Symmetry-breaking bifurcation of positive solutions to a one-dimensional Liouville type equation (Qualitative theory of ordinary differential equations in real domains)

田中 敏

数理解析研究所講究録 数理科学研究所講究録

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Symmetry-breaking bifurcation of positive solutions to a one-dimensional Liouville type equation

Satoshi Tanaka

Faculty of Science,
Okayama University of Science

We consider the two-point boundary value problem for the one-dimensional Liouville type equation

\[
\begin{align*}
\{ & u'' + \lambda|x|^l e^u = 0, \quad x \in (-1, 1), \\
& u(-1) = u(1) = 0,
\end{align*}
\]

(1)

where \( \lambda > 0 \) and \( l > 0 \).

Jacobsen and Schmitt [2] studied the exact multiplicity of radial solutions of the problem for the multi-dimensional Liouville type equation

\[
\begin{align*}
\{ & \Delta u + \lambda|x|^l e^u = 0 \text{ in } B, \\
& u = 0 \text{ on } \partial B,
\end{align*}
\]

(2)

where \( l \geq 0 \) and \( B := \{ x \in \mathbb{R}^n : |x| < 1 \} \). They proved the following (i)--(iii):

(i) if \( 1 \leq N \leq 2 \), then there exists \( \lambda_* > 0 \) such that (2) has exactly two radial solutions for \( 0 < \lambda < \lambda_* \), a unique radial solution for \( \lambda = \lambda_* \) and no radial solution for \( \lambda > \lambda_* \);

(ii) if \( 3 \leq N < 10 + 4l \), then (2) has infinitely many radial solutions for \( \lambda = (l + 2)(N-2) \) and a finite but large number of radial solutions when \( |\lambda-(l+2)(N-2)| \) is sufficiently small;

(iii) if \( N \geq 10 + 4l \), then (2) has a unique radial solution for \( 0 < \lambda < (l+2)(N-2) \) and no radial solution for \( \lambda \geq (l+2)(N-2) \).

We note here that every solution of (2) is positive in \( B \), by the strong maximum principle. Result (i)--(iii) were established by Joseph and Lundgren [3] for the case \( l = 0 \), that is, for the Liouville equation

\[
\begin{align*}
\{ & \Delta u + \lambda e^u = 0 \text{ in } \Omega, \\
& u = 0 \text{ on } \partial \Omega,
\end{align*}
\]

(3)

when \( \Omega = B \). Gidas, Ni and Nirenberg's theorem ([1]) shows that every positive solution of (3) is radially symmetric when \( \Omega = B \). However, when \( \Omega \) is an annulus \( A := \{ x \in \mathbb{R}^N : a < |x| < b \}, a > 0 \), problem (3) may have non-radial solutions. Indeed, Lin [4] proved that (3) has infinitely many symmetry-breaking bifurcation points when \( N = 2 \) and \( \Omega = A \). Nagasaki and Suzuki [6] found that large non-radial solutions of (3) when \( N = 2 \) and \( \Omega = A \). More precisely, for each sufficiently large \( \mu > 0 \), there exist \((\lambda, u)\) such that \( \lambda > 0 \), \( u \) is a non-radial solution of (3) and \( \int_A e^u dx = \mu \) when \( N = 2 \) and \( \Omega = A \).
Recently, Miyamoto [5] considered the problem for the Liouville type equation (2) and proved the following result.

**Theorem A** ([5]). Let $n_0$ be the largest integer that is smaller than $1 + \frac{1}{2}$ and let \( \alpha_n := 2 \log \frac{2l+4}{2l+2} \). All the radial solutions of (2) with $N = 2$ can be written explicitly as

\[
\lambda(\alpha) = 2(l + 2)^2(e^{-\alpha/2} - e^{-\alpha}), \quad U(r; \alpha) = \log \frac{e^{\alpha}}{(1 + (e^{\alpha/2} - 1)r^{l+2})^2}.
\]

The radial solutions can be parameterized by the $L^\infty$-norm, it has one turning point at \( \lambda = \lambda(\alpha_0) = (l + 2)/2 \), and it blows up as \( \lambda \downarrow 0 \). For each \( n \in \{1, 2, \cdots, n_0\} \), \( (\lambda(\alpha_n), U(r; \alpha_n)) \) is a symmetry breaking bifurcation point from which an unbounded branch consisting of non-radial solutions of (2) with $N = 2$ emanates, and $U(r; \alpha)$ is nondegenerate if $\alpha \neq \alpha_n$, $n = 0, 1, \cdots, n_0$. Each non-radial branch is in \( (0, \lambda(\alpha_0)) \times \{ u > 0 \} \subset \mathbb{R} \times H_0^2(B) \).

