ADDENDUM TO “COMMENSURABILITY BETWEEN ONCE-PUNCTURED TORUS GROUPS AND ONCE-PUNCTURED KLEIN BOTTLE GROUPS”

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1. INTRODUCTION

The main purpose of this addendum to [4] is to present a proof to [4, Proposition 3.7] which gives a classification of elliptic generator triples of the fundamental group of the quotient orbifold of the once-punctured Klein bottle (see Definition 2.1 and Proposition 2.2). We also prove the “converse” of [4, Theorem 5.1], namely, we give a condition for a faithful type-preserving $\text{PSL}(2, \mathbb{C})$-representation of the fundamental group of the once-punctured torus to be “commensurable” with that of the once-punctured Klein bottle by using Proposition 3.7 and Theorem 5.1 in the original paper (see Definitions 3.1, 3.2 and Theorem 3.13).

The rest of this paper is organized as follows. In Section 2, we give a proof to [4, Proposition 3.7] (see Proposition 2.2). In Section 3, we prove the “converse” of [4, Theorem 5.1] (see Theorem 3.13).

2. CLASSIFICATION OF ELLIPTIC GENERATOR TRIPLES

In this section, we give a proof to [4, Proposition 3.7]. To this end, we first introduce some notations and recall the definition of elliptic generators.

Let $N_{2,1}$ be the once-punctured Klein bottle and let $\iota_{N_{2,1}} : N_{2,1} \to N_{2,1}$ be the involution illustrated in Figure 1. Then we denote the quotient orbifold $N_{2,1}/\iota_{N_{2,1}}$ by $\mathcal{O}_{N_{2,1}}$ and denote the covering projection from $N_{2,1}$ to $\mathcal{O}_{N_{2,1}}$ by $p_{N_{2,1}}$. We identify $\pi_1(N_{2,1})$ with the image of the inclusion $\pi_1(N_{2,1}) \to \pi_1(\mathcal{O}_{N_{2,1}})$ induced by the projection $p_{N_{2,1}}$. Then $\pi_1(N_{2,1})$ is regarded as a normal subgroup of $\pi_1(\mathcal{O}_{N_{2,1}})$ of index 2,

$$\pi_1(N_{2,1}) = \langle Y_1, Y_2 \mid - \rangle \triangleleft \pi_1(\mathcal{O}_{N_{2,1}}) = \langle Q_0, Q_1, Q_2 \mid Q_0^2 = Q_1^2 = Q_2^2 = 1 \rangle,$$

such that $Y_1 = Q_0Q_1$ and $Y_2 = Q_0Q_2$. Set $K_{N_{2,1}} = (Y_1Y_2Y_1^{-1}Y_2)^{-1}$, $K_0 = Q_1^{Q_0}$ and $K_2 = Q_2^{Q_0}$, where $A^B = BAB^{-1}$. Then $K_{N_{2,1}}$ is represented by the puncture of $N_{2,1}$, and $K_0$ and $K_2$ are represented by the reflector lines which generate the corner reflector of order $\infty$. By the identification, we also obtain $K_{N_{2,1}} = K_2K_0$.

![Figure 1. The involution $\iota_{N_{2,1}}$ of $N_{2,1}$](image)

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Definition 2.1. An ordered triple \((Q_0, Q_1, Q_2)\) of elements of \(\pi_1(O_{N_2,1})\) is called an elliptic generator triple of \(\pi_1(O_{N_2,1})\) if its members generate \(\pi_1(O_{N_2,1})\) and satisfy \(Q_0^2 = Q_1^2 = 1\) and \(Q_1^2 Q_1 Q_0 = K_0 K_1\). A member of an elliptic generator triple of \(\pi_1(O_{N_2,1})\) is called an elliptic generator of \(\pi_1(O_{N_2,1})\).

Now we introduce Proposition 3.7 in the original paper.

Proposition 2.2. The elliptic generator triples of \(\pi_1(O_{N_2,1})\) are characterized as follows.
(1) For any elliptic generator triple \((Q_0, Q_1, Q_2)\) of \(\pi_1(O_{N_2,1})\), the following hold:

(1.1) The triples in the following bi-infinite sequence are also elliptic generator triples of \(\pi_1(O_{N_2,1})\).

\[
\ldots, (Q_0^{K_0 K_2}, Q_1^{K_0 K_2}, Q_2^{K_0 K_2}), (Q_2^{K_0}, Q_1^{K_0}, Q_0^{K_0}), (Q_0, Q_1, Q_2),
\]

\[(Q_2, Q_1, Q_2, Q_0^{K_0 K_2}), (Q_0^{K_0 K_2}, Q_1^{K_0 K_2}, Q_2^{K_0 K_2}), \ldots \]

To be precise, the following holds. Let \(\{Q_j\}\) be the sequence of elements of \(\pi_1(O_{N_2,1})\) obtained from \((Q_0, Q_1, Q_2)\) by applying the following rule:

\[Q_j^{K_0} = Q_{j-1}, \quad Q_j^{K_2} = Q_{j+5}\]

Then the triple \((Q_{3k}, Q_{3k+1}, Q_{3k+2})\) is also an elliptic generator triple of \(\pi_1(O_{N_2,1})\) for any \(k \in \mathbb{Z}\).

(1.2) \((Q_2, Q_1, Q_2, Q_0)\) is also an elliptic generator triple of \(\pi_1(O_{N_2,1})\).

(2) Conversely, any elliptic generator triple of \(\pi_1(O_{N_2,1})\) is obtained from a given elliptic generator triple of \(\pi_1(O_{N_2,1})\) by successively applying the operations in (1).

To prove Proposition 2.2, we need to introduce some definitions and notations. By a word in \(\{Q_0, Q_1, Q_2\}\), we mean a finite sequence \(Q_{i_1} Q_{i_2} \ldots Q_{i_t}\) where \(Q_{i_k} \in \{Q_0, Q_1, Q_2\}\). Here we call \(Q_{i_k}\) the \(k\)-th letter of the word. In particular, the first letter \(Q_{i_1}\) of the word is called the initial letter of the word and the last letter \(Q_{i_t}\) of the word is called the terminal letter of the word. The inverse of a word \(V = Q_{i_1} Q_{i_2} \ldots Q_{i_t}\) in \(\{Q_0, Q_1, Q_2\}\) is the word \(V^{-1} = Q_{i_t} Q_{i_{t-1}} \ldots Q_{i_1}\). The word length of \(V\) is denoted by \(l(V)\). A word \(V = Q_{i_1} Q_{i_2} \ldots Q_{i_t}\) in \(\{Q_0, Q_1, Q_2\}\) is reduced if \(Q_{i_k} \neq Q_{i_{k+1}}\) for any \(k = 1, \ldots, t-1\). Note that any element in \(\pi_1(O_{N_2,1})\) is uniquely represented by a reduced word. For two words \(U, V\) in \(\{Q_0, Q_1, Q_2\}\), by \(U \equiv V\) we denote the visual equality of \(U\) and \(V\), meaning that if \(U = Q_{i_1} Q_{i_2} \ldots Q_{i_t}\) and \(V = Q_{j_1} Q_{j_2} \ldots Q_{j_u}\) (\(Q_{i_k}, Q_{j_k} \in \{Q_0, Q_1, Q_2\}\)), then \(t = u\) and \(Q_{i_k} = Q_{j_k}\) for each \(k = 1, \ldots, t\). For example, two words \(Q_0 Q_1 Q_1 Q_2\) and \(Q_0 Q_2\) are not visually equal, though \(Q_0 Q_1 Q_1 Q_2\) and \(Q_0 Q_2\) are equal as elements of \(\pi_1(O_{N_2,1})\).

