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Positive Singular Solutions to a Nonlinear Elliptic Equation on the Unit Sphere

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1 Introduction

In this note, we consider the nonlinear elliptic equation of the scalar-field type on the whole sphere

$$\Lambda u - \lambda u + |u|^{p-1}u = 0 \quad \text{in } \mathbb{S}^n, \tag{1.1}$$

where Λ is the Laplace-Beltrami operator on the usual unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ $(n \geq 2)$, $p > 1$, $\lambda > 0$ and seek a positive solution which is singular at both the North pole $N = (0, 0, \dots, 1)$ and the South Pole $S = (0, 0, \dots, -1)$. Such a solution is called a positive singular-singular solution.

We introduce the polar coordinates to study (1.1). A point $(x_1, x_2, \dots, x_{n+1}) \in \mathbb{S}^n$ in the polar coordinates is expressed as

$$\begin{cases} x_1 = \sin\theta_1 \cos\theta_2, \\ x_k = \left(\prod_{j=1}^{k} \sin\theta_j\right) \cos\theta_{k+1}, \quad k = 2, .., n-2, \\ x_{n-1} = \left(\prod_{j=1}^{n-1} \sin\theta_j\right) \cos\phi, \\ x_n = \left(\prod_{j=1}^{n-1} \sin\theta_j\right) \sin\phi, \\ x_{n+1} = \cos\theta_1, \end{cases}$$

where $\theta_i \in [0, \pi]$ $(\imath = 1, 2, \dots, n)$ and $\phi \in [0, 2\pi)$. As the first step, we study (1.1) in the class of azimuthal solutions (depending only on the geodesic distance

$\theta := \theta_1$ from the North pole) which are *symmetric with respect to the equator*. Thus, we consider the following ordinary differential equation

$$(1.2)\qquad \begin{cases} \dfrac{1}{\sin^{n-1}\theta}\{(\sin^{n-1}\theta)u_\theta\}_\theta - \lambda u + |u|^{p-1}u = 0, & 0 < \theta < \pi/2, \\ u_\theta(\pi/2) = 0. \end{cases}$$

We prolong the solution u of (1.2) to the interval $\theta \in (\pi/2, \pi]$ by defining

$$\tilde{u}(\theta) = \begin{cases} u(\theta), & 0 \le \theta \le \pi/2, \\ u(\pi - \theta), & \pi/2 < \theta \le \pi \end{cases}$$

so that $\tilde{u}$ is a solution on $\mathbb{S}^n$.

Concerning nonlinear elliptic problems on the sphere, the existence or the nonexistence of positive regular solutions are discussed by Bandle and Benguria [3], Bandle and Peletier [4], Brezis and Peletier [5] and Kosaka [11]. In contrast with regular solutions, singular solutions seem not to be investigated intensively. The purpose of this note is to present sufficient conditions for the existence of a positive singular solution.

Here we note that there are several results on the existence of positive singular solutions to semilinear elliptic equations on the Euclidean space by Bae [1], Bae and Chang [2], Chern, Chen, Chen and Tang [6] and references therein.

Although the main topic in [8] by Dancer, Guo and Wei was on non-radial singular solutions to $\Delta u + u^p = 0$ in $\mathbb{R}^n$, they used a symmetric singular solution to (1.2) with a specific value of λ under suitable assumption on p. We discuss (1.2) for wider range of λ rather than [8].

We remark here that from (1.2), we see that

$$u_\theta = \int_\theta^{\pi/2} (-\lambda u + |u|^{p-1}u)\frac{\sin^{n-1} s}{\sin^{n-1}\theta}\, ds$$

for $\theta < \pi/2$. Thus, we cannot expect a positive singular solution if $\lambda \le 0$. Hence our restriction on λ is quite natural.

To find a singular solution to (1.2), we introduce two types of the Pohozaev (type) identities. One is directly constructed from (1.2) in Section 2, the other is constructed after transforming (1.2) to an ODE on the exterior of a ball in the Euclidean space in Section 4. This technique is suitable for analyzing the problem especially in the three dimensional case.

Before stating our main results, we enumerate symbols frequently used in this note. Let

$$(1.3)\qquad A_{\lambda,p} = [\frac{p+1}{2}(\lambda - \frac{n-2}{2})]^{1/(p-1)} \quad \text{if } \lambda > \frac{n-2}{2},$$

$$(1.4)\qquad B_{\lambda,p} = \left[\frac{(p+1)\{4\lambda - n(n-2)\}}{2\{2n-(p+1)(n-2)\}}\right]^{1/(p-1)} \quad \text{if } \lambda > \frac{n(n-2)}{4},$$

$$p_S := \frac{n+2}{n-2}, \quad m := \frac{2}{p-1}, \quad \bar{\lambda} = m(n-1-m).$$

To find a singular solution, we consider the initial value problem

$$\tag{1.5} \begin{cases} (\sin^{n-1}\theta \cdot u_\theta)_\theta + \sin^{n-1}\theta \cdot (-\lambda u + u^p) = 0, & \theta \in (0, \pi/2), \\ u(\pi/2) = \alpha > 0, \ u_\theta(\pi/2) = 0. \end{cases}$$

For any $\alpha > 0$, (1.5) has a unique solution, which is denoted by $u(\theta;\alpha)$. Analyzing (1.5), we have the following theorems concerning (1.2).