When $N = 2$, radial solutions of problems (2) and (3) can be written explicitly, and hence, Lin [4] and Miyamoto [5] succeeded to show the existence of bifurcation points. That is difficult even if we know exact solutions, much more difficult if we do not know them usually. When $N \neq 2$, we do not find exact radial solutions of (2). However, the structure of eigenvalues and eigenfunctions of the linearized problem in the dimension 1 is well-known, and then, by the comparison function introduced in [7], we can find the Morse indices of even solutions of (1). Then we obtain the existence of a symmetry-breaking bifurcation point of (1).

Let $m(U)$ be the Morse index of a solution $U$ to (1), that is, the number of negative eigenvalues $\mu$ of

\[
\left\{
\begin{array}{l}
\phi'' + \lambda|x|^l e^{U(x)} \phi + \mu \phi = 0, \quad x \in (-1, 1), \\
\phi(-1) = \phi(1) = 0,
\end{array}
\right.
\]

A solution $U$ of (1) is said to be degenerate if $\mu = 0$ is an eigenvalue of (4). Otherwise, it is said to be nondegenerate.

The main result is as follows.

**Theorem 1.** For each $\alpha > 0$, there exists a unique \( (\lambda(\alpha), U(x; \alpha)) \) such that (1) with \( \lambda = \lambda(\alpha) \) has a unique positive even solution $U = U(x; \alpha)$ such that $||U||_\infty = \alpha$. Moreover, there exist $\alpha_*, \alpha_1, \alpha_2$ and $\alpha_3$ such that $\alpha_* < \alpha_1 \leq \alpha_2 \leq \alpha_3$ and the following (i)-(vii) hold:

(i) if $0 < \alpha < \alpha_*$, then $m(U) = 0$ and $U(x; \alpha)$ is nondegenerate;
(ii) if $\alpha = \alpha_*$, then $m(U) = 0$ and $U(x; \alpha)$ is degenerate;
(iii) if $\alpha_* < \alpha < \alpha_1$, then $m(U) = 1$ and $U(x; \alpha)$ is nondegenerate;
(iv) if $\alpha = \alpha_1$, then $m(U) = 1$ and $U(x; \alpha)$ is degenerate;
(v) if $\alpha = \alpha_2$, then $m(U) = 1$, $U(x; \alpha)$ is degenerate and $(U, \lambda)$ is a non-even bifurcation point, that is, for each $\varepsilon > 0$, there exists $(\lambda, u)$ such that $u$ is a non-even positive solution of (1) and $|\lambda - \lambda(\alpha_2)| + ||u - U(\cdot, \alpha_2)||_\infty < \varepsilon$;
(vi) if $\alpha = \alpha_3$, then $m(U) = 2$ and $U(x; \alpha)$ is degenerate;
(vii) if $\alpha > \alpha_3$, then $m(U) = 2$ and $U(x; \alpha)$ is nondegenerate.
Here and Hereafter, we use the notation $\|U\|_\infty = \sup_{x \in [-1,1]} U(x)$.

For the proof of Theorem 1, see [8]. Here, we give a sufficient condition for the second eigenvalue of the linearized problem to be negative for the following problem

$$
\begin{align*}
\begin{cases}
   u'' + \lambda h(x)f(u) = 0, & x \in (-1,1), \\
   u(-1) = u(1) = 0.
\end{cases}
\end{align*}
$$

where $\lambda > 0$ and $h \in C^1([-1,0) \cup (0,1]) \cap C[-1,1]$, $h(-x) = h(x)$, $h(x) > 0$ and $h'(x) \geq 0$ for $x > 0$, $f \in C^1[0,\infty)$, $f(s) > 0$ and $f'(s) \geq 0$ for $s > 0$. Namely we will show the following result, which plays a crucial role in the proof of Theorem 1.

**Proposition 1.** Assume that, for each sufficiently large $\alpha > 0$, there exist $\lambda(\alpha) > 0$ and $U(x; \alpha)$ such that $U(x; \alpha)$ is a positive even solution of (5) at $\lambda = \lambda(\alpha)$ and $\|U(\cdot; \alpha)\|_\infty = \alpha$. Assume moreover that there exist $s_0 > 0$ and $\delta > 0$ such that

$$
\begin{align*}
   \frac{l(x)(g(s) - 1) - 4}{g(s) + l(x) + 3} \geq \delta, & \quad x \in (0,1], s \geq s_0,
\end{align*}
$$

where $l(x) = xh'(x)/h(x)$ and $g(s) = sf'(s)/f(s)$. Let $\mu_2(\alpha)$ be the second eigenvalue of

$$
\begin{align*}
\begin{cases}
   \phi'' + \lambda(\alpha)h(x)f'(U(x; \alpha))\phi + \mu \phi = 0, & x \in (-1,1), \\
   \phi(-1) = \phi(1) = 0.
\end{cases}
\end{align*}
$$

Then $\mu_2(\alpha) < 0$ for all sufficiently large $\alpha > 0$.