Proof of Proposition 2.2. The author got the idea of the proof from the proof of [2, Proposition 10.7] and [1, Lemma 2.1.7].

Since (1) can be proved by direct calculation, we give a proof of (2). For a given elliptic generator triple \((Q_0, Q_1, Q_2)\), set \(K_0 = Q_1^{Q_2}\) and \(K_2 = Q_1^{Q_2}\), and let \(\tau\) and \(\sigma\) be the automorphism of \(\pi_1(O)\) defined by

\[\tau(Q_0), \tau(Q_1), \tau(Q_2) = (Q_2^{K_2}, Q_1^{K_2}, Q_0^{K_2}),\]

\[\sigma(Q_0), \sigma(Q_1), \sigma(Q_2) = (Q_2, Q_1^{Q_2 Q_0}, Q_0^{Q_2}).\]

Then \(\tau\) and \(\sigma\) preserve \(K_{N_2,1}\) and hence they map elliptic generator triples to elliptic generator triples. Moreover, the operations in (1.1) is given by \(\tau^n\), and the operation in (1.2) is given by \(\sigma\). Hence we have only to show the following lemma.
**Lemma 2.3.** The group of automorphisms of $\pi_1(\mathcal{O}_{N_{2,1}})$ preserving $K_{N_{2,1}}$ is generated by $\sigma$ and $\tau$.

To prove this lemma, we prepare two claims.

**Claim 2.4.** Let $f$ be an automorphism of $\pi_1(\mathcal{O}_{N_{2,1}})$ which preserves $K_{N_{2,1}}$. Then for each $j = 0, 2$, we have

$$f(K_j) = K_j^{K_{N_{2,1}}^n}$$

for some $n \in \mathbb{Z}$ and some $j' \in \{0, 2\}$.

**Proof of Claim 2.4.** We first note that $\pi_1(\mathcal{O}_{N_{2,1}})$ is regarded as a subgroup of $\text{Isom}^+(\mathbb{H}^3)$. Then $\langle K_0, K_2 \rangle$ is regarded as the stabilizer of $\infty$ and $K_{N_{2,1}} = K_2 K_0$ is regarded as a parabolic transformation $K_{N_{2,1}}(z) = z + 2$. On the other hand, since $f(K_2) f(K_0) = K_2 K_0 = K_{N_{2,1}}$, we see that

$$f(K_0) K_{N_{2,1}}(f(K_0))^{-1} = f(K_0) f(K_2) f(K_0)(f(K_0))^{-1} = f(K_0) f(K_2) = K_{N_{2,1}}^{-1}.$$

This implies that $f(K_0) K_{N_{2,1}}(f(K_0))^{-1}$ is parabolic and that $\text{Fix}(f(K_0)) K_{N_{2,1}}(f(K_0))^{-1} = \{\infty\}$, where $\text{Fix}(A)$ denotes the fixed point set of $A$ in $\partial \mathbb{H}^3 = \mathbb{C} \cup \{\infty\}$. By $\text{Fix}(K_{N_{2,1}}) = \{\infty\}$ and $\text{Fix}(f(K_0) K_{N_{2,1}}(f(K_0))^{-1}) = f(K_0)(\text{Fix}(K_{N_{2,1}}))$, we have $f(K_0)(\infty) = \infty$. Hence $f(K_0) \in \langle K_0, K_2 \rangle$ and therefore $f(K_0) = K_j^{K_{N_{2,1}}^n}$ for some $n \in \mathbb{Z}$ and some $j' \in \{0, 2\}$. By a similar argument, we obtain the desired result for $f(K_2)$.

**Claim 2.5.** Let $f$ be an automorphism of $\pi_1(\mathcal{O}_{N_{2,1}})$ such that $f(K_j) = K_j$ for each $j = 0, 2$. Suppose that $f(Q_s) = W_s Q_s W_s^{-1}$ for each $s = 0, 1, 2$, where $W_s$ is a reduced word in $\{Q_0, Q_1, Q_2\}$ whose terminal letter is different from $Q_s$. Then the following hold.

1. If $W_1$ is a trivial word, then $W_j$ is also a trivial word for each $j = 0, 2$.
2. If $W_1$ is a non-trivial word, then one of the following holds for each $j = 0, 2$.
   a. $W_1 Q_1 Q_j = W_j Q_j W_j^{-1}$. In particular, the initial letter of $W_1$ is $Q_j$.
   b. $W_1 \equiv W_j Q_j W_j^{-1} Q_j$. In particular, the terminal letter of $W_1$ is $Q_j$.
   c. $W_1 Q_j \equiv W_j Q_j W_j^{-1}$. In particular, the terminal letter of $W_1$ is different from $Q_j$.

**Proof of Claim 2.5.** For each $j = 0, 2$, we have the following identity:

$$Q_j Q_1 Q_j = K_j = f(K_j) = f(Q_j Q_1 Q_j) = W_j Q_j W_j^{-1} \cdot W_1 Q_1 W_1^{-1} \cdot W_j Q_j W_j^{-1}.$$

This implies that $Q_j \cdot W_j Q_j W_j^{-1} \cdot W_1$ commutes with $Q_1$. Since $\pi_1(\mathcal{O}_{N_{2,1}})$ is isomorphic to the free product of three cyclic groups $\langle Q_s \rangle$ of order 2, we have

$$(\text{Eq1}) \quad Q_j \cdot W_j Q_j W_j^{-1} \cdot W_1 = Q_1 \text{ or } 1.$$ 

To show the assertion (1), we assume that $W_1$ is a trivial word. Then, by the identity (Eq1), we have $Q_j \cdot W_j Q_j W_j^{-1} = Q_1 \text{ or } 1$. By the abelianization of this identity, we have $Q_j \cdot W_j Q_j W_j^{-1} = 1$. This implies that $W_j$ commutes with $Q_j$, and hence $W_j = Q_j$ or $1$. Since the terminal letter of $W_j$ is different from $Q_j$, we have $W_j = 1$. So we obtain the desired result.

