $|J|$ will denote the length of the sequence $J$.

Let $\sigma$ be an $n$-string link. Given a sequence $I = i_1i_2\cdots i_m$ obtained from $12\cdots n$ by deleting some elements, and a subsequence $J = j_1j_2\cdots j_k$ of $I$, we define a knot $\sigma_{I,J}$ as the closure of the product $b_I \cdot \sigma_J$. Here $\sigma_J$ is the $m$-string link obtained from $\sigma$ by deleting the $i$-th string, for all $i \in \{1,2,\ldots,n\} \setminus \{i_1, i_2, \cdots, i_m\}$ and replacing the $i$-th string with a trivial string underpassing all other components, for all $i \in \{i_1, i_2, \cdots, i_m\} \setminus \{j_1, j_2, \cdots, j_k\}$, and $b_I$ is the $m$-braid associated with the permutation $b = (i_1 i_2 \cdots i_{m-1} i_m)$ and such that the arc with connecting $(b^k(i_1), 0)$ with $(b^{k+1}(i_1), 1)$ underpasses all arcs with connecting $(b^k(i_1), 0)$ with $(b^{k+1}(i_1), 1)$ in $[0,1] \times [0,1]$ of braid diagram for $k < k' < n$. See Figure 3 for an example. We then have the following Theorem.
Theorem 3.1. Let \( \sigma \) be an \( n \)-string link \( (n \geq 4) \) with vanishing Milnor's link-homotopy invariants of length \( \leq m - 2 \). Then for any sequence \( I \) obtained from \( 12 \cdots n \) by deleting \( n - m \) elements, we have

\[
\mu_{\sigma}(I) = \frac{(-1)^{m-1}}{(m-1)!2^{m-1}} \sum_{J \subset I} (-1)^{|J|} P_{0}^{(m-1)}(\sigma_{J}; 1),
\]

where \( P_{0}^{(m-1)}(\cdot; 1) \) is the \( (m-1) \)-th derivative of the 0-th coefficient \( P_{0}(\cdot; t) \) of the HOMFLYPT polynomial evaluated at \( t = 1 \).

Note that the above vanishing assumption for string link is equivalent to that any \( (m-2) \)-substring link is link-homotopic to the trivial string link.

Remark 3.2. Theorem 1.1 remains valid if we use one of the following two alternative definitions of \( b_{I} \). One is that we use “overpasses” instead of “underpasses”. The other is that we use “any \( i \in \{i_{1}, i_{2}, \cdots, i_{m}\} \)” instead of “\( i_{1} \)”.

We also give the case of \( \mu \)-invariants of length 3 without the assumption.

Theorem 3.3. Let \( \sigma \) be an \( n \)-string link and \( I = i_{1}i_{2}i_{3} \) be a length 3 sequence with distinct numbers in \( \{1, 2, \cdots, n\} \). We then have

\[
\mu_{\sigma}(I) = -\sum_{J < I} (-1)^{|J|} a_{2}(\overline{\sigma}_{I,J}) - lk_{\sigma}(i_{1}i_{2})lk_{\sigma}(i_{2}i_{3}) + A_{I},
\]

where \( a_{2} \) is the second coefficient of the Conway polynomial, \( lk_{\sigma}(ij) \) is the linking number of the \( i \)-th component and \( j \)-th component of \( \sigma \), and

\[
A_{I} = \begin{cases} lk_{\sigma}(i_{1}i_{2}) & (i_{2} < i_{3} < i_{1}) \\ -lk_{\sigma}(i_{1}i_{2}) & (i_{1} < i_{3} < i_{2}) \\ 0 & (otherwise). \end{cases}
\]

Remark 3.4. This operation from a string link to a knot corresponds to \( Y \)-graph sum of links defined by M. Polyak. By taking this formula modulo \( \Delta_{\overline{\sigma}_{I,J}}(I) \), we get Polyak's relation between Milnor's \( \overline{\mu} \)-invariants and Conway polynomials [17].

Remark 3.5. In [19], K. Taniyama gave a formula expressing Milnor’s \( \overline{\mu} \)-invariants of length 3 for links by the second coefficient of the Conway polynomial assuming that all linking numbers vanish.

Remark 3.6. In [12], J.B. Meilhan showed that all finite type invariants of degree 2 for string link was given by some invariants (Theorem 2.8). So the formula in Theorem 3.3 could also be derived from [12].

4. Examples

Example 4.1. Let \( \sigma \) be a 3-string link shown by Figure 3. Then \( \mu_{123}(\sigma) = -1 \), \( \mu_{132}(\sigma) = \mu_{213}(\sigma) = 1 \) and \( \mu_{231}(\sigma) = \mu_{312}(\sigma) = \mu_{321}(\sigma) = 0 \). And \( lk_{\sigma}(12) = lk_{\sigma}(23) = 1 \) and \( lk_{\sigma}(13) = 0 \).

On the other hand, \( \sigma_{123,123} \) and \( \sigma_{123,23} \) are the figure-eight knot, and \( \sigma_{I,J} \) \( (J \neq 123, 23) \) is the trivial knot. Therefore we obtain

\[
- \sum_{J < 123} (-1)^{|J|} a_{2}(\overline{\sigma}_{123,J}) - lk_{\sigma}(12)lk_{\sigma}(23) = a_{2}(4_{1}) - a_{2}(4_{1}) - 1 \cdot 1 = -1.
\]
Similarly, we have

\[- \sum_{J<231}(-1)^{|J|}a_{2}(\overline{\sigma_{231,J}}) - lk_{\sigma}(23)lk_{\sigma}(31) = a_{2}(3_{1}4_{1}) - a_{2}(3_{1}) - a_{2}(4_{1}) - 1 \cdot 0 = 0,\]

\[- \sum_{J<312}(-1)^{|J|}a_{2}(\overline{\sigma_{312,J}}) - lk_{\sigma}(31)lk_{\sigma}(12) + lk_{\sigma}(13) = a_{2}(3_{1}) - a_{2}(3_{1}) - 0 \cdot 1 + 0 = 0.\]

Moreover, $\overline{\sigma_{132,32}}$ is the figure-eight knot and $\overline{\sigma_{132,J}} (J \neq 32)$ is the trivial knot. Therefore we obtain

\[- \sum_{J<132}(-1)^{|J|}a_{2}(\overline{\sigma_{132,J}}) - lk_{\sigma}(13)lk_{\sigma}(32) - lk_{\sigma}(13) = -a_{2}(4_{1}) - 0 \cdot 1 - 0 = 1.\]

Similarly, we have

\[- \sum_{J<213}(-1)^{|J|}a_{2}(\overline{\sigma_{213,J}}) - tk_{\sigma}(21)lk_{\sigma}(13) = a_{2}(7_{6}) - a_{2}(3_{1}) - a_{2}(4_{1}) - 1 \cdot 0 = 1,\]

\[- \sum_{J<321}(-1)^{|J|}a_{2}(\overline{\sigma_{321,J}}) - lk_{\sigma}(32)lk_{\sigma}(21) = a_{2}(5_{2}) - a_{2}(3_{1}) - 1 \cdot 1 = 0.\]

\[\begin{array}{c}
\includegraphics{sigma} \\
\includegraphics{sigma123123} \\
\includegraphics{sigma12312} \\
\includegraphics{sigma12313} \\
\includegraphics{sigma12323}
\end{array}\]

FIGURE 3

REFERENCES


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