Theorem 1.1 *Let $n \geq 3$. Then equation* (1.2) *possesses a continuum of positive singular-singular solutions symmetric with respect to $\theta = \pi/2$ if one of the following conditions holds:*

(i) $1 < p < p_S$, $(n-2)/2 < \lambda < (n-2)(p+1)/2(p-1)$;

(ii) $p = p_S$, $(n-2)/2 < \lambda < n(n-2)/4$.

Remark 1.1 *The unique solution $u(\theta;\alpha)$ to* (1.5) *becomes positive and singular in the following range of the initial value:*

(i) $0 < \alpha \leq A_{\lambda,p}$ *if* $(n-2)/2 < \lambda \leq n(n-2)/4$ *and* $1 < p < p_S$;

(ii) $B_{\lambda,p} \leq \alpha \leq A_{\lambda,p}$ *if* $n(n-2)/4 < \lambda < (n-2)(p+1)/(p-1)$ *and* $1 < p < p_S$;

(iii) $0 < \alpha < A_{\lambda,p}$ *if* $p = p_S$.

If $p \leq p_S$, there exists a continuum of singular solutions. On the other hand, we prove the existence of a unique positive singular solution for the supercritical exponent case ($p > p_S$). This is one of the main differences in the structure of solutions.

Theorem 1.2 *Let $n \geq 3$, $p > p_S$ and $\lambda > -m$. Then a positive singular solution at $\theta = 0$ of* (1.2) *exists and it is unique.*

When $n = 3$, we have more accurate results than Theorem 1.1.

Theorem 1.3 *Let $n = 3$.*

(i) *If $\lambda \in (0, 1]$, then for each $p \in [2, 5]$,* (1.2) *has a continuum of positive singular-singular solutions.*

(ii) *If $\lambda > 1$, then for each $p \in [2, 5)$,* (1.2) *has a continuum of positive singular-singular solutions.*

Theorem 1.4 *Let $n = 3$.*

(i) *If $\lambda \in (0, 1]$, then for each $p > 5$,* (1.2) *has a continuum of crossing solutions.*

(ii) *If $\lambda > 1$, then for each $p \geq 5$, (1.2) has a continuum of crossing solutions.*

Note that $p = 5$ is excluded for $\lambda \in (0, 1]$ in (i) of Theorem 1.4.

This paper is organized as follows: In Section 2, symmetric solutions are treated and Theorem 1.1 is proved. Theorem 1.2 is proved in Section 3. Another approach on the three dimensional case is employed in Section 4. Concluding Remarks concerning asymmetric solutions are discussed in Section 5.

2 Positive Symmetric Singular Solutions for Subcritical and Critical Cases

In this section, we establish the existence of a continuum of positive singular-singular solutions of (1.2). We first observe the following Pohozaev identity for (1.2), which is verified by the direct calculations.

Lemma 2.1 [Pohozaev Identity] *Let*

$$P(\theta, u) := \sin^{n-1}\theta \cdot u_\theta \cdot (\sin\theta \cdot u_\theta + (n-2)\cos\theta \cdot u) + \sin^n\theta \cdot [(\frac{n-2}{2} - \lambda)u^2 + \frac{2}{p+1}u^{p+1}].$$

If $u(\theta)$ is a positive solution of (1.2), then there holds

$$\frac{d}{d\theta}P(\theta, u) = \sin^{n-1}\theta\cos\theta \cdot [\frac{n(n-2) - 4\lambda}{2}u^2 + (\frac{2n}{p+1} - (n-2))u^{p+1}]. \tag{2.1}$$

To prove Theorem 1.1, we need one lemma. Let $u(\theta; \alpha)$ be the solution of (1.5).

Lemma 2.2 *If one of the conditions (i) or (iii) in Theorem 1.1 holds, then, for any $0 < \alpha < A_{\lambda,p}$, there exists a singular solution to (1.5).*

Proof. Let $u = u(\theta; \alpha)$ be the solution of (1.5). Then u satisfies one of the following three cases:

(a) $u(\theta; \alpha)$ is a positive regular solution. Namely, $u(\theta; \alpha) > 0$ for all $\theta \in [0, \pi/2]$ and $u_\theta(0; \alpha) = 0$;

(b) $u(\theta; \alpha)$ is a sign-changing solution. There exists $\theta_1 \in (0, \pi/2)$ such that $u(\theta; \alpha) > 0$ for all $\theta \in (\theta_1, \pi/2]$ and $u(\theta_1; \alpha) = 0$;

(c) $u(\theta; \alpha)$ is a positive singular solution. That is, $u(\theta; \alpha) > 0$ for all $\theta \in (0, \frac{\pi}{2}]$ and $u(\theta; \alpha) \to \infty$ as $\theta \to 0$.