Next, we show the assertion (2). If either $W_0$ or $W_2$ is a trivial word, then the identity (Eq1) implies that $W_1 = Q_1 \text{ or } 1$. This is a contradiction. Hence $W_j$ is also a non-trivial word for any $j = 0, 2$.

Suppose first that $Q_j \cdot W_j$ is a reduced word. Then $Q_j \cdot W_j Q_j W_j^{-1}$ is also a reduced word. Hence the identity (Eq1) implies that the word $Q_j \cdot W_j Q_j W_j^{-1}$, except possibly for
the first letter $Q_j$, is cancelled out by the word $W_1$, and therefore one of the following holds.

- $W_1 \equiv W_j Q_j W_j^{-1} Q_j Q_1$,
- $W_1 \equiv W_j Q_j W_j^{-1}$ and $Q_j = Q_1$,
- $W_1 \equiv W_j Q_j W_j^{-1} Q_j$.

However, the first identity can not hold because the terminal letter of $W_1$ is different from $Q_1$ by the assumption, and second identity can not hold because $j = 0, 2$. Hence the third identity holds. So we obtain the identity in the condition (ii).

Suppose next that $Q_j \cdot W_j$ is not a reduced word, i.e., $W_j \equiv Q_j \cdot V_j$ for some reduced word $V_j$. Then, by the identity (Eq1), we have

\[(Eq2) \quad V_j Q_j W_j^{-1} \cdot W_1 = Q_1 \text{ or } 1.\]

Since $V_j Q_j W_j^{-1}$ is a reduced word, it must be cancelled out by $W_1$, except possibly for the initial letter of $V_j$, and therefore one of the following hold.

- $W_1 \equiv W_j Q_j V_j^{-1} Q_1$,
- $W_1 \equiv W_j Q_j V_j^{-1}$ and $V_j \equiv Q_1 V_j'$ for some reduced word $V_j'$.
- $W_1 \equiv W_j Q_j V_j^{-1}$.

The first identity can not hold by the fact that the terminal letter of $W_1$ is different from $Q_1$. If the second identity or the third identity holds, then the condition (i) or (iii) holds accordingly.

We now begin to prove Lemma 2.3 by using the above claims.

Let $f$ be an automorphism of $\pi_1(\mathcal{O}_{N_{2,1}})$ preserving $K_{N_{2,1}}$.

**Step 1.** For each $j = 0, 2$, we show that we may assume $f(K_j) = K_j$ by post composing a power of $\tau$ to $f$ if necessary. By Claim 2.4, we have $f(K_0) = K_0^{K_{N_{2,1}}} = K_0^{K_{N_{2,1}}}$ for some $n \in \mathbb{Z}$ and for some $j' \in \{0, 2\}$. Since $\tau^2$ is an inner-automorphism by $K_{N_{2,1}}$, we may assume $f(K_0) = K_{j'}$ by post composing a power of $\tau^2$ to $f$ if necessary. By the assumption $f(K_2)f(K_0) = f(K_{N_{2,1}})$, we have $f(K_2) = K_{N_{2,1}}f(K_0) = K_2 K_0 f(K_0)$. Hence

\[f(K_2) = K_2 K_0 K_{j'} = \begin{cases} K_2 & \text{if } j' = 0, \\ K_0 K_2 & \text{if } j' = 2. \end{cases}\]

Since $\tau$ maps $(K_0, K_2)$ to $(K_2^{K_0}, K_0)$, we may assume $f(K_j) = K_j$ for each $j = 0, 2$ by post composing $\tau$ to $f$ if necessary.

**Step 2.** For each $s = 0, 1, 2$, we show that we may assume $f(Q_s) = W_s Q_s W_s^{-1}$ by post composing $\sigma$ to $f$ if necessary. Since $f(Q_s)$ has order 2 and since $\pi_1(\mathcal{O}_{N_{2,1}})$ is isomorphic to the free product of three cyclic groups $\langle Q_s \rangle$ of order 2, we have $f(Q_s) = V_s Q_{\theta(s)} V_s^{-1}$ for some $\theta(s) \in \{0, 1, 2\}$, where $V_s$ is a reduced word whose terminal letter is different from $Q_{\theta(s)}$. By the abelianization of the identity

\[Q_s Q_t Q_s = K_2 = f(K_2) = f(Q_s Q_t Q_s) = f(Q_s) f(Q_t) f(Q_s),\]

we have $\theta(1) = 1$. By Step1, we have the following identities:

\[Q_0 Q_1 Q_0 = K_0 = f(K_0) = f(Q_0) f(Q_1) f(Q_0),\]
\[Q_2 Q_1 Q_2 = K_2 = f(K_2) = f(Q_2) f(Q_1) f(Q_2).\]
By these identities, we have the following identity:

$$Q_1 \cdot Q_2 f(Q_2) f(Q_0) Q_0 = Q_2 f(Q_2) f(Q_0) Q_0 \cdot Q_1.$$ 

This implies that $Q_2 f(Q_2) f(Q_0) Q_0 = Q_2 V_2 \theta(2) V_2^{-1} V_0 Q_0 V_0^{-1} Q_0$ commutes with $Q_1$. As in the proof of Claim 2.5, we see that

$$Q_2 V_2 \theta(2) V_2^{-1} V_0 Q_0 V_0^{-1} Q_0 = 1$$

Since the word length of the left hand side of the above identity is even, we have $Q_2 V_2 \theta(2) V_2^{-1} V_0 Q_0 V_0^{-1} Q_0 = 1$. By the abelianization of this identity, we have

$$Q_2 \theta(2) Q_0 Q_0 = 1.$$ 

This implies that $\theta(0), \theta(2) \in \{0, 2\}$ and $\theta(0) \neq \theta(2)$. Hence $\theta$ must be a permutation on the set $\{0, 1, 2\}$ such that $\theta(1) = 1$. Since $\sigma$ preserves $K_0$ and $K_2$ and since $\sigma$ maps $(Q_0, Q_1, Q_2)$ to $(Q_2, Q_2^Q_0, Q_0^Q_2)$, we may assume $\theta = id$ by post composing $\sigma$ to $f$ if necessary. Hence $f(Q_s) = W_s Q_s W_s^{-1}$ for each $s = 0, 1, 2$, where $W_s$ is a reduced word whose terminal letter is different from $Q_s$.