In the case where $h(x) = |x|^l$, $l > 0$ and $f(s) = e^s$, it follows that $l(x) = xh'(x)/h(x) = l$ for $x \in (0,1]$ and $g(s) = sf'(s)/f(s) = s$, and hence (6) is satisfied.

We conclude that if $U$ is a positive even solution of (1) and $\|U\|_\infty \leq 1$, then $m(U) = 0$. Indeed, let $\mu_1$ be the first eigenvalue of (4) and let $\phi_1$ be an eigenfunction corresponding to $\mu_1$. We may assume that $\phi_1(x) > 0$ on $(-1,1)$. Integrating the equality

$$
(\phi_1(x)U'(x) - \phi_1'(x)U(x))' = \mu_1 \phi_1(x)U(x) + \lambda|x|^l e^{U(x)}\phi_1(x)(U(x) - 1)
$$
on $[-1,1]$, we have

$$
\mu_1 \int_{-1}^{1} \phi_1(x)U(x)dx = \lambda \int_{-1}^{1} |x|^l e^{U(x)}\phi_1(x)(1 - U(x))dx > 0.
$$

Consequently, we have $\mu_1 > 0$, which means $m(U) = 0$. By applying Proposition 1, we can conclude that $m(U(\cdot; \alpha)) = 0$ for $0 < \alpha \leq 1$ and $m(U(\cdot; \alpha)) \geq 2$ for all sufficiently large $\alpha > 1$. Then, using the Leray-Schauder degree, we can find a bifurcation point.

To prove Proposition 1, we need the following two lemmas.

**Lemma 1.** Let $\phi_2$ be an eigenfunction corresponding to the second eigenvalue $\mu_2(\alpha)$ of (7). Then $\phi_2$ is odd, $\phi_2(0) = \phi_2(1) = 0$ and $\phi_2(x) \neq 0$ for $x \in (0,1)$.

**Proof.** Let $M_1$ be the first eigenvalue of

$$
\begin{align*}
\begin{cases}
   \Phi'' + \lambda(\alpha)h(x)f'(U(x; \alpha))\Phi + M_1 \Phi = 0, & x \in (0,1), \\
   \Phi(0) = \Phi(1) = 0,
\end{cases}
\end{align*}
$$
and let $\Phi_1$ be an eigenfunction corresponding to $M_1$. Then $\Phi_1(0) = \Phi_1(1) = 0$ and $\Phi_1(x) \neq 0$ on $(0, 1)$. Set

$$\Phi(x) = \begin{cases} 
\Phi_1(x), & x \in [0, 1], \\
-\Phi_1(-x), & x \in [-1, 0].
\end{cases}$$

Noting the fact that $\lim_{x \to 0^-} \Phi''(x) = \lim_{x \to 0^+} \Phi''(-x) = -\Phi_1''(0) = 0$, we easily check that $\Phi$ is a solution of

$$\begin{cases} 
\Phi'' + \lambda(\alpha)h(x)f'(U(x;\alpha))\Phi + M_1\Phi = 0, & x \in (-1, 1), \\
\Phi(-1) = \Phi(1) = 0,
\end{cases}$$

and $\Phi$ is odd, $\Phi(x) \neq 0$ on $(0, 1)$ and $\Phi(0) = 0$. Therefore, $M_1$ is an eigenvalue of (7) and $\Phi$ is an eigenfunction corresponding to $M_1$. Since $\Phi$ has exactly one zero in $(-1, 1)$, $M_1$ must be $\mu_2$ and hence $\phi_2(x)$ must be $c\Phi(x)$ for some $c \neq 0$. \hfill \square

**Lemma 2.** Assume that $w \in C[a, b]$ is positive and concave on $(a, b)$. Let $\rho \in (0, 1/2)$. Then $w(x) \geq \rho \max_{\zeta \in [a, b]} w(\xi)$ for $x \in [(1-\rho)a + \rho b, \rho a + (1-\rho)b]$.