Let $0 < \alpha < A_{\lambda,p}$, where $A_{\lambda,p}$ is defined by (1.3). If one of the conditions (i) and (iii) of Theorem 1.1 holds, then, by Lemma 2.1, we obtain

$$\begin{cases} P(\frac{\pi}{2}, u) = \dfrac{2}{p+1}(\alpha^{p-1} - A_{\lambda,p}^{p-1}) \cdot \alpha^2 < 0 \quad \text{for all} \quad 0 < \alpha < A_{\lambda,p}, \\ \dfrac{dP}{d\theta}(\theta, u) \geq 0 \quad \text{whenever} \quad u(\theta; \alpha) > 0. \end{cases} \tag{2.2}$$

If case (a) happens for some $\alpha \in (0, A_{\lambda,p})$, then by Lemma 2.1 and (2.2) we obtain

$$0 > P(\frac{\pi}{2}, u) \geq P(0, u) = 0,$$

a contradiction. Hence case (a) cannot hold.

If case (b) happens for some $\alpha \in (0, A_{\lambda,p})$, then similarly by Lemma 2.1 and (2.2) we obtain

$$0 > P(\frac{\pi}{2}, u) \geq P(\theta_1, u) = \sin^n \theta_1 \cdot u_\theta^2(\theta_1; \alpha) \geq 0.$$

We also get a contradiction, and hence case (b) cannot be true.

Therefore, for all $0 < \alpha < A_{\lambda,p}$, $u(\theta; \alpha)$ have to satisfy case (c). □

Proof of Theorem 1.1. By the reflection with respect to $\theta = \pi/2$, the existence of a continuum of positive singular-singular solutions immediately follows. By Lemma 2.2, (i) and (iii) of Theorem 1.1 are proved.

On the other hand, it is easy to see that

$$\frac{(n-2)(p+1)}{2(p-1)} \geq \frac{n(n-2)}{4} \quad \text{if } p \leq p_S.$$

Moreover, there holds

$$\frac{p+1}{2}(\lambda - \frac{n-2}{2}) > u^{p-1}(\frac{\pi}{2}) \geq \frac{(p+1)(4\lambda - n(n-2))}{2(2n-(p+1)(n-2))} = B_{\lambda,p}^{p-1}$$

equivalently, $\lambda < (n-2)(p+1)/2(p-1)$. Thus (ii) of Theorem 1.1 is proved. This completes the proof of Theorem 1.1. □

Remark 2.1 *The arguments in this section may be refined by using the method developed by Yanagida [14] and/or Shioji and Watanabe [13].*

3 Proof of Theorem 1.2

In this section, we prove Theorem 1.2 and show that the existence and uniqueness of positive singular solutions at $\theta = 0$ for the supercritical case. To this end, we need the following two lemmas.

Lemma 3.1 *Let $\lambda > -m$ and $u(\theta)$ be a positive singular solution at $\theta = 0$ of (1.2). Then there exists $\theta_0 > 0$ such that*

$$0 \leq \sin^m \theta \cdot u(\theta) < \left(\frac{m(p-1)}{2n(m+\lambda)}\right)^{-1/(p-1)} \quad \textit{for } 0 < \theta \leq \theta_0.$$

Proof. Since $\lim_{\theta\to 0} u(\theta) = \infty$, there exists $\theta_0 > 0$ such that $u_\theta(\theta) < 0$ and $u^{p-1}(\theta) > m + \lambda$ for $0 < \theta \le \theta_0$. From $(\sin^{n-1}\theta \cdot u_\theta)_\theta = -\sin^{n-1}\theta \cdot (-\lambda u + u^p)$ for $\theta > 0$, we have

$$\begin{aligned}
\sin^{n-1}\theta \cdot u_\theta(\theta) &= \sin^{n-1}\theta_1 \cdot u_\theta(\theta_1) - \int_{\theta_1}^{\theta} \sin^{n-1}\xi \cdot (-\lambda u + u^p)\, d\xi \\
&\le -u^p(\theta) \int_{\theta_1}^{\theta} \sin^{n-1}\xi \cdot (-\lambda u^{1-p} + 1)\, d\xi \\
&< -\frac{m}{m+\lambda} u^p(\theta) \int_{\theta_1}^{\theta} \sin^{n-1}\xi\, d\xi,
\end{aligned}$$

where $0 < \theta_1 < \theta < \theta_0$. For $\theta \in (0, \theta_0]$ and letting $\theta_1 \to 0$, we get

$$\frac{u'(\theta)}{u^p(\theta)} < -\frac{m}{m+\lambda} \frac{\int_0^\theta \sin^{n-1}\xi\, d\xi}{\sin^{n-1}\theta}.$$