**Step 3.** We show that $f = (\sigma^2)^{n+1}$. If $W_1$ is a trivial word, $W_j$ is a trivial word for any $j = 0, 2$ by Claim 2.5, and therefore $f = id$. So we assume that $W_1$ is a non-trivial word. Since the terminal letter of $W_1$ is different from $Q_1$, we assume that the terminal letter of $W_1$ is $Q_0$. (The other case is treated by a parallel argument.) Then the condition (2)-(i) or (2)-(ii) in Claim 2.5 holds for $j = 0$, and the condition (2)-(i) or (2)-(iii) in Claim 2.5 holds for $j = 2$. Note that the number of $Q_1$ contained $W_1$ is odd or even according to whether the condition (2)-(i) in Claim 2.5 holds or not. If the number of $Q_1$ contained $W_1$ is odd, then the condition (2)-(i) in Claim 2.5 holds for each $j = 0, 2$. In particular, the initial letter of $W_1$ is $Q_0$ and $Q_2$, a contradiction. Hence the number of $Q_1$ contained $W_1$ is even. Then the condition (2)-(ii) in Claim 2.5 holds for $j = 0$, and the condition (2)-(iii) in Claim 2.5 holds for $j = 2$, namely, we have $W_1 \equiv W_0 Q_0 W_0^{-1} Q_0$ and $W_1 Q_2 \equiv W_2 Q_2 W_2^{-1}$. Thus we see $W_0 Q_0 W_0^{-1} Q_0 Q_2 \equiv W_1 Q_2 \equiv W_2 Q_2 W_2^{-1}$.

This implies that $i$-th letter of $W_0 Q_0 W_0^{-1} Q_0 Q_2$ is equal to $(l-i+1)$-th letter of $W_0 Q_0 W_0^{-1} Q_0 Q_2$, where $l = l(W_0 Q_0 W_0^{-1} Q_0 Q_2)$. Hence $W_0 \equiv (Q_2 Q_2^n) Q_2$ for some $n \in N$, and therefore $W_1 \equiv (Q_2 Q_2^n)^{2(n+1)}$ and $W_2 \equiv (Q_2 Q_2^n)^{n+1}$. Thus we see

$$f(Q_0) = Q_0^{(Q_2 Q_2^n)^{2(n+1)}}, f(Q_1) = Q_1^{(Q_2 Q_2^n)^{2(n+1)}}$$

and $f(Q_2) = Q_2^{(Q_2 Q_2^n)^{n+1}}$.

On the other hand, $(\sigma^2 (Q_0), \sigma^2 (Q_1), \sigma^2 (Q_2)) = (Q_0^{Q_2 Q_2^n}, Q_1^{Q_2 Q_2^n}, Q_2^{Q_2 Q_2^n})$. Thus we have $f = (\sigma^2)^{n+1}$. Hence we obtain the desired result.

**Remark 2.6.** It should be noted that the proof of Proposition 2.2 does not use the condition that $(f(Q_0), f(Q_1), f(Q_2))$ generates $\pi_1(\mathcal{O}_{N_2,1})$. Hence, in Definition 2.1, the condition that members of the triple generate $\pi_1(\mathcal{O}_{N_2,1})$ is actually a consequence of the other conditions (cf. [4, Remark 3.6]).

**Definition 2.7.** For an elliptic generator triple $(Q_0, Q_1, Q_2)$ of $\pi_1(\mathcal{O}_{N_2,1})$, the bi-infinite sequence $\{Q_j\}$ in Proposition 2.2(1.1) is called the sequence of elliptic generators of $\pi_1(\mathcal{O}_{N_2,1})$ (associated with $(Q_0, Q_1, Q_2)$).

In preparation for the next section, we recall the definition of elliptic generators of the fundamental group of the quotient orbifold of the once-punctured torus.

Let $\Sigma_{1,1}$ be the once-punctured torus and let $\iota_{\Sigma_{1,1}}: \Sigma_{1,1} \to \Sigma_{1,1}$ be the involution illustrated in Figure 2. Then we denote the quotient orbifold $\Sigma_{1,1}/\iota_{\Sigma_{1,1}}$ by $\mathcal{O}_{\Sigma_{1,1}}$ and
denote the covering projection from $\Sigma_{1,1}$ to $\mathcal{O}_{\Sigma_{1,1}}$ by $p_{\Sigma_{1,1}}$. We identify $\pi_1(\Sigma_{1,1})$ with the image of the inclusion $\pi_1(\Sigma_{1,1}) \to \pi_1(\mathcal{O}_{\Sigma_{1,1}})$ induced by the projection $p_{\Sigma_{1,1}}$. Then $\pi_1(\Sigma_{1,1})$ is regarded as a normal subgroup of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ of index 2,

$$\pi_1(\Sigma_{1,1}) = \langle X_1, X_2 \mid - \rangle \triangleleft \pi_1(\mathcal{O}_{\Sigma_{1,1}}) = \langle P_0, P_1, P_2 \mid P_0^2 = P_1^2 = P_2^2 = 1 \rangle,$$

such that $X_1 = P_2P_1$ and $X_2 = P_0P_1$. Set $K_{\Sigma_{1,1}} = [X_1, X_2] = X_1X_2X_1^{-1}X_2^{-1}$, $K = (P_0P_1P_2)^{-1}$. Then $K_{\Sigma_{1,1}}$ and $K$ are represented by the punctures of $\Sigma_{1,1}$ and $\mathcal{O}_{\Sigma_{1,1}}$, respectively.

![Figure 2. The involution $\iota_{\Sigma_{1,1}}$ of $\Sigma_{1,1}$](image)

**Definition 2.8.** An ordered triple $(P_0, P_1, P_2)$ of elements of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ is called an **elliptic generator triple of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$** if its members generate $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ and satisfy $P_0^2 = P_1^2 = P_2^2 = 1$ and $(P_0P_1P_2)^{-1} = K$. A member of an elliptic generator triple of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ is called an **elliptic generator of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$**.

**Definition 2.9.** For an elliptic generator triple $(P_0, P_1, P_2)$ of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$, let \{P_2\} be the bi-infinite sequence defined as follows (see [1, Proposition 2.1.6(1.1)] and [4, Proposition 3.3(1.1)]).

$$\ldots, P_2, P_0K^{-1}, P_1K^{-1}, P_2K^{-1}, P_0, P_1, P_2, P_0K, P_1K, P_2K, P_0K^2, \ldots$$

We call the sequence \{P_2\} the **sequence of elliptic generators of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$** (associated with $(P_0, P_1, P_2)$).

### 3. Commensurability

In this section, we prove the "converse" of [4, Theorem 5.1], namely, we give a condition for a faithful type-preserving $\text{PSL}(2, \mathbb{C})$-representation of $\pi_1(\Sigma_{1,1})$ to be commensurable with that of $\pi_1(N_{2,1})$. We first introduce some notations and facts.