**Proof.** We take $c \in [a, b]$ for which $w(c) = \max_{\xi \in [a, b]} w(\xi)$. Then $w(c) > 0$. Since $w$ is positive and concave on $(a, b)$, we have

$$w(x) \geq \frac{w(c)(x-a)}{c-a} \geq \frac{w(c)(x-a)}{b-a} =: l_1(x), \quad x \in [a, c],$$

and

$$w(x) \geq \frac{w(c)(b-x)}{b-c} \geq \frac{w(c)(b-x)}{b-a} =: l_2(x), \quad x \in [c, b].$$

Hence $w(x) \geq \min\{l_1(x), l_2(x)\}$ on $[a, b]$. We conclude that if $x \in [(1-\rho)a + \rho b, (a+b)/2]$, then

$$\min\{l_1(x), l_2(x)\} = l_1(x) \geq l_1((1-\rho)a + \rho b) = \rho w(c),$$

and if $x \in [(a+b)/2, \rho a + (1-\rho)b]$, then

$$\min\{l_1(x), l_2(x)\} = l_2(x) \geq l_2(\rho a + (1-\rho)b) = \rho w(c).$$

The proof is complete. \hfill \square

Now we are ready to show Proposition 1.

**Proof of Proposition 1.** Let $\alpha > 0$ be sufficiently large. We use the following comparison function $y(x)$ introduced in [7]:

$$y(x) = xU(x;\alpha) - (x-1)^2U'(x;\alpha).$$

This function $y(x)$ satisfies $y(0) = y(1) = 0$, $y(x) > 0$ on $(0, 1)$, and

$$y'' + \lambda(\alpha)h(x)f'(U(x;\alpha))y = \lambda(\alpha)x^{-1}h(x)H(x;\alpha)f(U(x;\alpha))$$

for $x \in (0, 1)$, where

$$H(x;\alpha) = (1-x)^2l(x) + x(3x-4) + x^2g(U(x;\alpha)).$$
Let $\phi_2(x; \alpha)$ be an eigenfunction corresponding to $\mu_2(\alpha)$. From Lemma 1, it follows that $\phi_2(0; \alpha) = \phi_2(1; \alpha) = 0$ and $\phi_2(x; \alpha) \neq 0$ for $x \in (0, 1)$. Without loss of generality, we may assume that $\phi_2(x; \alpha) > 0$ for $x \in (0, 1)$ and $\max_{\zeta \in [0,1]} \phi_2(\zeta; \alpha) = 1$. We observe that

$$ (y \phi_2 - y \phi_2')' = \mu_2(\alpha) \phi_2 y + \lambda(\alpha) x^{-1} h(x) H(x; \alpha) f(U(x; \alpha)) \phi_2, \quad x \in (0, 1]. $$

Integrating this equality on $(0, 1)$, we obtain

$$ \mu_2(\alpha) \int_{0}^{1} \phi_2(x; \alpha) y(x) dx + \lambda(\alpha) \int_{0}^{1} x^{-1} h(x) H(x; \alpha) f(U(x; \alpha)) \phi_2(x; \alpha) dx = 0. \quad (8) $$

Since

$$ H(x) = \frac{[g(U(x; \alpha)) + l(x) + 3] (x - \frac{l(x) + 2}{g(U(x; \alpha)) + l(x) + 3})^2 + \frac{l(x) [g(U(x; \alpha)) - 1] - 4}{g(U(x; \alpha)) + l(x) + 3}} \geq \frac{l(x) [g(U(x; \alpha)) - 1] - 4}{g(U(x; \alpha)) + l(x) + 3}, $$

we have

$$ \int_{0}^{1} x^{-1} h(x) H(x; \alpha) f(U(x; \alpha)) \phi_2(x; \alpha) dx \geq \int_{0}^{1} x^{-1} h(x) \frac{l(x) [g(U(x; \alpha)) - 1] - 4}{g(U(x; \alpha)) + l(x) + 3} f(U(x; \alpha)) \phi_2(x; \alpha) dx. \quad (9) $$

Since $U''(x; \alpha) = -\lambda(\alpha) h(x) f(U(x; \alpha)) < 0$ on $(0, 1)$, we find that $U'(x; \alpha)$ is decreasing in $x \in (0, 1)$. From $U'(0; \alpha) = 0$ it follows that $U'(x; \alpha) < 0$ for $x \in (0, 1]$, which implies that $U(x; \alpha)$ is also decreasing in $x \in (0, 1]$. Then there exists $x(\alpha) \in (0, 1)$ such that $U(x; \alpha) \geq s_0$ for $x \in [0, x(\alpha)]$ and $U(x; \alpha) < s_0$ for $x \in (x(\alpha), 1]$. Since $U(x; \alpha)$ is concave on $(0, 1)$, we conclude that