For $\theta \in (0, \theta_0]$, we also have

$$\frac{\int_0^\theta \sin^{n-1}\xi\, d\xi}{\sin^{n-1}\theta} \ge \frac{\int_0^\theta \sin^{n-1}\xi \cos\xi\, d\xi}{\sin^{n-1}\theta} = \frac{\sin\theta}{n},$$

which implies

$$\frac{u_\theta(\theta)}{u^p(\theta)} < -\frac{m\sin\theta}{n(m+\lambda)}.$$

Hence, for $0 < \theta_2 \le \theta \le \theta_0$,

$$\frac{1}{1-p}\left(u^{1-p}(\theta) - u^{1-p}(\theta_2)\right) < -\frac{m}{n(m+\lambda)}(-\cos\theta + \cos\theta_2),$$

which implies, letting $\theta_2 \to 0$,

$$\frac{1}{1-p} u^{1-p}(\theta) < -\frac{m}{n(m+\lambda)}(1 - \cos\theta) \le -\frac{m\sin^2\theta}{2n(m+\lambda)} \quad \text{for } 0 < \theta \le \theta_0.$$

Therefore, we obtain

$$\sin^m\theta \cdot u(\theta) < \left(\frac{m(p-1)}{2n(m+\lambda)}\right)^{-1/(p-1)} \qquad \text{for } 0 < \theta \le \theta_0.$$

□

We also see that the limit $(\sin^m\theta) u(\theta)$ as $\theta \to 0$ exists although we do not give a proof here.

Lemma 3.2 *Let $p > p_S$. If $u(\theta)$ is a positive singular solution at $\theta = 0$ of* (1.2), *then there holds*

$$\lim_{\theta\to 0} \sin^m\theta \cdot u(\theta) = \big(m(n-2-m)\big)^{1/(p-1)} \equiv L.$$

Proof of Theorem 1.2. Suppose (1.2) has two distinct positive singular solutions, $\xi(\theta)$ and $\zeta(\theta)$. Then, by Lemma 3.2, we have

$$\lim_{\theta\to 0} \sin^m\theta \cdot \xi(\theta) = \lim_{\theta\to 0} \sin^m\theta \cdot \zeta(\theta) = \big(m(n-2-m)\big)^{1/(p-1)} \equiv L.$$

Let $v(\theta) = \zeta(\theta)/\xi(\theta)$. Then $v(\theta)$ satisfies

$$v_{\theta\theta} + \Big((n-1)\cot\theta + \frac{2\xi_\theta}{\xi}\Big)v_\theta + \xi^{p-1}(v^p - v) = 0.$$

Set $s = -\log(\sin\theta)$ and $w(s) = v(\theta) - 1$. Then $w(s)$ satisfies

$$\frac{d^2w}{ds^2} + f(\theta)\frac{dw}{ds} + g(\theta)w = 0, \tag{3.1}$$

where

$$f(\theta) = -(n-1) + \frac{1}{\cos^2\theta} - \frac{2\sin\theta\cdot\xi_\theta(\theta)}{\cos\theta\cdot\theta(\theta)}$$

and

$$g(\theta) = \begin{cases} \tan^2\theta\cdot\xi^{p-1}(\theta)\dfrac{v^p - v}{v-1} & \text{if } v(\theta) \neq 1, \\ (p-1)\tan^2\theta\cdot\xi^{p-1}(\theta) & \text{if } v(\theta) = 1. \end{cases}$$

By $\lim_{\theta\to 0}\big(\sin^{m+1}\theta\cdot\xi_\theta(\theta)\big) = -mL$, we obtain

$$\lim_{\theta\to 0}\frac{\sin\theta\cdot\xi_\theta(\theta)}{\cos\theta\cdot\xi(\theta)} = -m.$$

It follows that

$$\lim_{\theta\to 0} f(\theta) = -(n-2-2m) < 0 \quad\text{and}\quad \lim_{\theta\to 0} g(\theta) = (p-1)L^{p-1} > 0.$$

Thus, $f(\theta)$ and $g(\theta)$ are Lipschitz continuous in $[0,\infty)$. Hence, (3.1) can be solved uniquely for arbitrary given w and w' at $\theta = 0$.

Note that

$$\lim_{s\to\infty} w(s) = \lim_{\theta\to 0} v(\theta) - 1 = 0$$

and

$$\lim_{s\to\infty} w'(s) = \lim_{\theta\to 0}\Big(-\tan\theta\frac{\zeta_\theta}{\xi} + \tan\theta\frac{\zeta\xi_\theta}{\xi^2}\Big) = 0.$$

This reduces $w \equiv 0$ and we get a contradiction. The proof is complete. $\square$

4 Symmetric solutions for the case of $n = 3$

As we mentioned in Introduction, we give another method to show the existence of positive singular-singular solutions. First, we use a regular solution to the linear problem to erase the linear term. This process is called the Doob h-transform.