Let $\Sigma_{1,2}, \mathcal{O}_{\Sigma_{1,2}}, \mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ be the twice-punctured torus, the $(2, 2, 2, \infty)$-orbifold (i.e., the orbifold with underlying space a punctured sphere and with four cone points of cone angle $\pi$), the $(2; 2, \infty)$-orbifold (i.e., the orbifold with underlying space a disk and with a cone point of cone angle $\pi$ and with a corner reflector of order 2 and a corner reflector of order $\infty$) and the $[2, 2, 2, \infty]$-orbifold (i.e., the orbifold with underlying space a disk and with three corner reflectors of order 2 and a corner reflector of order $\infty$), respectively. Note that $\mathcal{O}_{\Sigma_{1,2}}$ is a quotient orbifold of $\Sigma_{1,2}$ by an involution and that both $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ are common quotient orbifolds of $\mathcal{O}_{\Sigma_{1,1}}$ and $\mathcal{O}_{N_{2,1}}$ by involutions (see [4, Section2] for
Their (orbifold) fundamental groups have the following presentations:

$$\pi_1(\Sigma_{1,2}) = \langle Z_1, Z_2, Z_3 \mid \rangle,$$
$$\pi_1(\mathcal{O}_{\Sigma_{1,2}}) = \langle R_0, R_1, R_2, R_3 \mid R_0^2 = R_1^2 = R_2^2 = R_3^2 = 1 \rangle,$$
$$\pi_1(\mathcal{O}_\alpha) = \langle S_0, S_1, S_2 \mid S_0^2 = S_1^2 = 1, (S_1 S_2)^2 = 1 \rangle,$$
$$\pi_1(\mathcal{O}_\beta) = \langle T_0, T_1, T_2, T_3 \mid T_0^2 = T_1^2 = T_2^2 = T_3^2 = 1 \rangle.$$

Here the generators satisfy the following conditions:

$$Z_1 = R_0 R_1, \quad Z_2 = R_2 R_1, \quad Z_3 = R_1 R_3, \quad K_{\Sigma_{1,2}} = K_{\mathcal{O}_{\Sigma_{1,2}}}, \quad K_{\Sigma_{1,2}}' = (K_{\overline{\mathcal{O}_{\Sigma_{1,2}}}})^{R_0}$$
$$P_0 = S_0^2, \quad P_1 = S_1 S_2, \quad P_2 = S_0,$$
$$Q_0 = S_0^2, \quad Q_1 = S_1, \quad Q_2 = S_0,$$
$$P_0 = T_0 T_1, \quad P_1 = T_1 T_2, \quad P_2 = T_2 T_3,$$
$$Q_0 = T_1 T_2, \quad Q_1 = T_3^{T_1}, \quad Q_2 = T_0 T_1,$$

where $K_{\Sigma_{1,2}} = Z_1 Z_2 Z_3$, $K_{\Sigma_{1,2}}' = Z_2 Z_1 Z_3$ and $K_{\mathcal{O}_{\Sigma_{1,2}}} = R_0 R_1 R_2 R_3$, which are represented by the punctures of $\Sigma_{1,2}$ and $\mathcal{O}_{\Sigma_{1,2}}$ (see Figure 3).

In summary, we have the commutative diagram of double coverings as shown in Figure 3. Every arrow represents a double covering (see [4, Section 2] for details).

**Figure 3**

**Definition 3.1.** (1) For $F = \Sigma_{1,1}, \Sigma_{1,2}, \Sigma_{2,1}, \mathcal{O}_{\Sigma_{1,1}}, \mathcal{O}_{\Sigma_{1,2}}, \mathcal{O}_{\Sigma_{2,1}}, \mathcal{O}_\alpha$ or $\mathcal{O}_\beta$, a representation $\rho : \pi_1(F) \to \text{PSL}(2, \mathbb{C})$ is *type-preserving* if it is irreducible (equivalently, it
does not have a common fixed point in $\partial \mathbb{H}^3$) and sends peripheral elements to parabolic transformations.

(2) Type-preserving $\text{PSL}(2, \mathbb{C})$-representations $\rho$ and $\rho'$ are equivalent if $i_g \circ \rho = \rho'$, where $i_g$ is the inner automorphism, $i_g(h) = ghg^{-1}$, of $\text{PSL}(2, \mathbb{C})$ determined by $g$.

In the above definition, if $F$ is an orbifold with reflector lines, an element of $\pi_1(F)$ is said to be peripheral if it is (the image of) a peripheral element of $\pi_1(\tilde{F})$, where $\tilde{F}$ is the orientation double covering of $F$.

**Definition 3.2.** Let $\rho_1$ and $\rho_2$ be type-preserving $\text{PSL}(2, \mathbb{C})$-representations of $\pi_1(\Sigma_{1,1})$ (resp. $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$) and $\pi_1(N_{2,1})$ (resp. $\pi_1(\mathcal{O}_{N_{2,1}})$), respectively. The representations $\rho_1$ and $\rho_2$ are commensurable if there exist a double covering $p_1$ from $\Sigma_{1,2}$ (resp. $\mathcal{O}_{\Sigma_{1,2}}$) to $\Sigma_{1,1}$ (resp. $\mathcal{O}_{\Sigma_{1,1}}$) and a double covering $p_2$ from $\Sigma_{1,2}$ (resp. $\mathcal{O}_{\Sigma_{1,2}}$) to $N_{2,1}$ (resp. $\mathcal{O}_{N_{2,1}}$) such that $\rho_1 \circ (p_1)_*$ and $\rho_2 \circ (p_2)_*$ are equivalent, namely $\rho_1 \circ (p_1)_* = i_g \circ \rho_2 \circ (p_2)_*$, for some $g \in \text{PSL}(2, \mathbb{C})$. After replacing $\rho_2$ with $i_g \circ \rho_2$, without changing the equivalence class, the last identity can be replaced with the identity $\rho_1 \circ (p_1)_* = \rho_2 \circ (p_2)_*$.

In this paper, we study the following problem which is a “converse” of [4, Problem 2.3].

**Problem 3.3.** For a given type-preserving $\text{PSL}(2, \mathbb{C})$-representation $\rho_1$ of $\pi_1(\Sigma_{1,1})$ (resp. $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$), when does there exist a type-preserving $\text{PSL}(2, \mathbb{C})$-representation $\rho_2$ of $\pi_1(N_{2,1})$ (resp. $\pi_1(\mathcal{O}_{N_{2,1}})$) which is commensurable with $\rho_1$?

To answer this problem, we recall the definitions of complex probabilities of type-preserving representations of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ and $\pi_1(\mathcal{O}_{N_{2,1}})$ (see [1, Section 2] and [4, Section 4] for details).

The following fact is well-known (cf. [5, Section 5.4] and [1, Proposition 2.2.2]).

**Proposition 3.4.** For $F = \Sigma_{1,1}$ or $N_{2,1}$, the following hold.

1. The restriction of any type-preserving $\text{PSL}(2, \mathbb{C})$-representation of $\pi_1(\mathcal{O}_F)$ to $\pi_1(F)$ is type-preserving.

2. Conversely, every type-preserving $\text{PSL}(2, \mathbb{C})$-representation of $\pi_1(F)$ extends to a unique type-preserving $\text{PSL}(2, \mathbb{C})$-representation of $\pi_1(\mathcal{O}_F)$.

By this proposition, the following are well-defined.