$$ U(x; \alpha) \geq \alpha(1 - x), \quad x \in [0, 1], $$

which shows that if $x \in [0, (\alpha - s_0)/\alpha]$, then $U(x; \alpha) \geq s_0$. Therefore, $x(\alpha) \geq (\alpha - s_0)/\alpha$, which implies

$$ \lim_{\alpha \to \infty} x(\alpha) = 1. \quad (10) $$

We take $s_1 \geq s_0$ for which $x(\alpha) \geq 3/4$ for $\alpha \geq s_1$. If $\alpha \geq s_1$, then (6) implies

$$ \int_{0}^{x(\alpha)} x^{-1} h(x) \frac{l(x) [g(U(x; \alpha)) - 1] - 4}{g(U(x; \alpha)) + l(x) + 3} f(U(x; \alpha)) \phi_2(x; \alpha) dx \geq \delta f(s_0) \int_{0}^{x(\alpha)} x^{-1} h(x) \phi_2(x; \alpha) dx \geq \delta f(s_0) \int_{1/4}^{3/4} x^{-1} h(x) \phi_2(x; \alpha) dx. \quad (11) $$
Recalling $\max_{\xi \in [0,1]} \phi_2(\xi) = 1$, we have

\begin{equation}
\int_{x_0}^1 x^{-1} h(x) \frac{l(x)[g(U(x; \alpha)) - l(x) + 3 f(U(x; \alpha))]\phi_2(x; \alpha)}{l(x) + 3} \, dx \\
\geq - \int_{x_0}^1 x^{-1} h(x) \frac{(l(x) + 4)f(U(x; \alpha))\phi_2(x; \alpha)}{g(U(x; \alpha)) + l(x) + 3} \, dx \\
\geq - f(s_0) \int_{x_0}^1 x^{-1} h(x) \frac{l(x) + 4}{l(x) + 3} \, dx.
\end{equation}

Now we will show that there exists $s_2 \geq s_1$ such that $\mu_2(\alpha) < 0$ for $\alpha \geq s_2$. Assume to the contrary that there exists $\{\alpha_n\}_{n=1}^{\infty}$ such that $\mu_2(\alpha_n) \geq 0$ and $\alpha_n \geq s_1$ for $n \in \mathbb{N}$ and $\lim_{n \to \infty} \alpha_n = \infty$.

Since $\phi_2(x; \alpha_n) > 0$ and

$$\phi'_2(x; \alpha_n) = -h(x)f'(U(x; \alpha_n))\phi_2(x; \alpha_n) - \mu_2(\alpha_n)\phi_2(x; \alpha_n) \leq 0, \quad x \in (0, 1),$$

we find that $\phi_2(x; \alpha_n)$ is concave on $(0, 1)$. From Lemma 2 with $\rho = 1/4$, $a = 0$ and $b = 1$, it follows that

$$\phi_2(x; \alpha_n) \geq \frac{1}{4} \max_{\xi \in [0,1]} \phi_2(\xi; \alpha_n) = \frac{1}{4}, \quad x \in \left[\frac{1}{4}, \frac{3}{4}\right].$$

By (11), we have

\begin{equation}
\int_{0}^{x_0(x)} x^{-1} h(x) \frac{l(x)[g(U(x; \alpha)) - l(x) + 3 f(U(x; \alpha))]\phi_2(x; \alpha)}{l(x) + 3} \, dx \\
\geq \frac{\delta f(s_0)}{4} \int_{1/4}^{3/4} x^{-1} h(x) \, dx.
\end{equation}

Combining (8) with (9), (12) and (13), we have

$$0 \geq -\mu_2(\alpha_n) \int_{0}^{1} \phi_2(x; \alpha_n) y(x) \, dx \\
\geq \lambda(\alpha_n) f(s_0) \left[ \frac{\delta}{4} \int_{1/4}^{3/4} x^{-1} h(x) \, dx - \int_{x_0(x)}^{1} x^{-1} h(x) \frac{l(x) + 4}{l(x) + 3} \, dx \right],$$

which implies

$$\int_{x_0(x_0)}^{1} x^{-1} h(x) \frac{l(x) + 4}{l(x) + 3} \, dx \geq \frac{\delta}{4} \int_{1/4}^{3/4} x^{-1} h(x) \, dx > 0, \quad n \in \mathbb{N}.$$ 

This contradicts the fact (10). Consequently, there exists $s_2 \geq s_1$ such that $\mu_2(\alpha) < 0$ for $\alpha \geq s_2$. This completes the proof.

**References**


岡山理科大学・理学部 田中 敏