Let $\Psi(\theta)$ be a regular solution to

$$\{(\sin^{n-1}\theta)\Psi_\theta\}_\theta - \lambda(\sin^{n-1}\theta)\Psi = 0. \tag{4.1}$$

In case of $n = 3$, let $\lambda = (1-\mu^2)$ with $\mu \in (0,1]$. Then $\Psi(\theta)$ is explicitly expressed as

$$\Psi(\theta) = \frac{\sin\mu\theta}{\sin\theta}$$

and

$$\Psi(\theta) = \frac{\theta}{\sin\theta}$$

for $\lambda = 1$ $(\mu = 0)$. Similarly, if $\lambda = (\nu^2 + 1)$ with $\nu > 0$, then

$$\Psi(\theta) = \frac{\sinh\nu\theta}{\sin\theta}.$$

Now, we transform (1.2) into the form which has no zeroth order term. Letting

$$g(\theta) = (\sin^{n-1}\theta)\Psi(\theta)^2,$$

$$\varrho = \frac{1}{\Psi(\pi/2)\Psi_\theta(\pi/2)}, \quad w(\tau) = \frac{u(\theta)}{\Psi(\theta)\tau}, \quad \tau = \int_\theta^{\pi/2}\frac{ds}{g(s)} + \varrho,$$

and

$$h(\theta) = g(\theta)\left(\int_\theta^{\pi/2}\frac{ds}{g(s)} + \varrho\right) = \tau g(\theta),$$

we see that (1.2) is transformed to

$$\begin{cases} \dfrac{1}{\tau^2}(\tau^2 w_\tau)_\tau + \hat{Q}(\tau)w_+^p = 0, & \tau \in (\varrho,\infty), \\ w > 0, \quad \tau \in (\varrho,\infty), \\ w_\tau(\varrho) = 0, \end{cases} \tag{4.2}$$

with

$$\hat{Q}(\tau) := Q(\theta) := \left(\frac{h(\theta)}{\sin^{(n-1)/2}\theta}\right)^{p-1}(g(\theta))^{-(p-5)/2}.$$

We remark that $\hat{Q}$ is a function of τ while Q is a function of θ.

For (4.2), we can apply the results in [9] to obtain the corresponding results. We introduce the functions $P_*(\tau;w)$, $G(\tau)$ and $H(\tau)$ as below:

$$P_*(\tau;w) := \frac{1}{2}\tau^2 w_\tau(\tau w_\tau + w) + \frac{1}{p+1}\tau^3 Q(\tau)w_+^{p+1}, \tag{4.3}$$

$$(4.4) \qquad G(\tau) := \frac{1}{p+1}\left\{\tau^3 - \frac{p+1}{2}\int_\varrho^\tau s^2 Q(s)\,ds\right\},$$

$$(4.5) \qquad H(\tau) := \frac{1}{p+1}\left\{\tau^{2-p} - \frac{p+1}{2}\int_\tau^\infty s^{1-p} Q(s)\,ds\right\}.$$

Then there holds

$$(4.6) \qquad \frac{d}{d\tau}P_*(\tau;w) = G_\tau(\tau) w_+^{p+1}$$

and

$$G_\tau(\tau) = \tau^{p+1}H(\tau) = \frac{1}{p+1}\tau^{(p+1)/2}(\tau^{(5-p)/2}Q(\tau))_\tau.$$

We note that $P_*(\varrho, w) = \varrho^3\hat{Q}(\varrho)w(\varrho)^{p+1}/(p+1) > 0$. Thus, if G_τ is monotone increasing, then $P_*(\tau;w) > 0$ for any $\tau \geq \varrho$. The locations of the smallest zero of $G(\tau)$ and the largest zero of $H(\tau)$ are important. A kind of general results to (4.2) is the following assertion based on Theorem 3.3 of Kabeya, Yanagida and Yotsutani [9]. We define

$$\hat{R}(\tau) := R(\theta) := \tau^{-(p-5)/2}\hat{Q}(\tau) = \frac{h^{(p+3)/2}}{(\sin\theta)^{p-1}}.$$

Theorem A *Suppose that $\hat{Q} \in C^1((\varrho,\infty))$, $\hat{Q} > 0$, $\tau\hat{Q} \in L^1([\varrho,\varrho+1])$ and $\tau^{1-p}\hat{Q}(\tau) \in L^1([\varrho+1,\infty))$.*

(i) *If $\hat{R}(\tau)_\tau \geq, \not\equiv 0$ in (ϱ,∞), then (1.1) has only a sign-changing solution.*

(ii) *If $\hat{R}(\tau)_\tau \leq, \not\equiv 0$ in (ϱ,∞) with $\varrho > 0$ and if $|\tau^{(p+1)/2}\hat{R}_\tau(\tau)| \notin L^1([\varrho+1,\infty))$, then there exists a continuum of positive slowly decaying solutions.*

(iii) *If there exists a number $\tau_* > \varrho$ such that $\hat{R}(\tau)_\tau \geq, \not\equiv 0$ in (ϱ,τ_*), that $\hat{R}_\tau(\tau) \leq, \not\equiv 0$ in (τ_*,∞) and that $|\tau^{(p+1)/2}\hat{R}_\tau(\tau)| \notin L^1([\varrho+1,\infty))$, then there exists a unique positive solution to (1.2) which decays at the rate τ^{-1} at $\tau = \infty$ and a continuum of positive solutions to (1.2) which decay more slowly than τ^{-1} at $\tau = \infty$.*

It is not easy to check the conditions in Theorem A. There hold weak versions of (i) and (ii) of Theorem A, which are essentially due to Theorems 2 and 3 in [17].