**Definition 3.5.** (1) For $F = \Sigma_{1,1}$ or $\mathcal{O}_{\Sigma_{1,1}}$, the symbol $\Omega(\Sigma_{1,1})$ denotes the space of all type-preserving $\text{PSL}(2, \mathbb{C})$-representations $\rho_1$ of $\pi_1(F)$.

(2) For $F = N_{2,1}$ or $\mathcal{O}_{N_{2,1}}$, the symbol $\Omega(N_{2,1})$ (resp. $\Omega'(N_{2,1})$) denotes the space of all type-preserving $\text{PSL}(2, \mathbb{C})$-representations $\rho_2$ of $\pi_1(F)$ such that $\text{tr}(\rho_2(K_{N_{2,1}})) = -2$ (resp. $\text{tr}(\rho_2(K_{N_{2,1}})) = +2$).

**Remark 3.6.** For any $\rho_2 \in \Omega'(N_{2,1})$, the isometries $\rho_2(Q_0Q_2) = \rho_2(Y_2)$ and $\rho_2(K_{N_{2,1}})$ have a common fixed point (see [3, Lemma 4.5(ii)]), and hence $\rho_2$ is indiscrete or non-faithful (see [3, Lemma 4.7]).

**Definition 3.7.** (1) Let $\rho_1$ be an element of $\Omega(\Sigma_{1,1})$. Fix a sequence of elliptic generators $\{P_j\}$ of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$. Set

$$(x_1, x_{12}, x_2) = (\text{tr}(\rho_1(X_1)), \text{tr}(\rho_1(X_1X_2)), \text{tr}(\rho_1(X_2))),$$

where $X_1 = P_2P_1$ and $X_2 = P_0P_1$. Suppose that $x_1x_{12}x_2 \neq 0$. Then we call the following triple $(a_0, a_1, a_2) \in (\mathbb{C}^*)^3$ the complex probability associated with $\{\rho_1(P_j)\}$, where $\mathbb{C}^* =$
\[ C - \{0\}. \]

\[
a_0 = \frac{x_1}{x_{12}x_2}, \quad a_1 = \frac{x_{12}}{x_2x_1}, \quad a_2 = \frac{x_2}{x_1x_{12}}.
\]

(2) Let \( \rho_2 \) be an element of \( \Omega(N_{2,1}) \). Fix a sequence of elliptic generators \( \{Q_j\} \) of \( \pi_1(O_{N_{2,1}}) \). Set

\[
(y_1, y_{12}, y_2) = (\text{tr}(\rho_2(Y_1))/i, \text{tr}(\rho_2(Y_1Y_2))/i, \text{tr}(\rho_2(Y_2)/i),
\]

where \( Y_1 = Q_0Q_1 \) and \( Y_2 = Q_0Q_2 \). Set \( y_{12} = \text{tr}(\rho_2(Y_1Y_2^{-1}))/i = \frac{y_1y_2 - y_{12}}{y_1y_2} \). Suppose that \( y_1y_2y_{12} \neq 0 \). Then we call the following triple \( (b_0, b_1, b_2) \in (\mathbb{C}^*)^3 \) the complex probability associated with \( \{\rho_2(Q_j)\} \).

\[
b_0 + b_1 + b_2 = 1, \quad b_0 = \frac{y_1}{y_2y_{12}}, \quad b_1 = \frac{4}{y_1y_2y_{12}}, \quad b_2 = \frac{y_{12}'}{y_1y_2}.
\]

**Remark 3.8.** (1) For any sequence of elliptic generators \( \{P_j\} \) of \( \pi_1(O_{\Sigma_{1,1}}) \) and any \( \rho_1 \in \Omega(\Sigma_{1,1}) \), the complex probability \( (a_0, a_1, a_2) \) associated with \( \{\rho_1(P_j)\} \) satisfies the following identity (see [1, Lemma 2.4.1(1)] for details):

\[
a_0 + a_1 + a_2 = 1.
\]

(2) For any sequence of elliptic generators \( \{Q_j\} \) of \( \pi_1(O_{N_{2,1}}) \) and any \( \rho_2 \in \Omega(N_{2,1}) \), the complex probability \( (b_0, b_1, b_2) \) associated with \( \{\rho_2(Q_j)\} \) satisfies the following identity (see [4, Section 4] for details):

\[
b_0 + b_1 + b_2 = 1.
\]

We introduce the following proposition (cf. [1, Proposition 2.4.4] and [4, Propositions 4.8 and 4.11]).

**Proposition 3.9.** (1) For any triple \( (a_0, a_1, a_2) \in (\mathbb{C}^*)^3 \) such that \( a_0 + a_1 + a_2 = 1 \) and for any sequence of elliptic generators \( \{P_j\} \) of \( \pi_1(O_{\Sigma_{1,1}}) \), there is an element \( \rho_1 \in \Omega(\Sigma_{1,1}) \) such that the complex probability associated with \( \{\rho_1(P_j)\} \) is equal to \( (a_0, a_1, a_2) \).

(2) For any triple \( (b_0, b_1, b_2) \in (\mathbb{C}^*)^3 \) such that \( b_0 + b_1 + b_2 = 1 \) and for any sequence of elliptic generators \( \{Q_j\} \) of \( \pi_1(O_{N_{2,1}}) \), there is an element \( \rho_2 \in \Omega(N_{2,1}) \) such that the complex probability associated with \( \{\rho_2(Q_j)\} \) is equal to \( (b_0, b_1, b_2) \).

**Notation 3.10.** (1) Let \( \rho_1 \) be an element of \( \Omega(\Sigma_{1,1}) \) and let \( \{P_j\} \) be a sequence of elliptic generators of \( \pi_1(O_{\Sigma_{1,1}}) \). Let \( \xi \) be the automorphism of \( \pi_1(O_{\Sigma_{1,1}}) \) given by the following (cf. [1, Proposition 2.1.6] and [4, Proposition 3.3]):

\[
(\xi(P_0), \xi(P_1), \xi(P_2)) = (P_2^{P_1}, P_1, P_0^K).
\]

If the complex probability associated with \( \{\rho_1(\xi^k(P_j))\} \) is well-defined, then we denote it by \( (a_0^{(k)}, a_1^{(k)}, a_2^{(k)}) \).

(2) Let \( \rho_2 \) be an element of \( \Omega(N_{2,1}) \) and let \( \{Q_j\} \) be a sequence of elliptic generators of \( \pi_1(O_{N_{2,1}}) \). Let \( \sigma \) be the automorphism of \( \pi_1(O_{N_{2,1}}) \) given by Proposition 2.2(1.2), namely,

\[
(\sigma(Q_0), \sigma(Q_1), \sigma(Q_2)) = (Q_2, Q_1^{Q_2Q_0}, Q_0^{Q_2}).
\]

If the complex probability associated with \( \{\rho_2(\sigma^k(Q_j))\} \) is well-defined, then we denote it by \( (b_0^{(k)}, b_1^{(k)}, b_2^{(k)}) \).

The following lemma can be verified by simple calculation (cf. [1, Lemma 2.4.1] and [4, Lemmas 4.10 and 4.13]).
Lemma 3.11. (1) Let $\rho_1$ be an element of $\Omega(\Sigma_{1,1})$ and let $\{P_j\}$ be a sequence of elliptic generators of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$. Suppose that the complex probability $(a_0^{(k)}, a_1^{(k)}, a_2^{(k)})$ associated with $\{\rho_1(\xi^k(P_j))\}$ is well-defined for any $k \in \mathbb{Z}$. Then we have the following identities (cf. Figure 4):

\[
\begin{align*}
(\text{a)} & \quad a_0^{(k+1)} = 1 - a_2^{(k)}, \quad a_1^{(k+1)} = \frac{a_1^{(k)} a_2^{(k)}}{1 - a_2^{(k)}}, \quad a_2^{(k+1)} = \frac{a_2^{(k)} a_0^{(k)}}{1 - a_2^{(k)}}, \\
(\text{b)} & \quad a_0^{(k-1)} = \frac{a_2^{(k)} a_0^{(k)}}{1 - a_0^{(k)}}, \quad a_1^{(k-1)} = \frac{a_0^{(k)} a_1^{(k)}}{1 - a_0^{(k)}}, \quad a_2^{(k-1)} = 1 - a_0^{(k)}.
\end{align*}
\]

(2) Let $\rho_2$ be an element of $\Omega(N_{2,1})$ and let $\{Q_j\}$ be a sequence of elliptic generators of $\pi_1(\mathcal{O}_{N_{2,1}})$. Suppose that the complex probability $(b_0^{(k)}, b_1^{(k)}, b_2^{(k)})$ associated with $\{\rho_2(\sigma^k(Q_j))\}$ is well-defined for any $k \in \mathbb{Z}$. Then we have the following identities (cf. Figure 5):

\[
\begin{align*}
(\text{a)} & \quad b_0^{(k+1)} = 1 - b_2^{(k)}, \quad b_1^{(k+1)} = \frac{b_1^{(k)} b_2^{(k)}}{1 - b_2^{(k)}}, \quad b_2^{(k+1)} = \frac{b_2^{(k)} b_0^{(k)}}{1 - b_2^{(k)}}, \\
(\text{b)} & \quad b_0^{(k-1)} = \frac{b_2^{(k)} b_0^{(k)}}{1 - b_0^{(k)}}, \quad b_1^{(k-1)} = \frac{b_0^{(k)} b_1^{(k)}}{1 - b_0^{(k)}}, \quad b_2^{(k-1)} = 1 - b_0^{(k)}.
\end{align*}
\]

Throughout this paper, we employ the following convention.

Convention 3.12. (1) For any element $\rho_1 \in \Omega(\Sigma_{1,1})$, after taking conjugate of $\rho_1$ by some element of $\text{PSL}(2, \mathbb{C})$, we always assume that $\rho_1$ is normalized so that the following identity is satisfied.

\[
\rho_1(K) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
\]
(2) For any element $\rho_2 \in \Omega(N_{2,1})$, after taking conjugate of $\rho_2$ by some element of $PSL(2, \mathbb{C})$, we always assume that $\rho_2$ is normalized so that the following identities are satisfied.

$$\rho_2(K_0) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \rho_2(K_0) = \begin{pmatrix} i & -2i \\ 0 & -i \end{pmatrix}. $$

Now we give a partial answer to Problem 3.3. By [4, Lemma 4.15], we may only consider the problem for the quotient orbifolds. Our partial answer to the commensurability problem for representations of the fundamental groups of the orbifolds $\mathcal{O}_{\Sigma_{1,1}}$ and $\mathcal{O}_{N_{2,1}}$ is given as follows.

**Theorem 3.13.** Under Convention 3.12, the following hold:

1. Let $\rho_1$ be an element of $\Omega(\Sigma_{1,1})$. Suppose that $\rho_1$ is faithful. Then the following conditions are equivalent.

   (i) There exists a faithful representation $\rho_2 \in \Omega(N_{2,1})$ which is commensurable with $\rho_1$.

   (ii) There exist a sequence of elliptic generators $\{P_j\}$ of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ and an integer $k_0$ such that the complex probability $(a_0, a_1, a_2)$ associated with $\{\rho_1(P_j)\}$ satisfies the following identity under Notation 3.10(1) (cf. Figure 6):

   $$(a_0^{(k_0)}, a_1^{(k_0)}, a_2^{(k_0)}) = (a_2, a_1, a_0).$$

   (iii) There exists a sequence of elliptic generators $\{P_j\}$ of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ such that the complex probability $(a_0, a_1, a_2)$ associated with $\{\rho_2(P_j)\}$ satisfies one of the following identities:

   (a) $(a_0^{(1)}, a_1^{(1)}, a_2^{(1)}) = (a_2, a_1, a_0),

   (b) $(a_0^{(2)}, a_1^{(2)}, a_2^{(2)}) = (a_2, a_1, a_0).$

2. If the conditions in (1) hold, the representation $\rho_2$ is unique up to precomposition by an automorphism of $\pi_1(\mathcal{O}_{N_{2,1}})$ preserving $K_{N_{2,1}}$.

3. Moreover, the following hold:

   (a) $\rho_1$ extends to a type-preserving $PSL(2, \mathbb{C})$-representation of $\pi_1(\mathcal{O}_\alpha)$ if and only if $\rho_1$ satisfies the condition (iii)-(a). Moreover, if these conditions are satisfied, the extension is unique.

   (b) $\rho_1$ extends to a type-preserving $PSL(2, \mathbb{C})$-representation of $\pi_1(\mathcal{O}_\beta)$ if and only if $\rho_1$ satisfies the condition (iii)-(b). Moreover, if these conditions are satisfied, the extension is unique.

**FIGURE 6.** $(a_0^{(k_0)}, a_1^{(k_0)}, a_2^{(k_0)}) = (a_2, a_1, a_0)$
Proof. We only show the implication $(1)-(i) \Rightarrow (1)-(ii)$ and the assertion (2) because the other assertions can be proved by an argument similar to [4, Theorem 5.1].