Theorem B *Suppose that $\hat{Q} \in C^1((\varrho,\infty))$, $\hat{Q} > 0$, $\tau\hat{Q} \in L^1([\varrho,\varrho+1])$ and $\tau^{1-p}\hat{Q}(\tau) \in L^1([\varrho+1,\infty))$.*

(i) *If $\hat{R}_\tau(\tau) \geq, \not\equiv 0$ near $\tau = \infty$ and if $|\tau^{(p+1)/2}\hat{R}_\tau(\tau)| \notin L^1([\varrho+1,\infty))$, then there exists a continuum of sign-changing solutions to (1.1).*

(ii) *If $\hat{R}_\tau(\tau) \leq, \not\equiv 0$ near $\tau = \infty$ and if $|\tau^{(p+1)/2}\hat{R}_\tau(\tau)| \notin L^1([\varrho+1,\infty))$, then there exists a continuum of positive slowly decaying solutions.*

In each case of Theorem B, we see that $u(\pi/2) > 0$ is always small due to Theorems 2 or 3 in [17].

(i) If (4.2) has only a sign-changing solution for any $w(\varrho) > 0$, we say that the structure of positive solutions to (4.2) is of type C .

(ii) If (iii) of Theorem A holds, then we say that the structure of positive solutions is of type M.

Theorem C (i) *If* (i) *of Theorem A or Theorem B holds, then* (1.1) *has a continuum of sign-changing solutions.*

(ii) *If* (ii) *of Theorem A or Theorem B holds, then* (1.1) *has a continuum of positive singular solutions.*

(iii) *If* (iii) *holds,* (1.1) *has a unique positive regular solution, which is indeed a constant, and has a continuum of positive singular solutions.*

In the following, we investigate the behavior of $h(\theta)/\sin\theta$ and check whether one of the statements in Theorems A and B is applicable or not. The important point is the sign of

(4.7)
$$\frac{d}{d\tau}\left\{\tau^{-(p-5)/2}\hat{Q}(\tau)\right\} = \frac{d}{d\theta}\left\{(\tau g)^{-(p-5)/2}\left(\frac{h(\theta)}{\sin\theta}\right)^{p-1}\right\}\frac{d\theta}{d\tau} = \frac{d}{d\theta}\left\{\frac{h^{(p+3)/2}}{(\sin\theta)^{p-1}}\right\}\frac{d\theta}{d\tau}.$$

Hence, we need to check the sign of

$$\frac{d}{d\theta}R(\theta) = \frac{h^{(p+1)/2}}{\sin^p\theta}\left\{\frac{p+3}{2}h_\theta\sin\theta - (p-1)h\cos\theta\right\}. \tag{4.8}$$

Proof of Theorem 1.3 for $\lambda \in (0,1)$. We need to check whether the statements of Theorem A hold or not for the case of $n = 3$ and $p > 1$. First, we consider the case $\lambda = 1 - \mu^2$ with $\mu \in (0, 1]$.

In this case, since $\Psi(\theta) = \sin\mu\theta/\sin\theta$, we have

$$\rho = \frac{1}{\mu\sin\frac{\mu\pi}{2}\cos\frac{\mu\pi}{2}} = \frac{2}{\mu\sin\mu\pi}.$$

Similarly, we have $g(\theta) = \sin^2\mu\theta$,

$$\begin{aligned}\tau &= \int_\theta^{\pi/2}\frac{ds}{g(s)} + \rho = \int_\theta^{\pi/2}\frac{ds}{\sin^2\mu s} + \rho \\ &= \left[-\frac{1}{\mu}\cot\mu s\right]_\theta^{\pi/2} + \rho = -\frac{1}{\mu}\cot\frac{\pi\mu}{2} + \frac{1}{\mu}\cot\mu\theta + \rho\end{aligned}$$

and

$$h(\theta) = (\rho - \frac{\cot(\frac{\mu\pi}{2})}{\mu})\sin^2\mu\theta + \frac{\sin 2\mu\theta}{2\mu}. \tag{4.9}$$

Concerning $\rho - \mu^{-1}\cot \mu\pi/2$, we get

$$\rho - \frac{1}{\mu}\cot\frac{\mu\pi}{2} = \frac{2}{\mu\sin\mu\pi} - \frac{1}{\mu}\cot\frac{\mu\pi}{2} = \frac{1}{\mu\sin\frac{\mu\pi}{2}}(\frac{1}{\cos\frac{\mu\pi}{2}} - \cos\frac{\mu\pi}{2}) = \frac{1}{\mu}\tan\frac{\mu\pi}{2}.$$

Thus, we see that

$$h(\theta) = \frac{1}{\mu}\tan\frac{\mu\pi}{2}\sin^2\mu\theta + \frac{\sin 2\mu\theta}{2\mu}.$$

Therefore, we have

$$h(0) = 0, \quad h_\theta(\theta) = \frac{1}{\mu}(\tan\frac{\mu\pi}{2})\sin 2\mu\theta + \cos 2\mu\theta, \quad h_\theta(0) = 1. \tag{4.10}$$

Since

$$\frac{d\tau}{d\theta} = -\frac{1}{\sin^2\mu\theta} < 0$$

on $(0,\pi/2)$, we have to check the sign of (4.8) by noting that $\tau^{(p+1)/2}$ is of order $\theta^{-(p+1)/2}$ as $\theta \to 0$. To check the condition of (i) or (ii) of Theorem B, we also note that

$$\int_\rho^\infty \tau^{(p+1)/2}|\frac{d}{d\tau}\hat{R}(\tau)|\,d\tau = \int_0^{\pi/2}\tau^{(p+1)/2}|\frac{d}{d\theta}R(\theta)|\frac{d\theta}{d\tau}\cdot\frac{d\tau}{d\theta}\,d\theta$$

$$= \int_0^{\pi/2}\tau^{(p+1)/2}|R_\theta(\theta)|\,d\theta.$$

If $|R_\theta(\theta)|$ is at most of order $\theta^{(p-1)/2}$, then we confirm the applicability of Theorem B. Then we see that

$$\frac{p+3}{2}h_\theta\sin\theta - (p-1)h\cos\theta = \frac{5-p}{2}\theta + \frac{4}{\mu}(\tan\frac{\mu\pi}{2})\theta^2 > 0 \tag{4.11}$$

if $1 < p \le 5$ and $\theta > 0$ is sufficiently small. If $1 < p < 5$, $\tau^{(p+1)/2}R(\theta)$ near $\theta = 0$ is of order $\theta^{-(p+1)/2}\theta^{-(p-1)/2}\theta = \theta^{1-p}$. If $p = 5$, then $\tau^3 R(\theta)$ is of order θ^{-3}. Thus, if $p \in [2,5]$, then$\tau^{(p+1)/2}R(\theta) \notin L^1([\varrho+1,\infty))$. Hence, (ii) of Theorem B holds and Theorem 1.3 holds for $\lambda \in (0,1)$. □

Proof of Theorem 1.3 for $\lambda = 1$. In this case, $\Psi(\theta)$ is taken as

$$\Psi(\theta) = \frac{\theta}{\sin\theta}.$$

Then we have

$$g(\theta) = \theta^2, \quad \rho = \frac{2}{\pi}$$

and

$$\tau = \int_\theta^{\pi/2}\frac{ds}{g(s)} + \rho = \frac{1}{\theta}, \quad h(\theta) = \tau g(\theta) = \theta.$$

Thus, we have

$$\frac{p+3}{2}h_\theta \sin\theta - (p-1)h\cos\theta = \frac{5-p}{2}\theta + \frac{5p-9}{6}\theta^3 > 0 \tag{4.12}$$

near $\theta = 0$ if $1 < p \leq 5$. We see that $\tau^{(p+1)/2}R(\theta)$ is of order θ^{1-p} if $1 < p < 5$ and is of θ^{-2} if $p = 5$. Thus, Theorem 1.3 holds for $\lambda = 1$. □

Proof of Theorem 1.3 for $\lambda > 1$. In this case, we put $\lambda = 1 + \nu^2$ with $\nu > 0$. In this case, $\Psi(\theta) = \sinh \nu\theta / \sin\theta$ and

$$\varrho = \frac{2}{\nu \sinh \nu\pi}$$

$$\begin{aligned}\tau &= \int_\theta^{\pi/2} \frac{ds}{g(s)} + \varrho = \int_\theta^{\pi/2} \frac{ds}{\sinh^2 \nu s} + \varrho \\ &= \Big[-\frac{1}{\nu}\coth \nu s\Big]_\theta^{\pi/2} + \varrho = -\frac{1}{\nu}\coth\frac{\pi\nu}{2} + \frac{1}{\nu}\coth\nu\theta + \varrho,\end{aligned}$$

$$\varrho - \frac{1}{\nu}\coth\frac{\pi\nu}{2} = -\frac{1}{\nu\tanh\frac{\pi\nu}{2}},$$

and

(4.13)

$$h(\theta) = -\frac{\tanh\frac{\nu\pi}{2}}{\nu}\sinh^2\nu\theta + \frac{\sinh 2\nu\theta}{2\nu} = -\frac{\tanh\frac{\nu\pi}{2}}{2\nu}\frac{\cosh 2\nu\theta - 1}{\sin\theta} + \frac{\sinh 2\nu\theta}{2\nu}.$$

Thus, we have

$$\frac{p+3}{2}h_\theta \sin\theta - (p-1)h\cos\theta = \frac{5-p}{2}\theta - 4(\nu\tanh\frac{\nu\pi}{2})\theta^2 > 0 \tag{4.14}$$

if $p < 5$. We see that $\tau^{(p+1)/2}R(\theta)$ is of order θ^{1-p} if $1 < p < 5$. Thus, (i) of Theorem B is applicable for $p \in [2, 5]$. Thus Theorem 1.3 is proved for $\lambda > 1$. □

Proof of Theorem 1.4. The proof immediately follows from (4.11), (4.12) and (4.14) with the order of θ near $\theta = 0$. □

5 Concluding remarks

In this section, we make concluding remarks in two subsections.