We first prove the implication $(1)-(i) \Rightarrow (1)-(ii)$. Suppose that there exists a faithful representation $\rho_2 \in \Omega(N_{2,1})$ which is commensurable with $\rho_1$. Then, by [4, Theorem 5.1(1)-(ii)], there exist a sequence of elliptic generators $\{Q_j\}$ of $\pi_1(\mathcal{O}_{N_{2,1}})$ and an integer $k_0$ such that the complex probability $(b_0, b_1, b_2)$ associated with $\{\rho_2(Q_j)\}$ satisfies the following identity under Notation 3.10(2):

$$\left(b_0^{(k_0)}, b_1^{(k_0)}, b_2^{(k_0)}\right) = (b_2, b_1, b_0).$$

By Proposition 3.9(1), there is an element $\rho'_1 \in \Omega(\Sigma_{1,1})$ such that the complex probability $(a'_0, a'_1, a'_2)$ associated with $\{\rho'_1(P'_j)\}$ is equal to $(b_0, b_1, b_2)$ for some sequence of elliptic generators $\{P'_j\}$ of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$. Moreover, we can prove that $\rho'_1$ and $\rho_2$ are commensurable (see proof of the implication $(1)-(ii) \Rightarrow (1)-(i)$ in [4, Theorem 5.1] for details). Hence, by [4, Theorem 5.1(2)], there is an automorphism $f$ of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ preserving $K$ such that $\rho_1 \circ f = \rho'_1$. Set $\{P_j\} = \{f(P'_j)\}$. Then $\{P_j\}$ is also a sequence of elliptic generators of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ and $\rho_1(P_j) = \rho'_1(P'_j)$. Hence the complex probability $(a_0, a_1, a_2)$ associated with $\{\rho_1(P_j)\}$ is equal to $(a'_0, a'_1, a'_2) = (b_0, b_1, b_2)$. By Lemma 3.11, the complex probability $(a_0^{(k_0)}, a_1^{(k_0)}, a_2^{(k_0)})$ associated with $\{\rho_1(\xi^{k_0}(P_j))\}$ is equal to the complex probability $(b_0^{(k_0)}, b_1^{(k_0)}, b_2^{(k_0)})$ associated with $\{\rho_2(\sigma^k(Q_j))\}$. Hence we have

$$(a_0^{(k_0)}, a_1^{(k_0)}, a_2^{(k_0)}) = (b_0^{(k_0)}, b_1^{(k_0)}, b_2^{(k_0)}) = (b_2, b_1, b_0) = (a_2, a_1, a_0).$$

Next we prove the assertion (2). Let $\rho_2$ and $\rho_2'$ be elements of $\Omega(N_{2,1})$ such that they are commensurable with $\rho_1$. Then there exist double coverings $p_1 : \mathcal{O}_{\Sigma_{1,2}} \to \mathcal{O}_{\Sigma_{1,1}}$ and $p_2, p_2' : \mathcal{O}_{\Sigma_{1,2}} \to \mathcal{O}_{N_{2,1}}$ such that $\rho_1 \circ (p_1)_* = \rho_2 \circ (p_2)_*$ and $\rho_1 \circ (p_1)_* = \rho_2' \circ (p_2')_*$. Pick an elliptic generator triple $(Q_0, Q_1, Q_2)$ of $\pi_1(\mathcal{O}_{N_{2,1}})$. Note that there is a unique covering from $\mathcal{O}_{\Sigma_{1,2}}$ to $\mathcal{O}_{N_{2,1}}$ up to equivalence which corresponds to the epimorphism $\phi_2 : \pi_1(\mathcal{O}_{N_{2,1}}) \to \mathbb{Z}/2\mathbb{Z}$ defined by the following formula (see [4, Section 2] for details):

$$\phi_2(Q_j) = \begin{cases} 0 & \text{if } j = 0 \text{ or } 2, \\ 1 & \text{if } j = 1. \end{cases}$$

Hence there is a self-homeomorphism $g$ of $\mathcal{O}_{\Sigma_{1,2}}$ such that $p_2' = g \circ p_2$, and $Q_0, Q_2 \in (p_2)_*(\pi_1(\mathcal{O}_{\Sigma_{1,2}}))$. Set $Q'_0 = (p'_2)_* \circ (p_2)^{-1}_*(Q_0)$ and $Q'_2 = (p'_2)_* \circ (p_2)^{-1}_*(Q_2)$.

Claim 3.14. $(Q'_0, Q_1, Q'_2)$ is also an elliptic generator triple of $\pi_1(\mathcal{O}_{N_{2,1}})$.

Proof. Note that $Q'_0$ and $Q'_2$ have order 2, because

$(1) \ (p'_2)_* \circ (p_2)^{-1}_* : (\pi_1(\mathcal{O}_{\Sigma_{1,2}})) \to (p'_2)_*(\pi_1(\mathcal{O}_{\Sigma_{1,2}}))$ is an isomorphism and

$(2) \ Q_0$ and $Q_2$ have order 2.

Since $\rho_1 \circ (p_1)_* = \rho_2 \circ (p_2)_*$ and $\rho_1 \circ (p_1)_* = \rho_2' \circ (p_2')_*$, we have $\rho_2 \circ (p_2)_* = \rho_2' \circ (p_2')_*$.

Hence we have

$\rho_2(Q_0) = \rho_2 \circ (p_2)_* ((p_2)^{-1}_*(Q_0))$

$= \rho_2' \circ (p_2')_* ((p_2)^{-1}_*(Q_0))$

by $\rho_2 \circ (p_2)_* = \rho_2' \circ (p_2')_*$.

$= \rho_2'(Q'_0)$

by $Q'_0 = (p'_2)_* \circ (p_2)^{-1}_*(Q_0)$. 

$Q'_2$ is also of order 2, as noted above, and

$$\phi_2(Q'_2) = 0$$

by Claim 3.11. Hence, the claim is proved.
Similarly, we have $\rho_2(Q_2) = \rho_2'(Q_2')$. Hence we have
\[
\rho_2'(Q_1^2Q_1^0) = \rho_2(Q_1^2Q_1^0) \quad \text{by } \rho_2(Q_j) = \rho_2'(Q_j') \quad \text{for } j = 0, 2
\]
\[
= \rho_2(K_{N_{2,1}}) \quad \text{by } Q_1^2Q_1^0 = K_{N_{2,1}}
\]
\[
= \rho_2'(K_{N_{2,1}}) \quad \text{by Convention 3.12}
\]
Since $\rho_2'$ is faithful, we have $Q_1^2Q_1^0 = K_{N_{2,1}}$. Thus, by Remark 2.6, the triple $(Q_0', Q_1, Q_2')$ is an elliptic generator triple of $\pi_1(\mathcal{O}_{N_{2,1}})$.

By this claim, there are elliptic generator triples $(Q_0, Q_1, Q_2)$ and $(Q_0', Q_1, Q_2')$ of $\pi_1(\mathcal{O}_{N_{2,1}})$ satisfying the following identity:
\[
(\rho_2(Q_0), \rho_2(Q_1), \rho_2(Q_2)) = (\rho_2'(Q_0'), \rho_2'(Q_1), \rho_2'(Q_2')).
\]
By Proposition 2.2(2), there is an automorphism $f$ of $\pi_1(\mathcal{O}_{N_{2,1}})$ preserving $K_{N_{2,1}}$ such that $f$ maps $(Q_0, Q_1, Q_2)$ to $(Q_0', Q_1, Q_2')$. Hence we have $\rho_2 = \rho_2' \circ f$.

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