5.1 Asymmetric solutions

In this subsection, we consider the following boundary value problem

$$\begin{cases} (\sin^{n-1}\theta \cdot u_\theta)_\theta + \sin^{n-1}\theta \cdot (-\lambda u + u^p) = 0, \ 0 < \theta < \pi \\ u(\pi/2) = \alpha > 0, \ u_\theta(\pi/2) = \beta \in \mathbb{R} \end{cases} \tag{5.1}$$

and investigate asymmetric singular solutions. Let $u(\theta;\alpha,\beta)$ be the unique solution of (5.1), and let

$$A_{\lambda,p} = [\frac{p+1}{2}(\lambda - \frac{n-2}{2})]^{1/(p-1)} \tag{5.2}$$

and define the set $S_{\lambda,p}$ as

$$S_{\lambda,p} = \left\{(\alpha,\beta)\,\Big|\, \alpha > 0 \text{ and } \beta^2 + \frac{2}{p+1}(\alpha^{p-1} - A_{\lambda,p}^{p-1}) < 0\right\}.$$

By the Pohozaev identity defined in (2.1), we see that if $\lambda > (n-2)/2$, then the set $S_{\lambda,p}(\neq \emptyset)$ satisfies:

(S1) $(\alpha, 0) \in S_{\lambda,p}$ if $0 < \alpha \leq A_{\lambda,p}$;

(S2) The set $S_{\lambda,p}$ is open. That is, if $(\alpha_0,\beta_0) \in S_{\lambda,p}$ and $(\alpha_0,\beta_0) \neq (A_{\lambda,p}, 0)$, then there exists $\delta > 0$ such that $(\alpha,\beta) \in S_{\lambda,p}$ for any (α,β) with $|\alpha-\alpha_0| + |\beta-\beta_0| < \delta$.

This observation leads the following observation. Intuitively, we can say that there exists an asymmetric positive singular-singular solution if the initial value and the initial slope are sufficiently close to zero.

Theorem 5.1 *If one of the conditions* (i)-(ii) *in Theorem 1.1 holds, then* $u(\theta;\alpha,\beta)$ *is a positive singular-singular solution for all* $(\alpha,\beta) \in S_{\lambda,p}$.

Here, we also remark that as in Section 4, the existence of an asymmetric positive singular-singular solution can be derived from the method developed in Yanagida and Yotsutani [18], Yotsutani [19] and Kabeya, Yanagida and Yotsutani [10].

5.2 Applicability of the method of transformation

The transformation used in Section 4 can be applied to the case $n \geq 4$. In general, the regular solution $\Psi(\theta)$ to (4.1) can be expressed by using the associated Legendre functions and the exact form is

$$\Psi(\theta) = \frac{P_\nu^{(n-2)/2}(\cos\theta)}{\sin^{(n-2)/2}\theta}$$

if n is even and

$$\Psi(\theta) = \frac{Q_\nu^{(n-2)/2}(\cos\theta)}{\sin^{(n-2)/2}\theta}$$

if n is odd, where P_ν^μ and Q_ν^μ are the associated Legendre functions, ν is taken as the positive root of

$$\nu(\nu+1) = \lambda + \frac{n(n-2)}{4}.$$

Note that $P_\nu^{(n-2)/2}(1) = 0$ when n is even and $Q_\nu^{(n-2)/2}(1) = 0$ when n is odd. Moreover, $P_\nu^\mu(x)$ and $Q_\nu^\mu(x)$ are independent solutions to

$$(1-x^2)\frac{d^2u}{dx^2} - 2x\frac{du}{dx} + \left\{\nu(\nu+1) - \frac{\mu^2}{1-x^2}\right\}u = 0$$

for $x \in (-1,1)$.

In the odd dimensional case, they can be written by a finite number of combinations of elementary functions, however, in the even dimensional case, they cannot be expressed by combinations of elementary functions.

In a general dimension, according to the value of λ, we divide the parametrization into two cases:

$$\lambda = -\{\frac{(n-1)^2}{4} - \mu^2\}, \quad \lambda = -\{\frac{(n-1)^2}{4} + \nu^2\}.$$

The parametrization coincides with that in Section 4 when $n = 3$.

The asymptotic behavior of the associated Legendre functions is known but very complicated. We do not consider it in this note.

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